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SOME PROBLEMS IN OPTION REPLICATIONS UNDER TRANSACTION COSTS

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OUTLINE

- **Leland conjectures and Leland–Lott theorem**
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- **Denis theorems**

Black–Scholes: pricing via replication

Classical model, under martingale measure, call option, maturity $T = 1$, price process S is gBM:

$$dS_t = \sigma S_t dW_t.$$

Let $C(t,x)$ be the solution of the Cauchy problem

$$C_t(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) = 0, \qquad C(1,x) = (x-K)^+. \tag{1}$$

That is

$$C(t,x) = C(t,x,\sigma) = x\Phi(d) - K\Phi(d - \sigma\sqrt{1-t}), \qquad t < 1, \tag{2}$$

where Φ is the Gaussian distribution function with the density φ,

$$d = d(t,x) = \frac{1}{\sigma\sqrt{1-t}} \ln\frac{x}{K} + \frac{1}{2}\sigma\sqrt{1-t}. \tag{3}$$

The process

$$V_t = C(0,S_0) + \int_0^t C_x(u,S_u)dS_u = C(t,S_t) \tag{4}$$

replicates the pay-off: $V_1 = (S_1 - K)^+$.

Leland approximate replication[a]

$$V_t^n = \widehat{C}(0, S_0) + \int_0^t \sum_{i=1}^n H_{t_{i-1}}^n I_{]t_{i-1},t_i]}(u) dS_u - \sum_{t_i<t} k_n S_{t_i} |H_{t_i}^n - H_{t_{i-1}}^n|, \tag{5}$$

where $H_{t_i}^n = \widehat{C}_x(t_i, S_{t_i})$, $t_i = i/n$, the positive parameter $k_n = k_0 n^{-\alpha}$ is the transaction costs coefficient, and $\widehat{C}(t,x)$ is the solution of the Cauchy problem (1) with σ replaced by $\widehat{\sigma} > 0$ with

$$\widehat{\sigma}^2 = \sigma^2 + \sigma k_0 n^{1/2-\alpha}\sqrt{8/\pi}. \tag{6}$$

That is $\widehat{C}(t,x) = C(t,x,\widehat{\sigma})$ and for such a strategy there is no need in a new software: traders can use their old one, just to change the input.

Note that for $\alpha = 1/2$

$$\widehat{\sigma}^2 = \sigma^2 + \sigma k_0 \sqrt{8/\pi} = consts.$$

Leland: $V_1^n \to V_1 = (S_1 - K)^+$ *in probability for* $\alpha = 0$ (*wrong!*) *and* $\alpha = 1/2$ (*true*).

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[a] **Leland H. Option pricing and replication with transactions costs, *Journal of Finance*, XL (1985), 5.**

LELAND–LOTT THEOREM[a][b]

Theorem 1 $V_1^n \to (S_1 - K)^+$ ***in probability for*** $\alpha =]0, 1/2]$**.**

K.–SAFARIAN THEOREM

Theorem 2 ***For*** $\alpha = 0$ ***(i.e. for*** $k_n = k_0$***)***

$$V_1^n \to (S_1 - K)^+ + S_1 \wedge K - S_1 F(\ln(S_1/K), k_0) \qquad \textit{in probability}$$

where

$$F(y, k_0) := \frac{1}{4} \int_0^\infty \frac{1}{\sqrt{v}} G(y, v, k_0) \exp\left\{ -\frac{v}{2} \left(\frac{y}{v} + \frac{1}{2} \right)^2 \right\} dv, \tag{7}$$

$$G(y, v, k_0) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left| x - \frac{2k_0 y}{\sqrt{2\pi v}} + \frac{k_0}{\sqrt{2\pi}} \right| e^{-x^2/2} dx. \tag{8}$$

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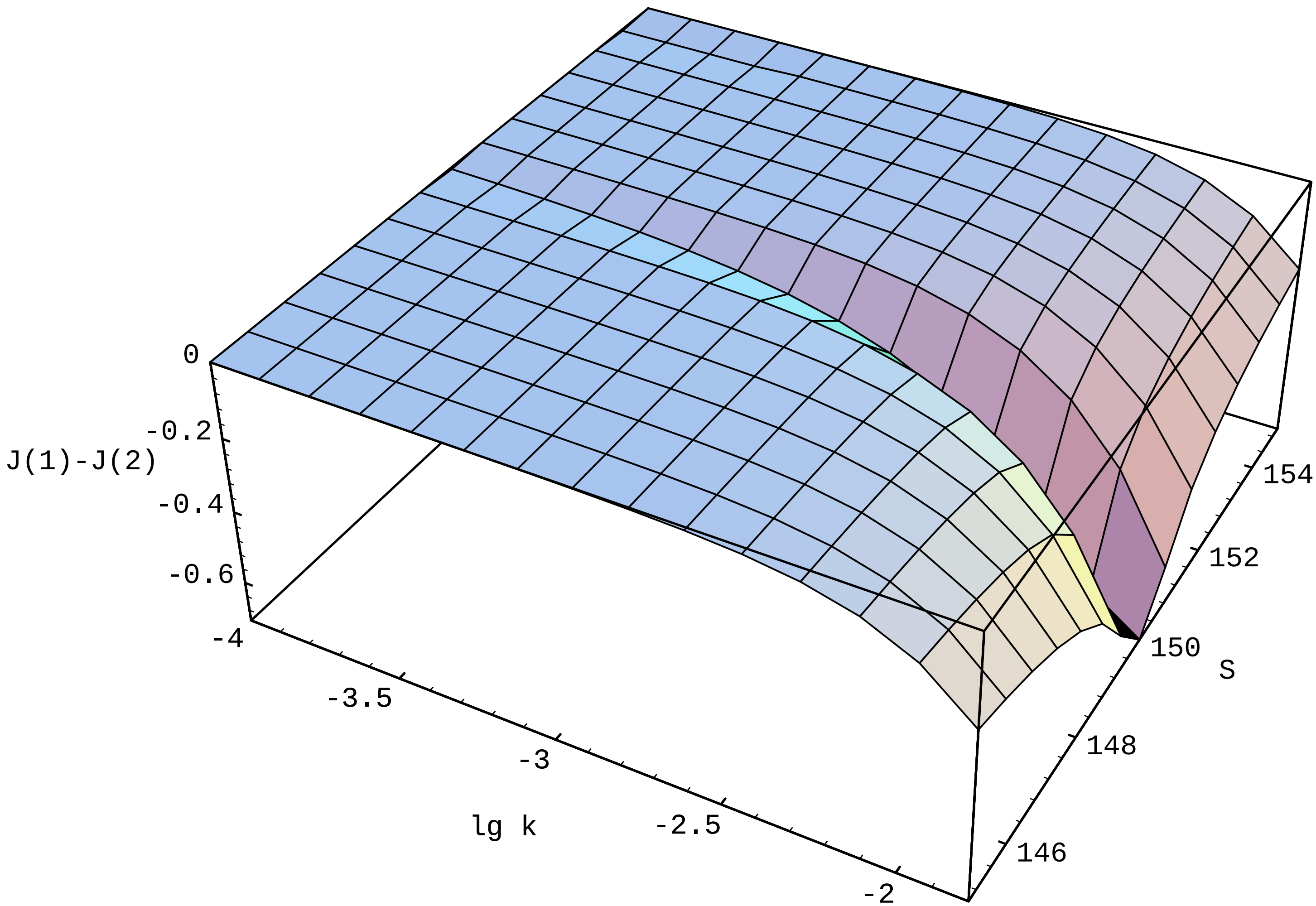
[a] **Lott K. Ein Verfahren zur Replikation von Optionen unter Transaktionkosten in stetiger Zeit, Dissertation, Universität der Bundeswehr München, 1993. (**$\alpha = 1/2$**)**

[b] **Kabanov Yu., Safarian M. On Leland's strategy of option pricing with transaction costs. *Finance and Stochastics*, 1, 3, 1997.**



Picture 1. Dependence of $J_1 - J_2$ on $k = k_0$ and $S = S_1$ for $K = 150$.

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PERGAMENSHCHIKOV THEOREM[a]

Theorem 3 ***Let*** $k = k_0 > 0$***. Then the sequence of random variables***

$$\xi_n := n^{1/4}(V_1^n - (S_1 - K)^+ - J_1 + J_2(k_0)) \tag{9}$$

where $J_1 := S_1 \wedge K$***,*** $J_2(k_0) := S_1 F(\ln(S_1/K), k_0)$ ***converges in law to a random variable*** ξ ***with a conditionally-Gaussian distribution.***

The amplifying factor $n^{1/4}$ **was found in the earlier paper[b].**

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[a] **Pergamenshchikov S. Limit theorem for Leland's strategy.** ***The Annals of Applied Probability*****, 13 (2003).**

[b] **Granditz P., Schachinger W. Leland's approach to option pricing: The evolution of discontinuity.** ***Mathematical Finance*****, 11 (2001), 3.**

GRENNAN–SWINDLE SCHEME[a]

A generalization of the Leland strategy with $\alpha = 1/2$ (i.e. $k_n = k_0/\sqrt{n}$) to the case of non-uniform revision intervals.

Let $f : [0,1] \to [0,1]$ be a function with $f' > 0$, $f(0) = 0$, $f(1) = 1$; $g := f^{-1}$. For fixed n the revision dates are $t_i = t_i^n = g(i/n)$, $1, ..., n$. The typical example: $g(t) = 1 - (1-t)^\beta$, $\beta \geq 1$. Now

$$\widehat{\sigma}_t^2 = \sigma^2 + \sigma k_0 \sqrt{8/\pi} \sqrt{f'(t)} \qquad \textbf{(10)}$$

the function $\widehat{C}(t,x)$ is the solution of the Cauchy problem

$$\widehat{C}_t(t,x) + \frac{1}{2}\widehat{\sigma}_t^2 x^2 \widehat{C}_{xx}(t,x) = 0, \qquad \widehat{C}(1,x) = h(x). \qquad \textbf{(11)}$$

Put $\Lambda_t = ES_t^4 \widehat{C}_{xx}^2(t, S_t)$. Then

$$E(V_1^n - V_1)^2 = A_1(f)n^{-1} + o(n^{-1}), \qquad n \to \infty, \qquad \textbf{(12)}$$

where the coefficient

$$A_1(f) = \int_0^1 \left[\frac{\sigma^4}{2} \frac{1}{f'(t)} + k_0 \sigma^3 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{f'(t)}} + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi} \right) \right] \Lambda_t dt. \qquad \textbf{(13)}$$

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[a] **Grannan E.R., Swindle G.H. Minimizing transaction costs of hedging strategies. *Mathematical Finance*, 6 (1996), 4.**

THEOREM[a]

Theorem 4 ***Let*** $h(x) = (x - K)^+$***. Suppose that*** $g' \in C([0,1])$ ***and*** $g'' \in C^2([0,1[)$ ***with***

$$\lim_{t \to 1} g''(t)(1-t) = 0. \tag{14}$$

Then

$$E(V_1^n - V_1)^2 = A_1(f)n^{-1} + o(n^{-1}), \qquad n \to \infty, \tag{15}$$

where the coefficient

$$A_1(f) = \int_0^1 \left[\frac{\sigma^4}{2} \frac{1}{f'(t)} + k_0 \sigma^3 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{f'(t)}} + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi}\right) \right] \Lambda_t dt, \tag{16}$$

with $\Lambda_t = E S_t^4 \widehat{C}_{xx}^2(t, S_t)$***. Explicitly, with the notation*** $\rho_t^2 = \int_t^1 \widehat{\sigma}_s^2 ds$***,***

$$\Lambda_t = \frac{1}{2\pi\rho_t} \frac{K^2}{\sqrt{2\sigma^2 t + \rho_t^2}} \exp\left\{ - \frac{\left(\ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 t - \frac{1}{2}\rho_t^2\right)^2}{2\sigma^2 t + \rho_t^2} \right\}. \tag{17}$$

[a] Gamys M., Kabanov Yu. Mean square error for the Leland–Lott hedging strategy. Preprint.

THEOREM[a]

Theorem 5 ***Suppose that*** $g' \in C([0,1])$ ***and*** $g'' \in C([0,1[)$ ***with***

$$\lim_{t\to 1} g''(t)(1-t) = 0. \tag{18}$$

Then the distributions of the process $X^n := n^{1/2}(V^n - V)$ ***in the Skorohod space*** $\mathcal{D}[0,1]$ ***converge weakly to the distributions of the diffusion process***

$$X_t = \int_0^t F(t, S_t)dW'_t \tag{19}$$

where W' ***is a Wiener process independent of*** W ***and***

$$F(t,x) = \left[\frac{\sigma^4}{2}\frac{1}{f'(t)} + k_0\frac{\sigma^3}{\sqrt{2\pi}\sqrt{f'(t)}} + k_0^2\sigma^2\left(1-\frac{2}{\pi}\right)\right]^{1/2}\widehat{C}_{xx}(t,x)x^2.$$

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[a] **Denis E., Kabanov Yu. Functional limit theorem for Leland–Lott hedging strategy. Preprint.**

Why the proof is difficult?

Lemma 1 ***We have the representation*** $V_1^n - V_1 = F_1^n + F_2^n$ ***where***

$$F_1^n := \sigma \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t)) S_t dW_t,$$

$$F_2^n := k_0 \sqrt{\frac{2}{\pi}} \sigma \int_0^1 S_t^2 \widehat{C}_{xx}(t, S_t) \sqrt{f'(t)} dt - \frac{k_0}{\sqrt{n}} \sum_{i=1}^{n-1} |\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})| S_{t_i}.$$

Lemma 2 ***Let***

$$P_1^n := \sum_{i=1}^{n-1} \sigma \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \int_{t_{i-1}}^{t_i} (1 - S_t/S_{t_{i-1}}) S_t/S_{t_{i-1}} dW_t,$$

$$P_2^n := k_0 \sum_{i=1}^{n-1} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \Big[\sigma \sqrt{2/\pi} \sqrt{f'(t_{i-1})} \Delta t_i - |S_{t_i}/S_{t_{i-1}} - 1| / \sqrt{n} \Big].$$

Then $nE(P_1^n + P_2^n)^2 \to A_1(f)$ ***as*** $n \to \infty$***.***

The most boring part of the proof is to check that for the residual terms $R_i^n := F_i^n - P_i^n$ **we have** $nE(R_i^n)^2 \to 0$**. These terms require a rather delicate estimates for** $\widehat{C}_{xxt}$**,** $\widehat{C}_{xxx}$**,** $\widehat{C}_{xxxx}$**, ...**

EXAMPLE: ESTIMATION OF THE SUMMAND R_{24}^n

"Telescoping", we can represent in a "natural" way R_2^n as $R_2^n = \; R_{2n}^n + R_{21}^n + R_{22}^n + R_{23}^n + R_{24}^n \;\; k_0$, where the summand

$$R_{24}^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \left[\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) |S_{t_i} - S_{t_{i-1}}| - |\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})| \right] S_{t_{i-1}}.$$

We apply the Ito formula to the function $\widehat{C}_x(t,x)$. Using the positivity of $\widehat{C}_{xx}(t,x)$ and the inequality $||a| - |b|| \leq |a - b|$ we dominate the absolute value of the square bracket $[\ldots]$ by the absolute value of

$$\int_{t_{i-1}}^{t_i} (\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_{xx}(t, S_t)) dS_t + \int_{t_{i-1}}^{t_i} \;\; \widehat{C}_{xt}(t, S_t) + \frac{\sigma^2}{2} S_t^2 \widehat{C}_{xxx}(t, S_t) \;\; dt.$$

The expression for $nE(R_{24}^n)^2$ involves the sum of expectations of squared terms and the sum of expectations of cross terms. By the Cauchy–Schwarz inequality the first one is dominated by a constant times

$$\sum_{i=1}^{n-1} ES_{t_{i-1}}^2 \int_{t_{i-1}}^{t_i} (\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_{xx}(t, S_t))^2 S_t^2 dt + \sum_{i=1}^{n-1} \Delta t_i ES_{t_{i-1}}^2 \int_{t_{i-1}}^{t_i} (\widehat{C}_{xt}^2(t, S_t) + S_t^4 \widehat{C}_{xxx}^2(t, S_t)) dt$$

and it is not difficult to show that these sums tends to zero. The really delicate work is with the sum of cross terms...

Comments on the Grennan–Swindle paper

Unfortunately, the formulations and arguments are rather embarrassing. In particular, the hypothesis that for any nonnegative integers m, n, p

$$||\widehat{C}||_{m,n,p} = \sup_{x>0,\ t\in[0,1]} \left[x^m \frac{\partial^{n+p}\widehat{C}(t,x)}{\partial x^n \partial t^p} \right] < \infty$$

is not fulfilled for the call-option with $h(x) = (x-K)^+$ (even for the uniform grid): explicit formulae show that derivatives of $\widehat{C}(t,x)$ have singularities at the point $(1,K)$. So, the mathematical results do not cover practically interesting cases. Confusingly, the obtained formula is used in numerical analysis of the approximate hedging of call-options. Moreover, even in such a restricted case the arguments are obscure: the authors do not care about eventual divergence of the integral (16) due to singularities of $1/f'$ which are not excluded by their assumptions. Neglecting the singularities may lead to an erroneous answer as in Leland's paper. An asymptotic analysis happens to be rather involved and we restrict ourselves to the case of the call option.

The Grannan-Swindle paper contains another interesting idea: to minimize the functional $A_1(f)$ with respect to the scale f in a hope to improve the performance of the strategy by an appropriate choice of the revision dates. Unfortunately, the reduction to a classical variational problem is not correct as well as the derived Euler–Lagrange equation. That is why the whole paper can be considered only as one giving useful heuristics but leaving open mathematical problems of practical importance.

APPROXIMATE HEDGING OF MORE GENERAL OPTIONS[a]

Assumption (G): $g'' \in C[0,1[$ **and there is a** $\lambda \in [0,1[$ **such that** $g''(t)(1-t)^{\lambda}$ **is bounded.**

Assumption (H): $h \in C(\mathbf{R}_+)$ **and two-times differentiable except the points** $K_1 < \cdots < K_p$ **where** h' **and** h'' **admit right and left limits;** $|h''(x)| \le Mx^{-\beta}$ **for** $x \ge K_p$ **where** $\beta \ge 3/2$**.**

Theorem 6 ***Let*** $\alpha \in]0,1/2]$***. Suppose that*** (G) ***and*** (H) ***hold. Then*** $P\text{-}\lim_n V_1^n = h(S_1) + \varepsilon_\alpha$***, with***

$$\varepsilon_\alpha = \frac{1}{2}\int_0^\infty \frac{1}{S_1}[\theta_1(x,S_1) - |\theta_1(x,S_1)|]dx, \quad \alpha \in [0,1/2[,$$

and

$$\varepsilon_{1/2} = \frac{1}{2}\sigma k_0 \sqrt{\frac{8}{\pi}} \int_0^1 \sqrt{f'(t)} \left(\widehat{C}_{xx}(t,S_t) - |\widehat{C}_{xx}(t,S_t)|\right) dt$$

where

$$\theta_1(x,S_1) := \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} h'(S_1 e^{\sqrt{x}y + x/2}) y \varphi(y) dy.$$

[a] Denis E. Approximate Hedging of Contingent Claims under Transaction Costs. Preprint.

Theorem 7 ***Let $k = k_0 \geq 0$ (i.e. $\alpha = 0$). Suppose the h is convex or concave and the assumptions (G) and (H) hold. Then***

$$P\text{-}\lim_n V_1^n = h(S_1) + J_1 - J_2(k_0) + \varepsilon_0 \qquad \textbf{(20)}$$

where J_1 is defined by the formula

$$J_1 = \frac{1}{2} S_1 \int_0^\infty \frac{1}{\sqrt{x}} \theta_1(S_1, x) dx$$

with

$$\theta_1(S, x) = \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} h'(S e^{\sqrt{x} y + x/2}) y \varphi(y) dy,$$

$$J_2(k_0) = \frac{1}{2} S_1 \int_0^\infty j_2(S_1, x) dx$$

where

$$j_2(S, x) = \theta_1(S, x) \exp\Big\{ - \frac{k_0^2 \theta_2^2(S, x)}{\pi \theta_1^2(S, x)} \Big\} + k_0 \Big[2\Phi\Big(k_0 \frac{\sqrt{2}}{\pi} \frac{\theta_2(S, x)}{\theta_1(S, x)} \Big) - 1 \Big] \theta_2(S, x),$$

$$\theta_2(S, x) = \frac{1}{x} \int_{-\infty}^{\infty} h'(S e^{\sqrt{x} y + x/2})(-y^2 - \sqrt{x} y + 1) \varphi(y) dy.$$