White noise generalization of the Clark-Ocone formula under change of measure

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Introduction

- Gaussian white noise theory was first introduced by Hida [Hida82].
- Contribution of white noise theory to finance, $F \in \mathcal{G}^* \supset L^2(P)$,

$$F(\omega) = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t] \diamond \dot{W}(t)dt.$$

  [AØPJ00]
- $F \in \mathbb{D}_{1,2}$

$$F(\omega) = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t]dW(t)$$

  [Ocone84]

Introduction

- $W(t)$ be a Brownian motion on the filtered white noise probability space $(\Omega, \mathcal{B}, \{\mathcal{F}_t\}_{t\geq 0}, P)$
- Let Q be the probability measure equivalent P such that $\hat{W}(t)$ is a Brownian motion with respect to Q, in virtue of the Girsanov theorem.
- $\hat{W}_t$ be defined as $d\hat{W}(t) = u(t) + dW(t)$, where $u(t)$ is an $\mathcal{F}_t$-measurable process satisfying certain conditions.

- $$L^2(P) = \{F : \Omega \to \mathbb{R} \; s.t. \; \| F \|^2_{L^2(P)} := \int_\Omega F^2(\omega) dP(\omega) < \infty\}.$$

- $$F \in L^2(\mathcal{F}_t, P), \quad F = \sum_{n=0}^{\infty} I_n(f_n)$$

  - $\| F \|_{L^2(P)} = \sum_{n=0}^{\infty} n! \| f_n \|_{L^2[0,T]^n}$
  - $\| F \|_{\mathbb{D}_{1,2}} := \sum_{n=0}^{\infty} nn! \| f_n \|_{L^2[0,T]^n}$

- $\mathbb{D}_{1,2} \subset L^2(P)$

$F \in L^2(P)$

$$F(\omega) = E_Q[F] + \int_0^T E_Q[(D_t F - F \int_t^T D_t u(s) d\hat{W}(s)) \mid \mathcal{F}_t] \, d\hat{W}(t).$$

[YO07]

White Noise Analysis

- $S = S(\mathbb{R})$ be the Schwartz space of rapidly decreasing smooth functions $\phi \in C^{\infty}(\mathbb{R})$ such that

$$\| \phi \|_{k,n} = \sup_{x \in \mathbb{R}} | \phi^{(j)}(x) x^n | < \infty \quad \text{for all } j \leq k,\ n \leq N.$$

  The space $S(\mathbb{R})$ equipped with the family of seminorms $\| \, . \, \|_{k,N}$ constitutes a Frechet space.
- $S^{'}(\mathbb{R})$ is called the space of tempered distributions equipped with the weak-star topology. If $\omega \in S^{'}(\mathbb{R})$ and $\phi \in S(\mathbb{R})$ we set

$$\omega(\phi) = < \omega, \phi >$$

  that denotes the action of the linear functional ω on the test function ϕ.

Bochner-Minlos

Theorem

Let $g : S(\mathbb{R}) \to \mathbb{R}$ be a given function. Then there exists a probability measure P on $\Omega := S'(\mathbb{R})$ s.t.

$$\int_{S'(\mathbb{R})} e^{i<\omega,\phi>} dP(\omega) = g(\phi) \;\; ; \quad \phi \in S(\mathbb{R}) \tag{1}$$

if and only if

- *g(0)=1*
- *g is positive definite*
- *g is continuous in Frechet topology.*

White noise analysis

$$g(\phi) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}$$

$$\int_{S'(\mathbb{R})} e^{i<\omega,\phi>} dP(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}$$

- $$E[<\omega,\phi>] := \int_{S'(\mathbb{R})} <\omega,\phi> dP(\omega) = 0 \qquad (2)$$
- $$E[<\omega,\phi>^2] := \int_{S'(\mathbb{R})} <\omega,\phi>^2 \, dP(\omega) = \| \phi \|^2_{L^2(\mathbb{R})} \qquad (3)$$

Construction of Brownian motion

- Extend the definition of $< \omega, \phi >$ from $\phi \in S(\mathbb{R})$ to $\phi \in L^2(\mathbb{R})$ by using the properties above and the fact that $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and $L^2(P)$ is complete.
- $$\tilde{W}(t,\omega) = \tilde{W}(t) =< \omega, \chi_{[0,t]}(.) >$$
  where $\chi_{[0,t]}$ belongs to $L^2(\mathbb{R})$ for all t and defined as follows:
  $$\chi_{[0,t]}(s) = \begin{cases} 1 & \text{if} \quad s \in [0,t] \ \text{ or } s \in [t,0] (t<0) \\ 0 & \text{otherwise} \end{cases}$$

Construction of Brownian motion cont.

- $\tilde{W}(t)$ is a gaussian process with mean o and variance t.
- By the Kolmogorov continuity theorem it has continuous version $W(t) = W(t, \omega)$[Loève78].
- Note that since $W(t, \omega)$, $\omega \in \Omega = S^{'}(\mathbb{R})$ is constructed in a special way ω can be regarded as a tempered distribution.

Hermite polynomials

- constitute an orthogonal basis for $L^2(\mathbb{R}, \mu(dx))$ if $\mu = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}$.
- $$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}),\ x \in \mathbb{R}, n = 0, 1, 2, \ldots$$
- $h_{n+1}(x) = xh_n(x) - nh_{n-1}(x),\ n \geq 1$
- $h_n^{'}(x) = nh_{n-1}(x),\ n \geq 1$

For the tensor power of $g \in L^2([0,T])$

$$n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1) \cdots g(t_n)\, dW(t_1) \cdots dW(t_n) = \| g \|^n \, h_n\left(\frac{\theta}{\| g \|}\right).$$

where

$$\theta = < \omega, g > = \int_0^T g(t) dW(t)$$

$$\| g \| = \| g \|_{L^2([0,T])}$$

[Itô51]

- $$J_n(g) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1, t_2, \cdots, t_n)\, dW(t_1) \cdots dW(t_n)$$
- $$n! J_n(g) = I_n(g)$$
- $$I_n(g) = \int_0^T \int_0^T \cdots \int_0^T g(t_1, t_2, t_3, \cdots, t_n) dW(t_1) \cdots dW(t_n).$$

[DØP]

Hermite Functions

- $$e_k(x) = \pi^{-1/4}((k-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_{k-1}(\sqrt{2}x), \; k = 1, 2, 3, \ldots \tag{4}$$
- $\{e_k\}_{k\geq 1}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$ and $e_k \in S(\mathbb{R})$ for all k.

Wiener-Itô Chaos Expansion I

Theorem

Let $F(\omega)$ be an $\mathcal{F}_T$-measurable random variable in $L^2(P)$

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

where $f_n \in \hat{L}^2([0,T]^n)$ and the convergence is in $L^2(P)$.

$$\| F \|^2_{L^2(P)} = E[F^2] = \sum_{n=0}^{\infty} n! \| f_n \|^2_{L^2([0,T]^n)}$$

[DØP]

Wiener-Itô Chaos Expansion II

Define

$$\theta_k(\omega) := <\omega, e_k> = \int_{\mathbb{R}} e_k(x) dW(x), \quad \omega \in \Omega \tag{5}$$

Let $\mathcal{I}$ be the set of all finite multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, where $\alpha_m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, m=1,2,... and $|\alpha| = \sum_{i=1}^{m} \alpha_i$. Define

$$H_\alpha := \prod_{j=1}^{m} h_{\alpha_j}(\theta_j(\omega)), \quad \omega \in \Omega. \tag{6}$$

The family $\{H_\alpha\}_{\alpha \in \mathcal{I}}$ is an orthogonal sequence that constitutes basis for the Hilbert space $L^2(P)$.

Wiener-Itô Chaos Expansion II

Theorem

For all $F \in L^2(P)$ there exists unique constants $c_\alpha \in \mathbb{R}$ such that

$$F = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha. \qquad (7)$$

Moreover, there exists the following equality

$$\| F \|^2_{L^2(P)} = \sum_{\alpha \in \mathcal{I}} \alpha! \, c_\alpha^2,$$

where $\alpha! = \alpha_1!\alpha_2! \dots \alpha_m!$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$.

[DØP]

Relation between Chaos Expansions

$$f_n = \sum_{\alpha \in \mathcal{I} : |\alpha| = n} c_\alpha e^{\hat{\otimes}\alpha}, \quad n = 0, 1, 2, \ldots,$$

where

$$e^{\hat{\otimes}\alpha} = e_1^{\otimes\alpha_1} \hat{\otimes} e_2^{\otimes\alpha_2} \hat{\otimes} \ldots \hat{\otimes} e_m^{\otimes\alpha_m}$$

and $\mid \alpha \mid = \sum_{i=1}^{m} \alpha_i$ then

$$F = \sum_{n=0}^{\infty} I_n(f_n) = \sum_{\alpha \in \mathcal{I}, |\alpha| = n} c_\alpha H_\alpha$$

Hida Stochastic Test Function Space

$(2\mathbb{N})^\alpha = \prod_{i=1}^m (2i)^{\alpha_i}$, for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathcal{I}$.

Definition

$f = \sum_{\alpha \in \mathcal{I}} a_\alpha H_\alpha \in L^2(P)$ belongs to the Hida test function Hilbert space $(S)_k$ for k $\in \{1, 2, \ldots\}$ if

$$\| f \|_k^2 := \sum_{\alpha \in \mathcal{I}} \alpha! \, a_\alpha^2 \, (2\mathbb{N})^{\alpha k} < \infty$$

Hida space of stochastic test functions (S) as

$$(S) = \bigcap_{k=1}^{\infty} (S)_k, \quad \text{(projective topology)} \tag{8}$$

Hida Stochastic Distribution Space

Let $(S)_{-q}$, $q = 1, 2, \ldots$, be the set of all expansions $F = \sum_{\alpha \in \mathcal{I}} b_\alpha H_\alpha$ such that

$$\| F \|_{-q}^2 := \sum_{\alpha \in \mathcal{I}} b_\alpha^2 \, \alpha! \, (2\mathbb{N})^{-\alpha q} < \infty,$$

Hida space of stochastic distributions (S) as

$$(S)^* = \bigcup_{q=1}^{\infty} (S)_{-q}, \quad \text{(inductive topology)} \tag{9}$$

Wick Product

- $(S)^*$ is the dual of (S) , the action of $F = \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha \in (S)^*$ on $f = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha \in L^2(P) \in (S)$
$$< F, f >= F(f) = \sum_{\alpha\in\mathcal{I}} a_\alpha\, b_\alpha \alpha!$$
- $$(S) \subset (S)_k \subset L^2(P) \subset (S)_{-q} \subset (S)^*, \quad \text{for all} \quad k, q.$$
- $X = \sum_\alpha a_\alpha\, H_\alpha \in (S)^*$, $Y = \sum_\beta b_\beta\, H_\beta \in (S)^*$
$$X \diamond Y := \sum_{\alpha,\,\beta} a_\alpha\, b_\beta\, H_{\alpha+\beta}$$

The (Hida)Malliavin Derivative

(i) Let $F \in L^2(P)$ be random variable and let $\gamma \in L^2(\mathbb{R}) \subset \Omega$. Then the directional (or Gateaux) derivative of F in the direction of γ

$$D_\gamma F(\omega) = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon} \tag{10}$$

if the limit exists in $(S)^*$.

(ii)

$$D_\gamma F(\omega) = \int_{\mathbb{R}} \psi(t)\gamma(t)dt \quad \text{for all} \quad \gamma \in L^2(\mathbb{R}),$$

then we say that F is (Hida) Malliavin differentible

$$D_t F = \psi(t) \quad t \in \mathbb{R}.$$

Chain Rule

Theorem

Suppose $F \in L^2(P)$ is (Hida) Malliavin differentiable. Let $\Phi \in \mathcal{C}^1(\mathbb{R})$ and assume that $D_t F \in L^2(P)$ and $\Phi'(F) D_t F \in L^2(\lambda \times P)$ where λ is the Lebesgue measure on $\mathbb{R}$. Then $\Phi(F)$ is Malliavin differentiable and

$$D_t(\Phi(F)) = \Phi'(F) D_t F \tag{11}$$

[DØP]

Results of Chain Rule

- $$D_t(F_1F_2) = F_1D_tF_2 + F_2D_tF_1$$
- $$D_t(h_n(<\omega, f>)) = nh_{n-1}(<\omega, f>)f(t).$$

Lemma

Let $F = I_n(f_n) \in L^2(P)$

$$D_tF = nI_{n-1}(f_n(., t)), \tag{12}$$

where $I_{n-1}(f_n(., t))$ *stands for the (n-1)-iterated Itô integral which is defined with respect to the* $n-1$ *first variables* $t_1, \ldots, t_{n-1}$ *of* $f_n(t_1, t_2, \ldots, t_{n-1}, t)$.

Lemma

(i) *Let* $G \in (S)^*$*. Then* $D_t G \in (S)^*$ *for a.a.* $t \in \mathbb{R}$*.*

(ii) *Suppose* $G, G_n \in (S)^*$ *for all* $n \in \mathbb{N}$ *and*

$$G_n \to G \ \ in \ \ (S)^*.$$

Then there exist a subsequence $\{G_{n_k}\}_{k \geq 1}$ *such that*

$$D_t G_{n_k} \to D_t G \ \ in \ \ (S)^*,$$

for almost all $t > 0$*.*

[YO07]

Skorohod Integral

Let $u(t) \in L^2(P), \mathcal{F}_T-$measurable stochastic process.

$$u(t) = \sum_{n=0}^{\infty} I_n(f_n(.,t))$$

$$\delta(u) := \int_0^T u(t)\delta W(t) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n), \tag{13}$$

where $\tilde{f}_n$ are the symmetric functions derived from $f_n(.,t)$ for $n = 1, 2, \ldots$. Note that

$$E[\delta(u)^2] = \sum_{n=0}^{\infty} (n+1)! \parallel \tilde{f}_n \parallel^2_{L^2([0,T]^{n+1})} < \infty.$$

Construction of spaces $\mathcal{G}$ and $\mathcal{G}^*$

(i) Let $\lambda \in \mathbb{R}$. $\mathcal{G}_\lambda$ consists of all formal expansions

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

$$\| F \|_{\mathcal{G}_\lambda}^2 := \sum_{n=0}^{\infty} n! \| f_n \|_{L^2(\mathbb{R}^n)}^2 e^{2\lambda n} < \infty.$$

(ii) Define

$$\mathcal{G} = \bigcap_{\lambda \in \mathbb{R}} \mathcal{G}_\lambda \quad \mathcal{G}^* = \bigcup_{\lambda \in \mathbb{R}} \mathcal{G}_\lambda$$

$\mathcal{G}^*$ is the dual of $\mathcal{G}$.

Lemma

(i) *Suppose* $F \in \mathcal{G}^*$. *Then* $D_t F \in \mathcal{G}^*$ *for a.a.* $t \in \mathbb{R}$.

(ii) *Suppose* $F, F_n \in \mathcal{G}^*$ *for all* $n \in \mathbb{N}$ *and*

$$F_n \to F \quad in \quad \mathcal{G}^*.$$

Then there exists a subsequence $\{F_{n_k}\}_{k \geq 1}$ *such that*

$$D_t F_{n_k} \to D_t F \quad in \quad \mathcal{G}^*, \tag{14}$$

for almost all $t > 0$.

[AØPJ00]

- $$(S) \subset \mathcal{G} \subset L^2(P) \subset \mathcal{G}^* \subset (S)^*$$

Theorem

Let $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(P)$. *Then*

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(., t)). \tag{15}$$

[YO07]

Sketch of the proof

- Let $F \in L^2(P)$. Define $F_m = \sum_{n=0}^{\infty} I_n(f_n)$
- $F_m \to F$ in $L^2(P)$ implies $F_m \to F$ in $\mathcal{G}^*$.
- By Lemma, there exists a subsequence F_{m_k} such that $D_t F_{m_k} \to D_t F$ in $\mathcal{G}^*$, i.e.,

$$\sum_{n=0}^{m_k} n I_{n-1}(f_n(., t)) \to D_t F$$

in $\mathcal{G}^*$.

Sketch of the proof

- We want to prove that

$$\sum_{n=0}^{m_k} nI_{n-1}(f_n(.,t)) \to \sum_{n=0}^{\infty} nI_{n-1}(f_n(.,t))$$

-

$$\| \sum_{m_{k+1}}^{\infty} nI_{n-1}(f_n(.,t)) \|^2_{\mathcal{G}_{-q-1}} = \sum_{m_{k+1}}^{\infty} (n-1)! n^2 \| f_n(.,t) \|^2_{L^2(\mathbb{R}^{n-1})} e^{-2q(n-1)}.$$

- Take integral of both sides

$$\int_{\mathbb{R}} \| \sum_{m_{k+1}}^{\infty} nI_{n-1}(f_n(.,t)) \|^2_{\mathcal{G}_{-q}} dt \le K \| F \|_{L^2(P)} < \infty$$

Conditional Expectation in $\mathcal{G}^*$

$F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^*$.

$$E[F \mid F_t] = \sum_{n=0}^{\infty} I_n(f_n \chi_{[0,t]^{\otimes n}}).$$

Note that this coincides with the usual expectation if $F \in L^2(P)$.

$$\| E[F \mid \mathcal{F}_t] \|_{\mathcal{G}_\lambda} \leq \| G \|_{\mathcal{G}_\lambda}, \text{ for all } \lambda \in \mathbb{R}$$

then

$$E[F \mid \mathcal{F}_t] \in \mathcal{G}^* \tag{16}$$

Fundamental Theorem of Calculus

Theorem

Let $u(s,\omega) \in L^2(P)$ be a stochastic process satisfying the following conditions:

(i) $E[\int_0^T u^2(s)ds] < \infty$

(ii) $s \to D_t u(s)$ *is Skorohod integrable in* $(S)^*$*, for all* $t \in [0, T]$

(iii) $E[\int_0^T (\int_0^T D_t(u(s))\delta W(s))^2 dt] < \infty$

then $\delta(u) = \int_0^T u(s)\delta W(s) \in L^2(P)$ *and*

$$D_t(\int_0^T u(s)\delta W(s)) = \int_0^T D_t\, u(s)\,\delta W(s) + u(t). \qquad (17)$$

Corollary

Let $u = u(t) \in L^2(P)$, $t \in [0, T]$, be an $\mathbb{F}$-adapted stochastic process. Then

(i) *$D_s u(t)$ is $\mathbb{F}$ adapted for all s,*

(ii) *$D_s u(t) = 0$ for $s > t$.*

Theorem

(The Clark-Ocone theorem for $L^2(P)$)

Let λ denote the Lebesgue measure on $\mathbb{R}$. Suppose $F(\omega) \in L^2(P)$ be $\mathcal{F}_T$-measurable. Then

$$(t, \omega) \to E[D_t F \mid \mathcal{F}_t](\omega) \in L^2(\lambda \times P)$$

and

$$F(\omega) = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t] dW(t) \tag{18}$$

The proof can be found in Aase et al [AØPJ00].

Girsanov Theorem

Theorem

Let $W(t)$ be a Brownian motion on the white noise probability space $(\Omega, \mathcal{F}, P)$. Let $(u(t))_{0\leq t\leq T}$ be an adapted measurable process satisfying $\int_0^T u^2(s)ds < \infty$ a.s. and such that $Z(T)$ defined by

$$Z(T) = \exp\{-\int_0^T u(s)dW(s) - \frac{1}{2}\int_0^T u^2(s)ds\} \tag{19}$$

is a martingale. Then under the white noise probability measure Q with density $Z(T)$ relative to P, the process $\hat{W}(t)$ defined by $\hat{W}(t) = W(t) + \int_0^t u(s)ds$ is a Brownian motion under measure Q.

Lemma

Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable, Q and $Z(T)$ is defined as in Girsanov theorem. Then

$$D_t(Z(T)F) = Z(T)\{D_t F - F[u(t) + \int_t^T D_t u(s) d\hat{W}(s)]\} \quad (20)$$

[YO07]

Main Result

Theorem

Suppose $F \in L^2(P)$ is $\mathcal{F}_T$-measurable

$$E_Q[\mid F \mid] < \infty$$

$$E_Q[\int_0^T \mid D_t F \mid^2 dt] < \infty$$

$$E_Q[\mid F \mid \int_0^T (\int_0^T D_t u(s) dW(s) + \int_0^T u(s) D_t u(s) ds)^2 dt] < \infty$$

$$F(\omega) = E_Q[F] + \int_0^T E_Q[(D_t F - F \int_t^T D_t u(s) d\hat{W}(s)) \mid \mathcal{F}_t] d\hat{W}(t).$$

[YO07]

Sketch of the proof

- $Y(t) = E_Q[F \mid \mathcal{F}_t]$
- $$\lambda(t) = Z^{-1}(t) = \exp\{\int_0^t u(s)dW(s) + \frac{1}{2}\int_0^t u^2(s)ds\}. \quad (21)$$
- By Itô Formula,
$$d\lambda(t) = \lambda(t)u(s)d\hat{W}(s).$$
- By Bayes' formula (Karatzas and Shreve Lemma 3.5.3. [KS87]) and the settlement of $Y(t)$,
$$\begin{aligned} Y(t) &= E[Z(T)F \mid \mathcal{F}_t]Z^{-1} \\ &= \lambda(t)E[Z(T)F \mid \mathcal{F}_t]. \end{aligned}$$

Sketch of the proof

- By Clark-Ocone formula,

$$E[Z(T)F \mid \mathcal{F}_t] = E[Z(T)F] + \int_0^t E[D_s(Z(T)F) \mid \mathcal{F}_s] dW(s) := U(t)$$

$$dY(t) = \{E_Q[D_t F \mid \mathcal{F}_t] - E_Q[Fu(t) \mid \mathcal{F}_t] - E_Q[F \int_t^T D_t u(s) \, d\hat{W}(s) \mid \mathcal{F}_t] + u(t) E_Q[F \mid \mathcal{F}_t]\} d\hat{W}(t).$$

Market

(i) Risk free asset (Bond)

$$dS_0(t) = \rho(t)S_0(t)dt,$$

$$S_0(0) = 1$$

(ii) Risky asset (Stock)

$$dS_1(t) = \mu(t)S_1(t)dt + \sigma(t)S_1(t)dW(t)$$

$$S_1(0) > 0$$

(iii) Contingent Claim (Digital Option)

$$F = \chi_{[K,\infty)}(W(T))$$

- $$dV^\theta = \theta_0(t)dS_0(t) + \theta_1 dS_1(t).$$

- $$\mu(t) - \rho(t) = \sigma(t)u(t). \tag{22}$$

- $$dV^\theta(t) = \rho(t)V^\theta(t)dt + \sigma(t)\theta_1(t)S_1(t)d\hat{W}(t).$$

- Discounted value of the portfolio

$$U^\theta(T) = V^\theta(0) + \int_0^T e^{-\int_0^t \rho(s)ds}\sigma(t)\theta_1(t)S_1(t)d\hat{W}(t).$$

Apply Clark-Ocone formula under change of measure for $U^\theta(T)$,

$$\begin{aligned} U^\theta(T) &= E_Q[U^\theta(T)] + \int_0^T E_Q[(D_t U^\theta(T) \\ &\quad - U^\theta(T) \int_t^T D_t u(s) d\hat{W}(s)) \mid \mathcal{F}_t] d\hat{W}(t). \end{aligned}$$

Application to Finance

$$\begin{aligned}\theta_1(t) &= e^{\int_0^t \rho(s)ds}\sigma^{-1}(t)S_1^{-1}(t)E_Q[(D_t(e^{-\int_0^T \rho(s)ds}F) \\ &\quad - e^{-\int_0^T \rho(s)ds}F\int_t^T D_t u(s)d\hat{W}(s)) \mid \mathcal{F}_t].\end{aligned}$$

Application to Finance

- Choose ρ constant and μ, σ deterministic functions
- $\theta_1(t) = e^{-\rho(T-t)}\sigma^{-1}(t)S_1^{-1}(t)E_Q[D_t F \mid \mathcal{F}_t]$
- $F = \chi_{[K,\infty)}(W(T))$ (Aase et al. [AØU01])

Donsker Delta Function

Definition

Let $Y : \Omega \to \mathbb{R}$ be a random variable which belongs to $\mathcal{G}^*$. Then the continuous function

$$\delta_Y(.) : \mathbb{R} \to \mathcal{G}^*$$

is called Donsker delta function of Y if it has the following property,

$$\int_{\mathbb{R}} g(y)\delta_Y(y)dy = g(Y) \quad \text{a.s.}$$

for all measurable function $g : \mathbb{R} \to \mathbb{R}$ such that the integral converges.

Theorem

Let $\phi : [0,T] \to \mathbb{R}$ *and* $\alpha : [0,T] \to \mathbb{R}$ *be deterministic functions such that* $||\phi||_{L^2([0,T])}$ *and* $||\alpha||_{L^2([0,T])} < \infty$. *Define*

$$Y(t) = Y(t,\omega) = \int_0^t \phi(s) d\hat{W}(s) + \int_0^t \phi(s)\alpha(s) ds.$$

Let $g : \mathbb{R} \to \mathbb{R}$ *be bounded. Then*

$$f(Y(T)) = V_0 + \int_0^T \Psi(t,\omega) \diamond (\alpha(t) + \dot{\hat{W}}_t) dt, \qquad (23)$$

where

$$V_0 = \int_{\mathbb{R}} \frac{g(y)}{\sqrt{2\pi}\,||\phi||_{L^2([0,T])}} \exp[-\frac{y^2}{2||\phi||_{L^2([0,T])}}]dy,$$

$$\Psi(t,\omega) = \phi(t) \int_{\mathbb{R}} \frac{g(y)}{\sqrt{2\pi}\,||\phi||_{L^2([0,T])}} \exp^{\diamond}[-\frac{(y - Y(t))^{\diamond 2}}{2||\phi||_{L^2([0,T])}}] \diamond \frac{y - Y(t)}{||\phi||_{L^2([0,T])}} dy$$

- Take $g(y) = \chi_{[K,\infty)}(y)$ and $Y(T) = W(T)$
- Then $\phi(t) = 1, \alpha(t) = -u(t)$ where $u(t)$ is defined in equation (22)
- Implies

$$\Psi(t,\omega) = \int_K^\infty (\sqrt{2\pi})^{-1/2}\, T^{-1/2} \exp^\diamond[-\frac{(y - W(t))^{\diamond 2}}{2T}] \diamond \frac{y - W(t)}{T} dy$$

By the Clark-Ocone formula,

$$\chi_{[K,\infty)}(W(T)) = E_Q[\chi_{[K,\infty)}(W(T))] + \int_0^T \underbrace{E_Q[D_t\{\chi_{[K,\infty)}(W(T))\}|\mathcal{F}_t]}_{*} \, d\hat{W}(t)$$

$$* = (2\pi T)^{-1/2} \int_K^\infty \exp^\diamond[-\frac{(y-W(t))^{\diamond 2}}{2T}] \diamond \frac{y-W(t)}{T} dy.$$

By Aase et al, Lemma 3.8 [AØU01]

$$* = (2\pi)^{-1/2}(T-t)^{-1/2}\int_K^\infty \exp[-\frac{(y-W(t))^2}{2(T-t)}]\,\frac{y-W(t)}{T-t}\,dy.$$

$$\theta_1(t) = e^{-\rho(T-t)}(2\pi(T-t))^{-1/2}\sigma^{-1}(t)S_1^{-1}(t)\int_K^\infty \exp[-\frac{(y-W(t))^2}{2(T-t)}]\frac{y-W(t)}{T-t}dy$$

Coordinates

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