

Course: Inverse and Ill-Posed Problems

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First three parts are taken from the book:

H.W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996.

Part 1: Wednesday, March 26, 16:00-18:00

1. Introduction

When using the term *inverse problem*, one immediately is tempted to ask “inverse to what?”. Following J.B. Keller, one calls *two* problems *inverse to each other* if the formulation of one problem involves the other one. For mostly historic reasons, one might call one of these problems (usually the simpler one or the one which was studied earlier) the *direct problem*, the other one the *inverse problem*. However, if there is a *real-world* problem behind the mathematical problem studied, there is, in most cases, a quite natural distinction between the direct and the inverse problem. E.g., if one wants to predict the future behaviour of a physical system from knowledge of its present state and the physical laws (including concrete values of all relevant physical parameters), one will call this the *direct problem*. Possible *inverse problems* are the determination of the present state of the system from future observations (i.e., the calculation of the evolution of the system backwards in time) or the identification of physical parameters from observations of the evolution of the system (*parameter identification*).

There are, from the applications point of view, two different motivations for studying such inverse problems: first, one wants to *know* past states or parameters of a physical system. Second, one wants to find out how to influence a system via its present state or via parameters in order to *steer* it to a desired state in the future.

Thus, one might say that *inverse problems are concerned with determining causes for a desired or an observed effect*.

Two mathematical problems inverse to each other are differentiation and integration. A-priori, it is not clear which of these problems should be the direct problem and which one the inverse problem. However, as we will now see, differentiation has (as opposed to integration) the properties of an ill-posed problem.

Let $f \in C^1[0, 1]$ be any function, $\delta \in (0, 1)$, $n \in \mathbb{N}$ ($n \geq 2$) be arbitrary, and define

$$f_n^\delta(x) := f(x) + \delta \sin \frac{nx}{\delta}, \quad x \in [0, 1].$$

Then

$$(f_n^\delta)'(x) = f'(x) + n \cos \frac{nx}{\delta}, \quad x \in [0, 1].$$

Now, in the uniform norm,

$$\|f - f_n^\delta\|_\infty = \delta,$$

but

$$\|f' - (f_n^\delta)'\|_\infty = n.$$

Hence, if we consider f and f_n^δ as the exact and perturbed data, respectively, then for an arbitrarily small data error δ , the error in the result, namely the derivative, can be arbitrarily large, namely n . Hence, the derivative does not depend continuously on the data with respect to the uniform norm.

The stability problems addressed must appear somehow when computing the derivative via difference quotients: let f be the function we want to differentiate, f^δ its noisy version with

$$\|f - f^\delta\|_\infty \leq \delta.$$

We want to use the central difference quotient with step size h . If $f \in C^2[0, 1]$, Taylor expansion yields

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h),$$

while for $f \in C^3[0, 1]$,

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2).$$

Thus, the accuracy of the central difference quotient depends on the smoothness of the exact data. Instead of f' , we are actually computing

$$\frac{f^\delta(x+h) - f^\delta(x-h)}{2h} \sim \frac{f(x+h) - f(x-h)}{2h} + \frac{\delta}{h}.$$

Thus, the total error behaves like

$$O(h^\nu) + \frac{\delta}{h},$$

where $\nu = 1$ or 2 if $f \in C^2[0, 1]$ or $f \in C^3[0, 1]$, respectively. For a fixed error level δ , it looks as in Figure 1.1.

If h becomes too small, the total error increases due to the error term δ/h , the propagated data error. Of course, if h is too large, then the approximation error becomes too large. There is an *optimal* discretization parameter h_0 , which can, however, not be computed explicitly, since it depends on unavailable information about the exact data, e.g., their smoothness. However, one can at least estimate the asymptotic behaviour of h_0 if h is chosen as a power of δ , i.e.,

$$h \sim \delta^\mu,$$

then one can minimize the total error estimate by taking $\mu = 1/2$ or $\mu = 1/3$, which results in a behaviour of the total error as $O(\sqrt{\delta})$ or $O(\delta^{2/3})$ for $f \in C^2[0, 1]$ or $f \in C^3[0, 1]$, respectively.

In this example, we saw some effects that are typical for ill-posed problems:

- amplification of *high frequency* errors

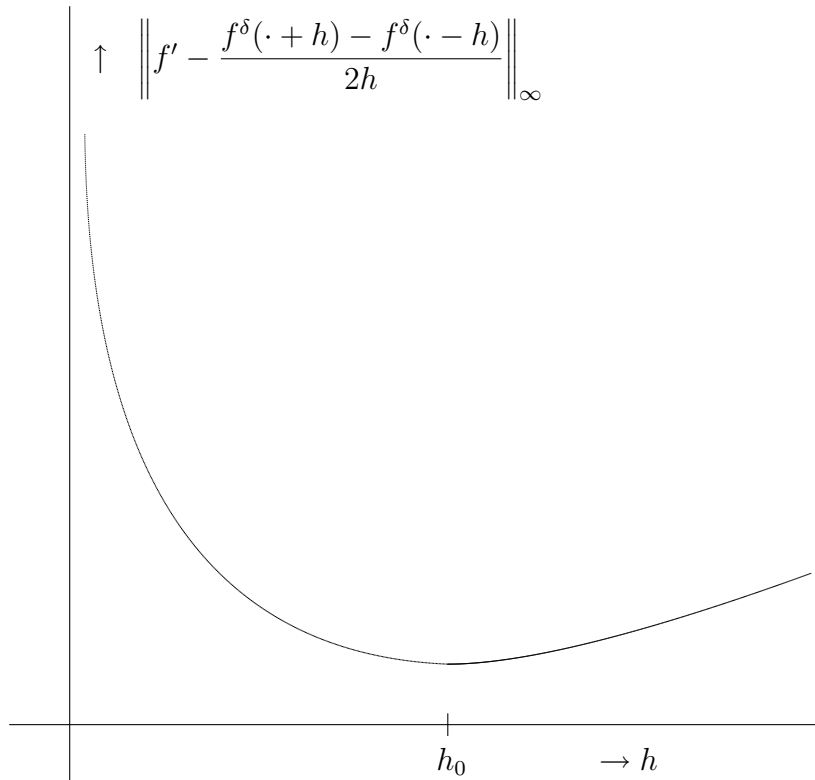


Figure 1.1: Total error depending on h

- restoration of stability by using *a-priori information*
- two error terms of different nature, one for the approximation error, the other one for the propagation of the data error, adding up to a total error as in Figure 1.1
- the appearance of an optimal discretization parameter, whose choice depends on a-priori information
- loss of information even under optimal circumstances.

For other examples of inverse problems like X-ray tomography, or inverse problems in signal and image processing or in heat conduction, parameter identification and inverse scattering see the book.

2. Ill-posed linear operator equations

We consider linear operator equations of the form

$$Tx = y,$$

where T is a bounded linear operator between Hilbert spaces \mathcal{X} and \mathcal{Y} .

Inverse problems frequently lead to mathematical problems which do not fulfill Hadamard's definition of *well-posedness*:

$$\begin{aligned} \mathcal{R}(T) & \quad \text{For all admissible data, a solution exists.} \\ \mathcal{N}(T) = \{0\} & \quad \text{For all admissible data, the solution is unique.} \\ T^{-1} \in L(\mathcal{Y}, \mathcal{X}) & \quad \text{The solution depends continuously on the data.} \end{aligned}$$

Violation of the last condition creates serious numerical problems: if one wants to approximate a problem whose solution does not depend continuously on the data by a “traditional” numerical method as one would use for a well-posed problem, then one has to expect that the numerical method becomes unstable. A (partial) remedy for this is the use of “regularization methods”, although one has to keep in mind that no mathematical trick can make an inherently unstable problem stable. All that a regularization method can do is to recover partial information about the solution as stably as possible.

Uniqueness of solutions can be achieved if one generalizes the notion of solution.

2.1. The Moore-Penrose generalized inverse

Here, we provide the basic theory of best-approximate solutions of linear operator equations and of the (Moore-Penrose) generalized inverse of linear operators between Hilbert spaces.

Definition 2.1. *Let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded linear operator.*

(i) *$x \in \mathcal{X}$ is called least-squares solution of $Tx = y$ if*

$$\|Tx - y\| = \inf\{\|Tz - y\| \mid z \in \mathcal{X}\}.$$

(ii) *$x \in \mathcal{X}$ is called best-approximate solution of $Tx = y$ if x is a least-squares solution of $Tx = y$ and*

$$\|x\| = \inf\{\|z\| \mid z \text{ is least-squares solution of } Tx = y\}$$

holds.

The best-approximate solution is unique. It is defined as the least-squares solution of minimal norm. One could as well minimize $\|Lx\|$, where L is usually a differential operator.

The notion of a best-approximate solution is closely related to the *Moore-Penrose (generalized) inverse* of T , which is the *solution operator* mapping y onto the best-approximate solution of $Tx = y$.

Definition 2.2. *The Moore-Penrose (generalized) inverse T^\dagger of $T \in L(\mathcal{X}, \mathcal{Y})$ is defined as the unique linear extension of T^{-1} to*

$$\mathcal{D}(T^\dagger) := \mathcal{R}(T) \dot{+} \mathcal{R}(T)^\perp$$

with

$$\mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp,$$

where

$$\tilde{T} := T|_{\mathcal{N}(T)^\perp} : \mathcal{N}(T)^\perp \rightarrow \mathcal{R}(T).$$

T^\dagger is well-defined: since $\mathcal{N}(\tilde{T}) = \{0\}$ and $\mathcal{R}(\tilde{T}) = \mathcal{R}(T)$, \tilde{T}^{-1} exists. For any $y \in \mathcal{D}(T^\dagger)$ with the unique representation $y = y_1 + y_2$, $y_1 \in \mathcal{R}(T)$, $y_2 \in \mathcal{R}(T)^\perp$, $T^\dagger y$ has to be $\tilde{T}^{-1}y_1$.

Proposition 2.3. *Let P and Q be the orthogonal projectors onto $\mathcal{N}(T)$ and $\overline{\mathcal{R}(T)}$, respectively. Then $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$, and the four “Moore-Penrose equations” hold:*

$$\begin{aligned} TT^\dagger T &= T, \\ T^\dagger TT^\dagger &= T^\dagger, \\ T^\dagger T &= I - P, \\ TT^\dagger &= Q|_{\mathcal{D}(T^\dagger)}. \end{aligned}$$

Proposition 2.4. *The Moore-Penrose generalized inverse T^\dagger has a closed graph $gr(T^\dagger)$. Furthermore, T^\dagger is bounded (i.e., continuous) if and only if $\mathcal{R}(T)$ is closed.*

We will now provide the connection between least-squares solutions and the Moore-Penrose inverse:

Theorem 2.5. *Let $y \in \mathcal{D}(T^\dagger)$. Then, $Tx = y$ has a unique best-approximate solution, which is given by*

$$x^\dagger := T^\dagger y.$$

The set of all least-squares solutions is $x^\dagger + \mathcal{N}(T)$.

Theorem 2.6. *Let $y \in \mathcal{D}(T^\dagger)$. Then $x \in \mathcal{X}$ is a least-squares solution of $Tx = y$ if and only if the normal equation*

$$T^*Tx = T^*y$$

holds.

Operator equations with a *compact* linear operator will be of special importance. Before discussing properties of their Moore-Penrose inverse, we need the concept of the *singular value expansion* of such operators.

2.2. Compact linear operators: singular value expansion

Compact linear operators are of special interest, since many problems in practice may be formulated as integral equations and integral operators are compact if the kernel is

in L^2 :

$$\begin{aligned} K : L^2(\Omega) &\rightarrow L^2(\Omega) \\ x &\mapsto (Kx)(s) := \int_{\Omega} k(s, t)x(t) dt \end{aligned}$$

is compact.

If we assume a bounded linear operator to be compact, we will use the symbol K instead of T for this operator as a reminder of this assumption.

If $K : \mathcal{X} \rightarrow \mathcal{Y}$ is not selfadjoint, no eigensystem need to exist. However, one can construct a substitute for an eigensystem, a *singular system* $(\sigma_n; v_n, u_n)$ defined as follows:

Let $\{\sigma_n^2\}_{n \in \mathbb{N}}$ be the non-zero eigenvalues of the selfadjoint operator K^*K (and also of KK^* , $K^* : \mathcal{Y} \rightarrow \mathcal{X}$ denotes the adjoint of K), written down in decreasing order with multiplicity, $\sigma_n > 0$. The $\{v_n\}_{n \in \mathbb{N}}$ are a corresponding complete orthonormal system of eigenvectors of K^*K (which spans $\overline{\mathcal{R}(K^*K)} = \overline{\mathcal{R}(K^*)} = \mathcal{N}(K)^\perp$), and the $\{u_n\}_{n \in \mathbb{N}}$ are defined via $u_n := Kv_n / \|Kv_n\|$. The $\{u_n\}_{n \in \mathbb{N}}$ are a complete orthonormal system of eigenvectors of KK^* and span $\overline{\mathcal{R}(KK^*)} = \overline{\mathcal{R}(K)}$.

Then the following formulas hold:

$$\begin{aligned} Kv_n &= \sigma_n u_n, \\ K^*u_n &= \sigma_n v_n, \\ Kx &= \sum_{n=1}^{\infty} \sigma_n \langle x, v_n \rangle u_n, \quad x \in \mathcal{X}, \\ K^*y &= \sum_{n=1}^{\infty} \sigma_n \langle y, u_n \rangle v_n, \quad y \in \mathcal{Y}, \end{aligned}$$

where these infinite series converge in the Hilbert space norms of \mathcal{X} and \mathcal{Y} , respectively; the series are called *singular value expansion*.

If (and only if) K has a finite-dimensional range, K has only finitely many singular values, so that all infinite series involving singular values degenerate to finite sums. If K is an integral operator (as above) this happens if and only if the kernel k is *degenerate*.

If there are infinitely many singular values, they accumulate (only) at 0, i.e., $\lim_{n \rightarrow \infty} \sigma_n = 0$.

The range $\mathcal{R}(K)$ is closed if and only if it is finite-dimensional, so that in the (generic) case of infinitely many singular values, $\mathcal{R}(K)$ is non-closed. Together with Proposition 2.4, this implies

Proposition 2.7. *Let $K : \mathcal{X} \rightarrow \mathcal{Y}$ be compact, $\dim \mathcal{R}(K) = \infty$. Then K^\dagger is a densely defined unbounded linear operator with closed graph.*

Hence, for a compact linear operator with non-closed range (e.g., for an integral operator with a non-degenerate L^2 -kernel), the best-approximate solution of $Tx = y$ does not depend continuously on the right-hand side; the equation is ill-posed.

Using a singular system, one can find a series representation of the Moore-Penrose inverse of a compact linear operator. Note that, if the singular system contains only finitely many elements (i.e., if $\mathcal{R}(K)$ is finite-dimensional), then all infinite series containing singular functions have to be interpreted as finite sums.

Theorem 2.8. Let $(\sigma_n; v_n, u_n)$ be a singular system for the compact linear operator K , $y \in \mathcal{Y}$. Then we have:

$$(i) \quad y \in \mathcal{D}(K^\dagger) \iff \sum_{n=1}^{\infty} \frac{|\langle y, u_n \rangle|^2}{\sigma_n^2} < \infty.$$

(ii) For $y \in \mathcal{D}(K^\dagger)$,

$$K^\dagger y = \sum_{n=1}^{\infty} \frac{\langle y, u_n \rangle}{\sigma_n} v_n. \quad (2.1)$$

The condition in Theorem 2.8 (i) for the existence of a best-approximate solution is called the *Picard criterion*. It says that a best-approximate solution of $Kx = y$ exists only if the (generalized) Fourier coefficients $(\langle y, u_n \rangle)$ with respect to the singular functions u_n decay fast enough relative to the singular values σ_n .

(2.1) shows how errors in y affect the result $K^\dagger y$: error components (with respect to the basis $\{u_n\}$) which correspond to large singular values are harmless. However, error components which correspond to small singular values σ_n are amplified by the (then large) factors $1/\sigma_n$, so that those are dangerous. The instability becomes the more severe the faster the singular values decay. A problem is called *mildly (modestly) ill-posed* if $\sigma_n = O(n^{-\alpha})$, and *severely ill-posed* otherwise, e.g., if $\sigma_n = O(e^{-n})$ holds.

2.3. Spectral theory and functional calculus

For the construction and analysis of regularization methods, we will need the notion of a *function of a selfadjoint operator*. The definition and properties of such operator functions are done in the framework of *functional calculus*, which we motivate for the case of a compact operator:

Let $(\sigma_n; v_n, u_n)$ be a singular system for a compact linear operator K . Since $(\sigma_n^2; v_n)$ is an eigensystem for the selfadjoint compact operator K^*K ,

$$K^*Kx = \sum_{n=1}^{\infty} \sigma_n^2 \langle x, v_n \rangle v_n$$

holds, which will be written as

$$K^*Kx = \int \lambda dE_\lambda x,$$

where for $\lambda \in \mathbb{R}$ and $x \in \mathcal{X}$, we define

$$E_\lambda x := \sum_{\substack{n=1 \\ \sigma_n^2 < \lambda}}^{\infty} \langle x, v_n \rangle v_n \quad (+Px), \quad (2.2)$$

where P is the orthogonal projector onto $\mathcal{N}(K^*K)$ and is meant to appear in (2.2) only for $\lambda > 0$. For all λ , E_λ is an orthogonal projector and projects onto

$$\mathcal{X}_\lambda := \text{span}\{v_n \mid n \in \mathbb{N}, \sigma_n^2 < \lambda\} \quad (+\mathcal{N}(K^*K), \text{ if } \lambda > 0).$$

For $\lambda \leq 0$, $E_\lambda = 0$ and $E_\lambda = I$ for $\lambda > \sigma_1^2$. For all $\lambda \leq \mu$ and $x \in \mathcal{X}$,

$$\langle E_\lambda x, x \rangle \leq \langle E_\mu x, x \rangle,$$

which is a kind of monotonicity of this so-called *spectral family* E_λ . Note that E_λ is piecewise constant and has jumps at $\lambda = \sigma_n^2$ (and at $\lambda = 0$ if and only if $\mathcal{N}(K) = \mathcal{N}(K^*K) \neq \{0\}$) of “height”

$$\sum_{\substack{n=1 \\ \sigma_n^2 = \lambda}}^{\infty} \langle \cdot, v_n \rangle v_n.$$

Now, recall that the integral with respect to a piecewise constant weight function is defined as the sum over all function values of the integrand where the weight function has jumps, multiplied with the heights of these jumps. This fits into the general measure-theoretic concept of an integral, all important results of integration theory like the Lebesgue Dominated Convergence Theorem and integration by parts hold. This motivates the notations

$$\begin{aligned} f(K^*K)x &:= \int_0^\infty f(\lambda) dE_\lambda x &:= \sum_{n=1}^{\infty} f(\sigma_n^2) \langle x, v_n \rangle v_n, \\ \|f(K^*K)x\|^2 &:= \int_0^\infty f^2(\lambda) d\|E_\lambda x\|^2 &:= \sum_{n=1}^{\infty} f^2(\sigma_n^2) |\langle x, v_n \rangle|^2 \end{aligned}$$

for a (piecewise) continuous function f and $x, y \in \mathcal{X}$. The upper limit of integration could be replaced by $\sigma_1^2 + \varepsilon = \|K\|^2 + \varepsilon$ for any $\varepsilon > 0$.

If F_λ is a spectral family of KK^* , then one can show that

$$f(K^*K)K^* = K^*f(KK^*).$$

One can show that also for non-compact (possibly unbounded) linear selfadjoint operators, T , in a Hilbert space \mathcal{X} , a spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of orthogonal projectors in \mathcal{X} exists such that the integrals above still make sense.

There is a strong connection between the spectral family of T and its spectrum $\sigma(T)$:

Proposition 2.9. *Let T be a selfadjoint operator in \mathcal{X} with spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$.*

- (i) $\lambda_0 \in \sigma(T)$ if and only if $E_{\lambda_0} \neq E_{\lambda_0 + \varepsilon}$ for all $\varepsilon > 0$.
- (ii) λ_0 is an eigenvalue of T if and only if $E_{\lambda_0} \neq E_{\lambda_0 + 0} = \lim_{\varepsilon \rightarrow 0} E_{\lambda_0 + \varepsilon}$; the corresponding eigensubspace is then given by $(E_{\lambda_0 + 0} - E_{\lambda_0})(\mathcal{X})$.

Part 2: Thursday, March 27, 10:00-12:00

3. Regularization operators

In general terms, *regularization* is the approximation of an ill-posed problem by a family of neighbouring well-posed problems.

We want to approximate the best-approximate solution $x^\dagger = T^\dagger y$ of $Tx = y$ for a specific right-hand side y in the situation that the “exact data” y are not known precisely, but that only an approximation y^δ with

$$\|y^\delta - y\| \leq \delta$$

is available; we will call y^δ the “noisy data” and δ the “noise level”.

In the ill-posed case, $T^\dagger y^\delta$ is certainly not a good approximation of $T^\dagger y$ due to the unboundedness of T^\dagger even if it exists (which will, in general, also not be the case, since $\mathcal{D}(T^\dagger)$ is only dense in \mathcal{Y}). We are looking for some approximation, say x_α^δ , of x^\dagger which does, on the one hand, depend continuously on the (noisy) data y^δ , so that it can be computed in a stable way, and has, on the other hand, the property that as the noise level δ decreases to zero and the *regularization parameter* α is chosen appropriately (whatever this means), then x_α^δ tends to x^\dagger .

Definition 3.1. *Let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded linear operator between the Hilbert spaces \mathcal{X} and \mathcal{Y} , $\alpha_0 \in (0, +\infty]$. For every $\alpha \in (0, \alpha_0)$, let*

$$R_\alpha : \mathcal{Y} \rightarrow \mathcal{X}$$

be a continuous (not necessarily linear) operator. The family $\{R_\alpha\}$ is called a regularization or a regularization operator (for T^\dagger), if, for all $y \in \mathcal{D}(T^\dagger)$, there exists a parameter choice rule $\alpha = \alpha(\delta, y^\delta)$ such that

$$\limsup_{\delta \rightarrow 0} \{ \|R_{\alpha(\delta, y^\delta)} y^\delta - T^\dagger y\| \mid y^\delta \in \mathcal{Y}, \|y^\delta - y\| \leq \delta \} = 0 \quad (3.1)$$

holds. Here,

$$\alpha : \mathbb{R}^+ \times \mathcal{Y} \rightarrow (0, \alpha_0)$$

is such that

$$\limsup_{\delta \rightarrow 0} \{ \alpha(\delta, y^\delta) \mid y^\delta \in \mathcal{Y}, \|y^\delta - y\| \leq \delta \} = 0. \quad (3.2)$$

For a specific $y \in \mathcal{D}(T^\dagger)$, a pair (R_α, α) is called a (convergent) regularization method (for solving $Tx = y$) if (3.1) and (3.2) hold.

Thus, a regularization method consists of a regularization operator and a parameter choice rule which is convergent in the sense that, if the regularization parameter is chosen according to that rule, then the regularized solutions converge (in the norm) as the noise level tends to zero; this is assured for any collection of noisy right-hand sides compatible with the noise level and thus is a “worst-case” concept of convergence with respect to the right-hand side.

For linear operators T , the following holds:

Proposition 3.2. *Let, for all $\alpha > 0$, R_α be a continuous (possibly nonlinear) operator. Then, the family $\{R_\alpha\}$ is a regularization for T^\dagger if*

$$R_\alpha \rightarrow T^\dagger \quad \text{pointwise on } \mathcal{D}(T^\dagger) \text{ as } \alpha \rightarrow 0.$$

In this case, there exists, for every $y \in \mathcal{D}(T^\dagger)$, an a-priori parameter choice rule α such that (R_α, α) is a convergent regularization method for solving $Tx = y$.

4. Continuous regularization methods

4.1. A-priori Parameter Choice Rules

Let $\{E_\lambda\}$ be a spectral family for T^*T . If T^*T is continuously invertible, then $(T^*T)^{-1} = \int \frac{1}{\lambda} dE_\lambda$. Since the best-approximate solution $x^\dagger = T^\dagger y$ can be characterized by the normal equation $T^*Tx = T^*y$,

$$x^\dagger = \int \frac{1}{\lambda} dE_\lambda T^*y \quad (4.1)$$

holds in this case. Now, if $\mathcal{R}(T)$ is non-closed and $y \notin \mathcal{D}(T^\dagger)$, i.e., if $Tx = y$ is ill-posed, then the integral in (4.1) does not exist. The idea is now to replace the integrand $1/\lambda$ by a parameter-dependent family of functions $g_\alpha(\lambda)$ which are at least piecewise continuous on $[0, \|T\|^2]$ and, for convenience, continuous from the right in points of discontinuity and to replace (4.1) by

$$x_\alpha := \int g_\alpha(\lambda) dE_\lambda T^*y. \quad (4.2)$$

By construction, the operator on the right-hand side of (4.2) acting on y is continuous, so that, for noisy data y^δ with $\|y - y^\delta\| \leq \delta$, one can bound the error between x_α and

$$x_\alpha^\delta := \int g_\alpha(\lambda) dE_\lambda T^*y^\delta, \quad (4.3)$$

i.e., x_α^δ can be computed in a stable way.

First we study the question under which condition the family $\{R_\alpha\}$ with

$$R_\alpha := \int g_\alpha(\lambda) dE_\lambda T^*$$

is a regularization operator for T^\dagger .

If x_α is defined by (4.2), then the residual has the representation

$$x^\dagger - x_\alpha = x^\dagger - g_\alpha(T^*T)T^*y = (I - g_\alpha(T^*T)T^*T)x^\dagger = \int (1 - \lambda g_\alpha(\lambda)) dE_\lambda x^\dagger.$$

Hence, if we define, for all (α, λ) for which $g_\alpha(\lambda)$ is defined,

$$r_\alpha(\lambda) := 1 - \lambda g_\alpha(\lambda), \quad (4.4)$$

then

$$x^\dagger - x_\alpha = r_\alpha(T^*T)x^\dagger. \quad (4.5)$$

Theorem 4.1. *Let, for all $\alpha > 0$, $g_\alpha : [0, \|T\|^2] \rightarrow \mathbb{R}$ fulfill the following assumptions: g_α is piecewise continuous, and there is a $C > 0$ such that*

$$|\lambda g_\alpha(\lambda)| \leq C, \quad (4.6)$$

and

$$\lim_{\alpha \rightarrow 0} g_\alpha(\lambda) = \frac{1}{\lambda} \quad (4.7)$$

for all $\lambda \in (0, \|T\|^2]$. Then, for all $y \in \mathcal{D}(T^\dagger)$,

$$\lim_{\alpha \rightarrow 0} x_\alpha = \lim_{\alpha \rightarrow 0} g_\alpha(T^*T)T^*y = x^\dagger$$

holds with $x^\dagger = T^\dagger y$.

Proof. Because of (4.2) and (4.5),

$$\|x^\dagger - x_\alpha\|^2 = \int_0^\infty r_\alpha^2(\lambda) d\|E_\lambda x^\dagger\|^2. \quad (4.8)$$

The integrand in (4.8) is bounded by the constant $(C + 1)^2$, which is integrable with respect to the measure $d\|E_\lambda x^\dagger\|^2$. Hence, by the Dominated Convergence Theorem,

$$\lim_{\alpha \rightarrow 0} \int_0^\infty r_\alpha^2(\lambda) d\|E_\lambda x^\dagger\|^2 = \int_0^\infty \lim_{\alpha \rightarrow 0} r_\alpha^2(\lambda) d\|E_\lambda x^\dagger\|^2 \quad (4.9)$$

holds. Since, by (4.7), $\lim_{\alpha \rightarrow 0} r_\alpha^2(\lambda) = 0$ for $\lambda > 0$ and $\lim_{\alpha \rightarrow 0} r_\alpha^2(0) = 1$, the integral on the right-hand side of (4.9) equals the “jump” of $\lambda \mapsto \|E_\lambda x^\dagger\|^2$ at $\lambda = 0$, i.e., $\lim_{\lambda \rightarrow 0^+} \|E_\lambda x^\dagger\|^2 - \|E_0 x^\dagger\|^2 = \|Px^\dagger\|^2$, where P is the orthogonal projector onto $\mathcal{N}(T)$. Since $x^\dagger \in \mathcal{N}(T)^\perp$, $Px^\dagger = 0$, so that we finally obtain from (4.8) and (4.9) that the assertion holds. ■

While Theorem 4.1 was about convergence of regularized solutions with exact data, the next theorem is concerned with the stability question, i.e., with the propagation of data noise:

Theorem 4.2. *Let g_α and C be as in Theorem 4.1, x_α and x_α^δ be defined by (4.2) and (4.3), respectively. For $\alpha > 0$, let*

$$G_\alpha := \sup\{|g_\alpha(\lambda)| \mid \lambda \in [0, \|T\|^2]\}. \quad (4.10)$$

Then

$$\|x_\alpha - x_\alpha^\delta\| \leq \delta \sqrt{CG_\alpha}.$$

Proof. Noting that $g_\alpha(T^*T)T^* = T^*g_\alpha(TT^*)$, it follows from (4.6) and (4.10) that

$$\begin{aligned} \|x_\alpha - x_\alpha^\delta\|^2 &= \|(TT^*)^{\frac{1}{2}}g_\alpha(TT^*)(y - y^\delta)\|^2 \\ &= \int_0^\infty \lambda g_\alpha^2(\lambda) d\|F_\lambda Q(y - y^\delta)\|^2 \leq CG_\alpha \delta^2. \end{aligned}$$

$\{F_\lambda\}$ denotes the spectral family of TT^* . ■

Thus, for the total error we have the estimate

$$\|x_\alpha^\delta - x^\dagger\| \leq \|x_\alpha - x^\dagger\| + \delta\sqrt{CG_\alpha}. \quad (4.11)$$

By Theorem 4.1, the first term in this estimate goes to 0 if $y \in \mathcal{D}(T^\dagger)$. However, since $g_\alpha(\lambda) \rightarrow 1/\lambda$ as $\alpha \rightarrow 0$,

$$\lim_{\alpha \rightarrow 0} G_\alpha = \infty,$$

so that for fixed $\delta > 0$, the second term in the estimate (4.11) explodes.

We now give an estimate for the convergence rate in $\|x_\alpha - x^\dagger\|$ in terms of an estimate for the function r_α defined by (4.4):

Theorem 4.3. *Let g_α fulfill the assumptions of Theorem 4.1, r_α be defined by (4.4), $\mu > 0$. Let for all $\alpha \in (0, \alpha_0)$ and $\lambda \in [0, \|T\|^2]$,*

$$\lambda^\mu |r_\alpha(\lambda)| \leq c_\mu \alpha^\mu \quad (4.12)$$

holds for some $c_\mu > 0$ and assume that G_α as defined in (4.10) fulfills

$$G_\alpha = O(\alpha^{-1}) \quad \text{as } \alpha \rightarrow 0.$$

Then, with the parameter choice rule

$$\alpha \sim \delta^{\frac{2}{2\mu+1}},$$

we obtain for $x^\dagger \in \mathcal{R}((T^*T)^\mu)$

$$\|x_\alpha^\delta - x^\dagger\| = O(\delta^{\frac{2\mu}{2\mu+1}}).$$

The largest number $\mu = \mu_0$ for which (4.12) holds, is called *qualification* of the regularization method.

Example 4.4. Let, for $\alpha \in (0, \alpha_0)$, $\lambda \in [0, \|T\|^2]$,

$$g_\alpha(\lambda) := \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha, \\ 0, & \lambda < \alpha. \end{cases}$$

Then, the assumptions of Theorems 4.1, 4.2, and 4.3 hold with $C = 1$, $G_\alpha = 1/\alpha$, $c_\mu(\alpha) = 1$ for all $\mu > 0$, i.e., $\mu_0 = \infty$. For a compact operator K with singular system $(\sigma_n; v_n, u_n)$ this yields:

$$x_\alpha^\delta = \sum_{\substack{n=1 \\ \sigma_n^2 \geq \alpha}}^{\infty} \frac{\langle y^\delta, u_n \rangle}{\sigma_n} v_n.$$

This method is called *truncated singular value expansion*.

4.2. The discrepancy principle

For a-priori parameter choice rules the μ has to be known. However, this information is usually not available. There are also a-posteriori order-optimal rules. Well known is the *discrepancy principle* due to Morozov: the regularization parameter is defined as

$$\alpha(\delta, y^\delta) := \sup\{\alpha > 0 \mid \|Tx_\alpha^\delta - y^\delta\| \leq \tau\delta\}, \quad (4.13)$$

where

$$\tau > \sup\{|r_\alpha(\lambda)| \mid \alpha > 0, \lambda \in [0, \|T\|^2]\}.$$

If $\|Tx_\alpha^\delta - y^\delta\| \leq \tau\delta$ for all $\alpha > 0$, then $\alpha(\delta, y^\delta) = \infty$.

The idea of this method is obvious: since $\|y - y^\delta\| \leq \delta$, it does not make sense to ask for an approximate solution \tilde{x} with a discrepancy $\|T\tilde{x} - y^\delta\| < \delta$, a residual in the order of δ is the best we should ask for.

Theorem 4.5. *The regularization method (R_α, α) , where α is defined via the discrepancy principle (4.13), is convergent for all $y \in \mathcal{R}(T)$, and of optimal order in $\mathcal{R}((T^*T)^\mu)$ for $\mu \in (0, \mu_0 - 1/2]$, i.e., $\|x_{\alpha(\delta, y^\delta)}^\delta - x^\dagger\| = O(\delta^{\frac{2\mu}{2\mu+1}})$. For $\mu < \mu_0 - 1/2$, $O(\cdot)$ may be replaced by $o(\cdot)$.*

Unfortunately, optimal rates are not obtained for $\mu > \mu_0 - 1/2$ if μ_0 is finite as for Tikhonov regularization. However, there are improved a-posteriori rules that yield optimal rates up to μ_0 .

5. Tikhonov regularization

5.1. The classical theory

A special choice for g_α which fulfills the assumptions of Theorem 4.1 (with $C = 1$) is

$$g_\alpha(\lambda) := \frac{1}{\lambda + \alpha}.$$

With this choice, x_α^δ as defined by (4.3) has the form

$$x_\alpha^\delta = (T^*T + \alpha I)^{-1}T^*y^\delta, \quad (5.1)$$

i.e., is defined via the linear equation

$$T^*Tx_\alpha^\delta + \alpha x_\alpha^\delta = T^*y^\delta,$$

which can be thought of a regularized form of the normal equation.

This method is called *Tikhonov regularization*, sometimes it is also called *Tikhonov-Phillips regularization*. For a compact operator K with singular system $(\sigma_n; v_n, u_n)$, (5.1) has the form

$$x_\alpha^\delta = \sum_{n=1}^{\infty} \frac{\sigma_n}{\sigma_n^2 + \alpha} \langle y^\delta, u_n \rangle v_n.$$

A comparison with (2.1) clearly shows the stabilization: errors in $\langle y, u_n \rangle$ are not propagated with the factors $1/\sigma_n$, but only with the factors $\sigma_n/(\sigma_n^2 + \alpha)$ into the result; these factors stay bounded as $n \rightarrow \infty$.

Tikhonov regularization has the following *variational characterization*:

Theorem 5.1. *Let x_α^δ be as in (5.1). Then x_α^δ is the unique minimizer of the Tikhonov functional*

$$x \mapsto \|Tx - y^\delta\|^2 + \alpha\|x\|^2. \quad (5.2)$$

The Tikhonov functional in (5.2) sheds more light on the role of the regularization parameter. Minimization of (5.2) is a compromise between minimizing the residual norm, and keeping the “penalty term” $\|x\|$ small, i.e., enforcing stability.

While (5.1) makes sense only for a linear operator T , (5.2) can be formulated also for nonlinear operators.

Theorem 5.2. *Let x_α^δ be defined by (5.1), $y \in \mathcal{D}(T^\dagger)$, $\|y - y^\delta\| \leq \delta$. If $\alpha = \alpha(\delta)$ is such that*

$$\lim_{\delta \rightarrow 0} \alpha(\delta) = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} \frac{\delta^2}{\alpha(\delta)} = 0,$$

then

$$\lim_{\delta \rightarrow 0} x_{\alpha(\delta)}^\delta = T^\dagger y.$$

We now apply Theorem 4.3: G_α as defined by (4.10) has the form $G_\alpha = \alpha^{-1}$ and (4.12) holds with $c_\mu = 1$ for all $0 < \mu \leq 1 = \mu_0$. The best possible rate $\|x_\alpha^\delta - x^\dagger\| = O(\delta^{\frac{2}{3}})$ is obtained if $x^\dagger \in \mathcal{R}(T^*T)$ and if $\alpha \sim \delta^{\frac{2}{3}}$. Thus, Tikhonov regularization for an ill-posed linear problem *saturates* at this rate.

We now turn to a-posteriori parameter choice methods: it follows from Theorem 4.5 that $O(\sqrt{\delta})$ is the best possible rate that can be obtained with the discrepancy principle (4.13) if $\mu = 1/2$.

One can show that the a-posteriori method

$$\alpha^3 \|(TT^* + \alpha I)^{-\frac{3}{2}} Qy^\delta\|^2 = \tau \delta^2 \quad (5.3)$$

yields optimal rates for all $\mu \in (0, 1]$, where τ has to fulfill

$$\tau \in (1, \|Qy^\delta\|^2 \delta^{-2}) \quad (5.4)$$

Such a choice for τ is possible as long as the signal-to-noise-ratio condition $\|Qy^\delta\| > \delta$ is fulfilled.

For *iterated Tikhonov regularization*, which is defined as follows

$$\begin{aligned} x_{\alpha,0}^\delta &:= 0, \\ T^*Tx_{\alpha,i}^\delta + \alpha x_{\alpha,i}^\delta &= T^*y^\delta + \alpha x_{\alpha,i-1}^\delta, \quad i \in \mathbb{N}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} z_{\alpha,0}^\delta &:= 0, \\ TT^*z_{\alpha,i}^\delta + \alpha z_{\alpha,i}^\delta &= Qy^\delta + \alpha z_{\alpha,i-1}^\delta, \quad i \in \mathbb{N}, \\ x_{\alpha,i}^\delta &:= T^*z_{\alpha,i}^\delta, \end{aligned}$$

one can show that the qualification is $\mu_0 = n$, i.e., the best possible rate is then $\|x_{\alpha,n}^\delta - x^\dagger\| = O(\delta^{\frac{2n}{2n+1}})$ if $x^\dagger \in \mathcal{R}((T^*T)^n)$ and if α is appropriately chosen, e.g., by the a-psoetriori rule

$$\alpha^{2n+1} \langle (TT^* + \alpha I)^{-(2n+1)} Qy^\delta, Qy^\delta \rangle = \tau \delta^2$$

or equivalently

$$\alpha^2 \langle z_{\alpha,n+1}^\delta - z_{\alpha,n}^\delta, z_{\alpha,n}^\delta - z_{\alpha,n-1}^\delta \rangle = \tau \delta^2$$

with τ as in (5.4).

5.2. Numerical realization

For the numerical realization of Tikhonov regularization we have to approximate the space \mathcal{X} by a finite-dimensional subspace \mathcal{X}_n . This can be done in many ways. One is the following:

We assume that $\{\mathcal{Y}_n\}$ is a sequence of finite-dimensional subspaces of $\mathcal{N}(T^*)^\perp$ with $\mathcal{Y}_n \subset \mathcal{Y}_{n+1}$ and that $\bigcup_{n \in \mathbb{N}} \mathcal{Y}_n$ is dense in $\mathcal{N}(T^*)^\perp$. Let $Q_n : \mathcal{Y} \rightarrow \mathcal{Y}_n$ denote the orthogonal projector, then Q_n converges pointwise towards I on $\mathcal{N}(T^*)^\perp$ as $n \rightarrow \infty$. Actually, for the convergence analysis one only needs this pointwise convergence and not the inclusion $\mathcal{Y}_n \subset \mathcal{Y}_{n+1}$.

We consider the equation

$$T_n x = y_n, \quad T_n := Q_n T, \quad y_n := Q_n y,$$

and approximate x^\dagger by

$$x_{\alpha,n}^\delta := (T_n^* T_n + \alpha I)^{-1} T_n^* y_n^\delta,$$

with $\alpha > 0$. Note that $x_{\alpha,n}^\delta$ is like x_α^δ an element of $\mathcal{R}(T^*)$, due to the special choice of T_n . The definition of $x_{\alpha,n}^\delta$ may be extended to the case $\alpha = 0$ by $x_{0,n}^\delta := T_n^\dagger y_n^\delta$. Moreover, for the noise free case, $\alpha = 0$ is the best possible choice, since then $x_{\alpha,n}^\delta = x_{0,n}^\delta = P_n x^\dagger$, where P_n is the orthogonal projector onto $\mathcal{X}_n = T^* \mathcal{Y}_n$.

For noisy data y^δ , we will choose the regularization parameter α similar to the infinite-dimensional case (cf. (5.3)), i.e., as the solution of

$$\alpha^3 \langle (T_n T_n^* + \alpha I)^{-3} Q_n y_n^\delta, Q_n y_n^\delta \rangle = \tau \delta^2, \quad (5.5)$$

where $\tau \geq 1$. One can show that (5.5) has a unique solution, $\alpha_n^\delta > 0$, if

$$\|Q_n(y - y^\delta)\|^2 \leq \delta^2 < \frac{1}{\tau} \|Q_n y_n^\delta\|^2. \quad (5.6)$$

Let us assume that \mathcal{Y}_n is spanned by the functions $\{\psi_1, \dots, \psi_n\}$, then

$$x_{\alpha,n}^\delta = \sum_{i=1}^n \xi_i T^* \psi_i, \quad \mathbf{x} = [\xi_1, \dots, \xi_n]^T,$$

where \mathbf{x} solves the linear system

$$(M + \alpha H) \mathbf{x} = \mathbf{y},$$

with $M = [\langle T^* \psi_i, T^* \psi_j \rangle_{\mathcal{X}}]$, $\mathbf{y} = [\langle y, \psi_i \rangle_{\mathcal{Y}}]$, and $H = [\langle \psi_i, \psi_j \rangle_{\mathcal{Y}}]$.

The parameter rule (5.5) is realized via

$$\phi(\alpha) := \alpha^3 \mathbf{x}^T H(M + \alpha H)^{-1} H \mathbf{x} = \tau \delta^2.$$

This equation can be solved via Newton's method applied to $\psi(\beta) := \phi(\beta^{-1})$. Since ψ is convex, the convergence is quadratic and monoton provided the initial guess α_0 is chosen larger than the solution of (5.5).

For convergence rates we have the following result:

Theorem 5.3. *Let $\tau > 1$, let (5.6) hold and let α_n^δ be the solution of (5.5). Moreover, let*

$$x^\dagger = (T^*T)^\mu v, \quad v \in \mathcal{N}(T)^\perp,$$

for some $\mu \in [0, 1]$. Then, as $\delta \rightarrow 0$ and $n \rightarrow \infty$,

$$\|x_{\alpha_n^\delta, n}^\delta - x^\dagger\| = \begin{cases} o(1), & \text{if } \mu = 0, \\ o(\delta^{\frac{2\mu}{2\mu+1}}) + O(\eta_n^\mu(v)), & \text{if } 0 < \mu < 1, \\ O(\delta^{\frac{2}{3}}) + O(\gamma_n \|(I - Q_n)Tv\|), & \text{if } \mu = 1, \end{cases}$$

where

$$\eta_n(v) := \sup_{t>0} t \|T^*(I - Q_n)T(T^*T + tI)^{-1}v\| \|v\|^{-1} \leq \gamma_n^2$$

and

$$\gamma_n := \|(I - Q_n)T\|.$$

Note that for compact operators $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$.

One can show that the rate is optimal with respect to δ and the finite dimensional approximation.

The next result shows for $\tau = 1$ in (5.5) the smallest error is obtained.

Proposition 5.4. *Let n, δ be fixed and let (5.6) (with $\tau = 1$) hold. Moreover, let α_n^δ be the unique solution of (5.5) (with $\tau = 1$). Then for all $\alpha \geq \alpha_n^\delta$*

$$\|x_{\alpha_n^\delta, n}^\delta - x^\dagger\| \leq \|x_{\alpha, n}^\delta - x^\dagger\|$$

holds.

Part 3: Thursday, March 27, 16:00-18:00

6. Iterative regularization methods

6.1. Landweber iteration

Most iterative methods for approximating $T^\dagger y$ are based on a transformation of the normal equation into equivalent fixed point equations like

$$x = x + T^*(y - Tx).$$

If $\|T\|^2 < 2$ then the corresponding fixed point operator $I - T^*T$ is nonexpansive and one may apply the method of successive approximations. Note that $I - T^*T$ is no contraction if our basic problem is ill-posed, since the spectrum of T^*T clusters at the origin.

Besides Landweber iteration, other iterative methods are used for inverse problems that do not fit into the framework of this chapter. Most prominent is the method of conjugate gradients which is a nonlinear regularization method even for linear operators T .

The Landweber iteration $\{x_k^\delta\}$ is defined recursively:

$$x_k^\delta = x_{k-1}^\delta + T^*(y^\delta - Tx_{k-1}^\delta), \quad k \in \mathbb{N}. \quad (6.1)$$

Throughout the analysis it we assume that the initial guess $x_0^\delta = 0$ and that $\|T\| \leq 1$. If this were not the case, then one would have to introduce a relaxation parameter $0 < \omega \leq \|T\|^{-2}$ in front of T^* in (6.1).

Note that for this method also the theory for continuous regularization methods can be applied with:

$$g_k(\lambda) = \sum_{j=0}^{k-1} (1 - \lambda)^j \quad \text{and} \quad r_k(\lambda) = (1 - \lambda)^k,$$

Thus, we can show:

Theorem 6.1. *If $y \in \mathcal{D}(T^\dagger)$, then $x_k \rightarrow T^\dagger y$ as $k \rightarrow \infty$. If $y \notin \mathcal{D}(T^\dagger)$, then $\|x_k\| \rightarrow \infty$ as $k \rightarrow \infty$.*

Since $\|x_k - x_k^\delta\| \leq \sqrt{k}\delta$, we obtain convergence for x_k^δ as $\delta \rightarrow 0$ if the stopping index k is chosen such that $\sqrt{k}\delta \rightarrow 0$. As an a-posteriori stopping method we consider the discrepancy principle for the case $y \in \mathcal{R}(T)$: the iteration is terminated with $k = k(\delta, y^\delta)$ when for the first time

$$\|y^\delta - Tx_{k(\delta, y^\delta)}^\delta\| \leq \tau\delta, \quad (6.2)$$

with $\tau > 1$ fixed. The following observation due to Defrise and de Mol motivates its usefulness.

Proposition 6.2. *Let $y \in \mathcal{R}(T)$ and consider any solution x of $Tx = y$. If $\|y^\delta - Tx_k^\delta\| > 2\delta$, then x_{k+1}^δ is a better approximation of x^\dagger than x_k^δ .*

Proof. We estimate

$$\begin{aligned} \|x^\dagger - x_{k+1}^\delta\|^2 &= \|x^\dagger - x_k^\delta - T^*(y^\delta - Tx_k^\delta)\|^2 \\ &= \|x^\dagger - x_k^\delta\|^2 - 2\langle x^\dagger - x_k^\delta, T^*(y^\delta - Tx_k^\delta) \rangle \\ &\quad + \langle y^\delta - Tx_k^\delta, TT^*(y^\delta - Tx_k^\delta) \rangle \\ &= \|x^\dagger - x_k^\delta\|^2 - 2\langle y - y^\delta, y^\delta - Tx_k^\delta \rangle - \|y^\delta - Tx_k^\delta\|^2 \\ &\quad + \langle y^\delta - Tx_k^\delta, (TT^* - I)(y^\delta - Tx_k^\delta) \rangle. \end{aligned}$$

As $TT^* - I$ is negative semidefinite we obtain

$$\|x^\dagger - x_k^\delta\|^2 - \|x^\dagger - x_{k+1}^\delta\|^2 \geq \|y^\delta - Tx_k^\delta\| (\|y^\delta - Tx_k^\delta\| - 2\delta).$$

If $\|y^\delta - Tx_k^\delta\| > 2\delta$, then the right-hand side is positive which was to be shown. ■

In other words, the iteration should not be terminated before (6.2) with $\tau = 2$ is violated; the discrepancy principle with $\tau = 2$ determines the beginning of the transition of the iteration from convergence to divergence. If the discrepancy principle is employed with some $\tau \in (1, 2)$, then it depends on the particular noise sample whether the chosen stopping index is at the beginning or at the end of the transition phase.

Proposition 6.3. *If $\tau > 1$ in (6.2) is fixed, then the discrepancy principle determines a finite stopping index $k(\delta, y^\delta)$ for the Landweber iteration, with $k(\delta, y^\delta) = O(\delta^{-2})$.*

Theorem 6.4. *If $y \in \mathcal{R}(T)$, then the Landweber iteration with the discrepancy principle (6.2) ($\tau > 1$ fixed) is an order optimal regularization method, i.e., if $x^\dagger \in \mathcal{R}((T^*T)^\mu)$ with $\mu > 0$, then $\|x_{k(\delta, y^\delta)}^\delta - x^\dagger\| = O(\delta^{\frac{2\mu}{2\mu+1}})$.*

This shows that Landweber iteration has qualification $\mu_0 = \infty$.

Unfortunately, Landweber iteration (for linear operators T) usually requires far too many iterations until the stopping criterion (6.2) is met. Therefore, is rarely used in practice.

6.2. Accelerated Landweber Methods

The major drawback of Landweber iteration is its comparatively slow rate of convergence. Far too many iterations are required to reduce the residual to the order of the noise level. In the past decade, more sophisticated iteration methods have been developed on the basis of so-called *semiiterative methods*: a basic step of a semiiterative method consists of one step of iteration (6.1), followed by an averaging process over all or some of the previously obtained approximations. These methods fit in our spectral theoretic framework. The functions g_k and r_k (the regularization parameter remains the iteration index, as in Landweber iteration) are now polynomials of degree $k - 1$ and k , respectively.

It can be shown that semiiterative methods, where r_k satisfies

$$\|\lambda^\mu r_k(\lambda)\|_{C[0,1]} = \omega_\mu(k) = O(k^{-2\mu}), \quad k \in \mathbb{N}.$$

are order optimal and need much fewer iterations than the classical Landweber iteration (asymptotically only the square root).

Examples for such methods are the so-called ν -methods introduced by Brakhage. He introduced them to obtain theoretical estimates for the performance of the conjugate gradient method. In the meantime they have been established as a promising alternative to the conjugate gradient method.

7. Tikhonov regularization of nonlinear problems

7.1. Introduction

We have seen in the chapters above that the theory for linear ill-posed problems is very well developed and can be considered as rather complete. Now we want to deal with the fully nonlinear case, where the theory is by far not so well developed as in the linear one, i.e., we want to solve

$$F(x) = y, \tag{7.1}$$

where $F : \mathcal{D}(F) \subset \mathcal{X} \rightarrow \mathcal{Y}$ is a nonlinear operator between Hilbert spaces \mathcal{X} and \mathcal{Y} . Under ill-posedness of the nonlinear problem we will always mean that the solutions do not depend continuously on the data. Throughout this chapter we assume that

- (i) F is continuous and
- (ii) F is weakly (sequentially) closed, i.e., for any sequence $\{x_n\} \subset \mathcal{D}(F)$, weak convergence of x_n to x in \mathcal{X} and weak convergence of $F(x_n)$ to y in \mathcal{Y} imply that $x \in \mathcal{D}(F)$ and $F(x) = y$.

More general than in the linear case, where we looked for the minimum-norm solution, for the nonlinear problem (7.1), we choose the concept of an x^* -minimum-norm solution x^\dagger , i.e.:

$$F(x^\dagger) = y$$

and

$$\|x^\dagger - x^*\| = \min\{\|x - x^*\| \mid F(x) = y\}.$$

The reason for treating the general x^* case is that for nonlinear problems $x^* = 0$ plays no special role. In the contrary, the choice of x^* will be very crucial, especially for the local results about convergence rates in the next section. Of course, available a-priori information about the solutions of $F(x) = y$ has to enter into the selection of x^* . In the case of multiple solutions x^* plays the role of a selection criterion. As for linear problems, an x^* -minimum-norm solution need not exist, and even if it does, it need not be unique, since F is nonlinear. In the following we assume the existence of an x^* -minimum-norm solution x^\dagger for the data $y \in \mathcal{Y}$, which, by the weak closedness of F , follows from the attainability assumption that equation (7.1) has an exact solution.

As for linear problems, we want to find a criterion to decide whether a nonlinear problem is ill-posed or not. If the nonlinear operator F is compact, one can give a sufficient condition for ill-posedness of (7.1) being similar to its compact linear counterpart. There injectivity of the linear operator is such a sufficient condition, provided that \mathcal{X} is infinite-dimensional (see Proposition 2.7). This condition carries over to the nonlinear case in the sense that local injectivity around x^\dagger of the nonlinear compact operator F implies the ill-posedness of problem (7.1) provided that $\mathcal{D}(F)$ is infinite-dimensional around x^\dagger (see (7.2)).

Proposition 7.1. *Let F be a (nonlinear) compact and weakly closed operator, and let $\mathcal{D}(F)$ be weakly closed. Moreover, assume that $F(x^\dagger) = y$ and that there exists an $\varepsilon > 0$ such that $F(x) = \bar{y}$ has a unique solution for all $\bar{y} \in \mathcal{R}(F) \cap \mathcal{B}_\varepsilon(y)$. If there exists a sequence $\{x_n\} \subset \mathcal{D}(F)$ satisfying*

$$x_n \rightharpoonup x^\dagger \quad \text{but} \quad x_n \not\rightarrow x^\dagger, \quad (7.2)$$

then F^{-1} (defined on $\mathcal{R}(F) \cap \mathcal{B}_\varepsilon(y)$) is not continuous in y .

Since there is an easy way for linear problems to characterize their stability via the closedness of the range of the linear operator (cf. Proposition 2.4), it were helpful if one could characterize the stability of a nonlinear problem through conditions on its linearization (if F is Fréchet-differentiable). However, although F' is compact if F is compact, it might happen that $\mathcal{R}(F'(x^\dagger))$ is finite-dimensional so that the generalized inverse of the linearization is bounded. It is even possible to construct a nonlinear operator such that the corresponding nonlinear problem is everywhere ill-posed, whereas the linearized operator has a finite-dimensional range at a dense set of points, i.e., the linearization is well-posed almost everywhere. On the other hand, well-posed nonlinear problems may have ill-posed linearizations.

7.2. Convergence Analysis

As in the linear case, we replace problem (7.1) by the minimization problem

$$\|F(x) - y^\delta\|^2 + \alpha \|x - x^*\|^2 \rightarrow \min, \quad x \in \mathcal{D}(F), \quad (7.3)$$

where $\alpha > 0$, $y^\delta \in \mathcal{Y}$ is an approximation of the exact right-hand side y of problem (7.1) and $x^* \in \mathcal{X}$. Under the given assumptions on F problem (7.3) admits a solution. Since F is nonlinear, the solution will not be unique, in general. As in the linear case, any solution to (7.3) will be denoted by x_α^δ . Before we address the question of convergence, we will first prove that the problem of solving (7.3) is stable in the sense of continuous dependence of the solutions on the data y^δ .

Theorem 7.2. *Let $\alpha > 0$ and let $\{y_k\}$ and $\{x_k\}$ be sequences where $y_k \rightarrow y^\delta$ and x_k is a minimizer of (7.3) with y^δ replaced by y_k . Then there exists a convergent subsequence of $\{x_k\}$ and the limit of every convergent subsequence is a minimizer of (7.3).*

Proof. By definition of x_k we have

$$\|F(x_k) - y_k\|^2 + \alpha \|x_k - x^*\|^2 \leq \|F(x) - y_k\|^2 + \alpha \|x - x^*\|^2 \quad (7.4)$$

for any $x \in \mathcal{D}(F)$. Hence, $\{\|x_k\|\}$ and $\{\|F(x_k)\|\}$ are bounded. Therefore, a subsequence $\{x_m\}$ of $\{x_k\}$ and \bar{x} exist such that

$$x_m \rightharpoonup \bar{x} \quad \text{and} \quad F(x_m) \rightharpoonup F(\bar{x}).$$

By weak lower semicontinuity of the norm we have

$$\|\bar{x} - x^*\| \leq \liminf \|x_m - x^*\|$$

and

$$\|F(\bar{x}) - y^\delta\| \leq \liminf \|F(x_m) - y_m\|. \quad (7.5)$$

Moreover, (7.4) implies

$$\begin{aligned} \|F(\bar{x}) - y^\delta\|^2 + \alpha \|\bar{x} - x^*\|^2 &\leq \liminf (\|F(x_m) - y_m\|^2 + \alpha \|x_m - x^*\|^2) \\ &\leq \limsup (\|F(x_m) - y_m\|^2 + \alpha \|x_m - x^*\|^2) \\ &\leq \lim_{m \rightarrow \infty} (\|F(x) - y_m\|^2 + \alpha \|x - x^*\|^2) \\ &= \|F(x) - y^\delta\|^2 + \alpha \|x - x^*\|^2 \end{aligned}$$

for all $x \in \mathcal{D}(F)$. This implies that \bar{x} is a minimizer of (7.3) and that

$$\lim_{m \rightarrow \infty} (\|F(x_m) - y_m\|^2 + \alpha \|x_m - x^*\|^2) = \|F(\bar{x}) - y^\delta\|^2 + \alpha \|\bar{x} - x^*\|^2. \quad (7.6)$$

Now, assume that $x_m \not\rightarrow \bar{x}$. Then $c := \limsup \|x_m - x^*\| > \|\bar{x} - x^*\|$ and there exists a subsequence $\{x_n\}$ of $\{x_m\}$ such that $x_n \rightharpoonup \bar{x}$, $F(x_n) \rightharpoonup F(\bar{x})$ and $\|x_n - x^*\| \rightarrow c$. As a consequence of (7.6), we obtain

$$\lim_{n \rightarrow \infty} \|F(x_n) - y_n\|^2 = \|F(\bar{x}) - y^\delta\|^2 + \alpha (\|\bar{x} - x^*\|^2 - c^2) < \|F(\bar{x}) - y^\delta\|^2$$

in contradiction to (7.5). This argument shows that $x_m \rightarrow \bar{x}$. ■

In the next theorem we show that under the same conditions on $\alpha(\delta)$ as in the linear case solutions of (7.3) converge towards an x^* -minimum-norm solution of (7.1).

Theorem 7.3. *Let $y^\delta \in \mathcal{Y}$ with $\|y - y^\delta\| \leq \delta$ and let $\alpha(\delta)$ be such that $\alpha(\delta) \rightarrow 0$ and $\delta^2/\alpha(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Then every sequence $\{x_{\alpha_k}^{\delta_k}\}$, where $\delta_k \rightarrow 0$, $\alpha_k := \alpha(\delta_k)$ and $x_{\alpha_k}^{\delta_k}$ is a solution of (7.3), has a convergent subsequence. The limit of every convergent subsequence is an x^* -minimum-norm solution. If in addition, the x^* -minimum-norm solution x^\dagger is unique, then*

$$\lim_{\delta \rightarrow 0} x_{\alpha(\delta)}^\delta = x^\dagger.$$

Proof. Let α_k and $x_{\alpha_k}^{\delta_k}$ be as above and let x^\dagger be an x^* -minimum-norm solution. Then by the definition of $x_{\alpha_k}^{\delta_k}$

$$\|F(x_{\alpha_k}^{\delta_k}) - y^{\delta_k}\|^2 + \alpha_k \|x_{\alpha_k}^{\delta_k} - x^*\|^2 \leq \delta_k^2 + \alpha_k \|x^\dagger - x^*\|^2$$

and hence

$$\lim_{k \rightarrow \infty} F(x_{\alpha_k}^{\delta_k}) = y \quad (7.7)$$

and

$$\limsup_{k \rightarrow \infty} \|x_{\alpha_k}^{\delta_k} - x^*\| \leq \|x^\dagger - x^*\|. \quad (7.8)$$

Therefore, $\{x_{\alpha_k}^{\delta_k}\}$ is bounded. Hence, there exist an element $x \in \mathcal{X}$ and a subsequence again denoted by $\{x_{\alpha_k}^{\delta_k}\}$ such that

$$x_{\alpha_k}^{\delta_k} \rightharpoonup x \quad \text{as } k \rightarrow \infty. \quad (7.9)$$

This together with (7.7) implies that $x \in \mathcal{D}(F)$ and that $F(x) = y$. Now by the weak lower semicontinuity of the norm, (7.8), (7.9), and the definition of x^\dagger ,

$$\|x - x^*\| \leq \limsup_{k \rightarrow \infty} \|x_{\alpha_k}^{\delta_k} - x^*\| \leq \|x^\dagger - x^*\| \leq \|x - x^*\|.$$

This shows that $\|x - x^*\| = \|x^\dagger - x^*\|$, i.e., x is also an x^* -minimum-norm solution. This together with (7.8), (7.9) and the fact that

$$\|x_{\alpha_k}^{\delta_k} - x\|^2 = \|x_{\alpha_k}^{\delta_k} - x^*\|^2 + \|x^* - x\|^2 + 2\langle x_{\alpha_k}^{\delta_k} - x^*, x^* - x \rangle$$

yields that

$$\limsup_{k \rightarrow \infty} \|x_{\alpha_k}^{\delta_k} - x\|^2 \leq 2\|x^* - x\|^2 + 2 \lim_{k \rightarrow \infty} \langle x_{\alpha_k}^{\delta_k} - x^*, x^* - x \rangle = 0$$

and hence that

$$\lim_{k \rightarrow \infty} x_{\alpha_k}^{\delta_k} = x.$$

If x^\dagger is unique, the assertion about the convergence of $x_{\alpha(\delta)}^\delta$ follows from the fact that every sequence has a subsequence converging towards x^\dagger . ■

We now turn to the convergence rate analysis. In the next theorem, we will give sufficient conditions for the rate $\|x_\alpha^\delta - x^\dagger\| = O(\sqrt{\delta})$. These conditions will also imply the rate $\|F(x_\alpha^\delta) - y\| = O(\delta)$ for the residuals. Note that one can not use spectral-theoretic techniques as for linear problems.

Theorem 7.4. *Let $\mathcal{D}(F)$ be convex, let $y^\delta \in \mathcal{Y}$ with $\|y - y^\delta\| \leq \delta$ and let x^\dagger be an x^* -minimum-norm solution. Moreover, let the following conditions hold:*

- (i) *F is Fréchet-differentiable,*
- (ii) *there exists $\gamma \geq 0$ such that $\|F'(x^\dagger) - F'(x)\| \leq \gamma\|x^\dagger - x\|$ for all $x \in \mathcal{D}(F)$ in a sufficiently large ball around x^\dagger ,*
- (iii) *there exists $w \in \mathcal{Y}$ satisfying $x^\dagger - x^* = F'(x^\dagger)^*w$ and*
- (iv) *$\gamma\|w\| < 1$.*

Then for the choice $\alpha \sim \delta$, we obtain

$$\|x_\alpha^\delta - x^\dagger\| = O(\sqrt{\delta}) \quad \text{and} \quad \|F(x_\alpha^\delta) - y^\delta\| = O(\delta).$$

Remark 7.5. Even if the x^* -minimum-norm solution is not unique, it is a consequence of Theorem 7.4 that only one x^* -minimum-norm solution x^\dagger can fulfill conditions (ii)–(iv).

In the linear case, the best possible convergence rate for $\|x_\alpha^\delta - x^\dagger\|$ is $O(\delta^{\frac{2}{3}})$, and this rate is obtained under the condition $x^\dagger - x^* \in \mathcal{R}(F^*F)$. The same is true in the nonlinear case if we replace $\mathcal{R}(F^*F)$ by $\mathcal{R}(F'(x^\dagger)^*F'(x^\dagger))$. With some $\mu \in [1/2, 1]$, it is even possible to prove results for $x^\dagger - x^* \in \mathcal{R}((F'(x^\dagger)^*F'(x^\dagger))^\mu)$.

Theorem 7.6. *Under the assumptions of Theorem 7.4, if x^\dagger is an element in the interior of $\mathcal{D}(F)$ and if $x^\dagger - x^* = (F'(x^\dagger)^*F'(x^\dagger))^\mu v$ for some $\mu \in [1/2, 1]$, then, for the choice $\alpha \sim \delta^{\frac{2}{2\mu+1}}$, we obtain*

$$\|x_\alpha^\delta - x^\dagger\| = O(\delta^{\frac{2\mu}{2\mu+1}}).$$

We have already mentioned that in the linear case $O(\delta^{\frac{2}{3}})$ is the best possible rate if $\alpha \sim \delta^{\frac{2}{3}}$, except for the trivial case that $x^\dagger = x^*$. It is also known that the condition $x^\dagger - x^* \in \mathcal{R}(F^*F)$ is necessary and sufficient for the rate $O(\delta^{\frac{2}{3}})$. Analogous assertions also hold in the nonlinear case.

Finally, we want to mention that in computational reality it is, in general, not possible to solve the nonlinear minimization problem (7.3) exactly. So instead of solving (7.3) one is really looking for a solution $x_\alpha^{\delta,\eta}$ of the problem

$$\|F(x_\alpha^{\delta,\eta}) - y^\delta\|^2 + \alpha \|x_\alpha^{\delta,\eta} - x^*\|^2 \leq \|F(x) - y^\delta\|^2 + \alpha \|x - x^*\|^2 + \eta$$

for all $x \in \mathcal{D}(F)$, where $\eta \geq 0$ is a small parameter. All results concerning convergence and convergence rates remain valid for this approach, as long as it is guaranteed that $\eta/\alpha \rightarrow 0$.

Part 4: Friday, March 28, 10:00-12:00

Fourth part is taken from the book:

B. Kaltenbacher, A. Neubauer, and O. Scherzer, *Iterative Regularization Methods for Nonlinear Ill-posed Problems*, Radon Series on Computational and Applied Mathematics, Vol. 6, de Gruyter, Berlin, 2008.

8. Nonlinear Landweber iteration

We assume throughout this chapter that F is a map between Hilbert spaces \mathcal{X} and \mathcal{Y} , and that F has a continuous Fréchet-derivative $F'(\cdot)$, the *nonlinear Landweber iteration* is defined via

$$x_{k+1}^\delta = x_k^\delta + F'(x_k^\delta)^*(y^\delta - F(x_k^\delta)), \quad k \in \mathbb{N}_0. \quad (8.1)$$

By $x_0^\delta = x_0$ we again denote an initial guess.

In case of noisy data, the iteration procedure has to be combined with a stopping rule in order to act as a regularization method. We will employ the *discrepancy principle*, i.e., the iteration is stopped after $k_* = k_*(\delta, y^\delta)$ steps with

$$\|y^\delta - F(x_{k_*}^\delta)\| \leq \tau\delta < \|y^\delta - F(x_k^\delta)\|, \quad 0 \leq k < k_*, \quad (8.2)$$

where τ is an appropriately chosen positive number.

For nonlinear problems, iteration methods like (8.1) will in general not converge globally. We are able to prove local convergence if we impose some conditions on F . As in the linear case, the Landweber iteration can only converge if problem $F(x) = y$ is properly scaled. For our analysis we assume that

$$\|F'(x)\| \leq 1, \quad x \in \mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F), \quad (8.3)$$

where $\mathcal{B}_{2\rho}(x_0)$ denotes a closed ball of radius 2ρ around x_0 .

In addition to this scaling property, we need the following local condition:

$$\begin{aligned} \|F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x})\| &\leq \eta \|F(x) - F(\tilde{x})\|, \quad \eta < \frac{1}{2}, \\ x, \tilde{x} &\in \mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F). \end{aligned} \quad (8.4)$$

Both conditions are strong enough to ensure local convergence to a solution of the equation $F(x) = y$ if it is solvable in $\mathcal{B}_\rho(x_0)$. They also guarantee that all iterates x_k^δ , $0 \leq k \leq k_*$, remain in $\mathcal{D}(F)$, which makes the Landweber iteration well defined.

From (8.4) it follows immediately with the triangle inequality that

$$\frac{1}{1+\eta} \|F'(x)(\tilde{x} - x)\| \leq \|F(\tilde{x}) - F(x)\| \leq \frac{1}{1-\eta} \|F'(x)(\tilde{x} - x)\| \quad (8.5)$$

for all $x, \tilde{x} \in \mathcal{B}_{2\rho}(x_0)$. Thus, condition (8.4) seems to be rather restrictive. However, the condition is quite natural as the following argument shows: if $F'(\cdot)$ is Lipschitz continuous and $x, \tilde{x} \in \mathcal{D}(F)$, then the error bound

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \leq c \|\tilde{x} - x\|^2 \quad (8.6)$$

holds for the Taylor approximation of F . For ill-posed problems, however, it turns out that this estimate carries too little information about the local behaviour of F around x to draw conclusions about convergence of the nonlinear Landweber iteration, since the left hand side of (8.6) can be much smaller than the right hand side for certain pairs of points \tilde{x} and x , whatever close to each other they are. For example, fix $x \in \mathcal{D}(F)$, and assume that F is weakly closed and compact. Then $F'(x)$ is compact, and hence, for every sequence $\{\tilde{x}_n\}$ with $\tilde{x}_n \in \mathcal{D}(F)$, $\|\tilde{x}_n - x\| = \varepsilon$ for all $n \in \mathbb{N}$, and $\tilde{x}_n \rightharpoonup x$ as $n \rightarrow \infty$, the left hand side of (8.6) goes to zero as $n \rightarrow \infty$ whereas the right hand side remains $c\varepsilon^2$ for all n .

For several examples one can even prove the stronger condition

$$\|F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x})\| \leq c\|x - \tilde{x}\| \|F(x) - F(\tilde{x})\| .$$

Provided $\|x - \tilde{x}\|$ is sufficiently small, this implies condition (8.4).

To prove local convergence, we that the equation $F(x) = y$ is solvable in $\mathcal{B}_\rho(x_0)$. Note that then (8.5) even implies the existence of a unique solution of minimal distance to x_0 , denoted as above by x_0 -*minimum-norm solution*.

Before we turn to a convergence analysis of the nonlinear Landweber iteration we want to emphasize that for fixed iteration index k the iterate x_k^δ depends continuously on the data y^δ , since x_k^δ is the result of a combination of continuous operations.

To begin with, we formulate the following monotonicity property that gives us a clue how to choose the number τ in the stopping rule (8.2) (compare Proposition 6.2).

Proposition 8.1. *Assume that the conditions (8.3) and (8.4) hold and that the equation $F(x) = y$ has a solution $x_* \in \mathcal{B}_\rho(x_0)$. If $x_k^\delta \in \mathcal{B}_\rho(x_*)$, a sufficient condition for x_{k+1}^δ to be a better approximation of x_* than x_k^δ is that*

$$\|y^\delta - F(x_k^\delta)\| > 2 \frac{1 + \eta}{1 - 2\eta} \delta . \quad (8.7)$$

Moreover, it then holds that $x_k^\delta, x_{k+1}^\delta \in \mathcal{B}_\rho(x_*) \subset \mathcal{B}_{2\rho}(x_0)$.

Proof. Let us assume that $x_k^\delta \in \mathcal{B}_\rho(x_*)$. Then, due to the triangle inequality, $x_*, x_k^\delta \in \mathcal{B}_{2\rho}(x_0)$. Hence, (8.3) and (8.4) are applicable and we obtain together with definition (8.1) that

$$\begin{aligned} & \|x_{k+1}^\delta - x_*\|^2 - \|x_k^\delta - x_*\|^2 \\ &= 2\langle x_{k+1}^\delta - x_k^\delta, x_k^\delta - x_* \rangle + \|x_{k+1}^\delta - x_k^\delta\|^2 \\ &= 2\langle y^\delta - F(x_k^\delta), F'(x_k^\delta)(x_k^\delta - x_*) \rangle + \|F'(x_k^\delta)^*(y^\delta - F(x_k^\delta))\|^2 \\ &\leq 2\langle y^\delta - F(x_k^\delta), y^\delta - F(x_k^\delta) - F'(x_k^\delta)(x_* - x_k^\delta) \rangle - \|y^\delta - F(x_k^\delta)\|^2 \\ &\leq \|y^\delta - F(x_k^\delta)\| (2\delta + 2\eta\|y - F(x_k^\delta)\| - \|y^\delta - F(x_k^\delta)\|) \\ &\leq \|y^\delta - F(x_k^\delta)\| (2(1 + \eta)\delta - (1 - 2\eta)\|y^\delta - F(x_k^\delta)\|) . \end{aligned}$$

The assertions now follow since the right hand side is negative if (8.7) holds. \blacksquare

In view of this proposition, the number τ in the stopping rule (8.2) should be chosen subject to the following constraint depending on η , with η as in (8.4):

$$\tau > 2 \frac{1 + \eta}{1 - 2\eta} > 2 . \quad (8.8)$$

From the proof of Proposition 8.1 we can easily extract an inequality that guarantees that the stopping index k_* in (8.2) is finite and hence well defined.

Corollary 8.2. *Let the assumptions of Proposition 8.1 hold and let k_* be chosen according to the stopping rule (8.2), (8.8). Then*

$$k_*(\tau\delta)^2 < \sum_{k=0}^{k_*-1} \|y^\delta - F(x_k^\delta)\|^2 \leq \frac{\tau}{(1-2\eta)\tau - 2(1+\eta)} \|x_0 - x_*\|^2.$$

In particular, if $y^\delta = y$ (i.e., if $\delta = 0$), then

$$\sum_{k=0}^{\infty} \|y - F(x_k)\|^2 < \infty.$$

Theorem 8.3. *Assume that the conditions (8.3) and (8.4) hold and that $F(x) = y$ is solvable in $\mathcal{B}_\rho(x_0)$. Then the nonlinear Landweber iteration applied to exact data y converges to a solution of $F(x) = y$.*

If $k_ = k_*(\delta, y^\delta)$ is chosen according to the stopping rule (8.2), (8.8). Then the Landweber iterates $x_{k_*}^\delta$ converge to a solution of $F(x) = y$.*

If $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$ for all $x \in \mathcal{B}_\rho(x^\dagger)$, then x_k and $x_{k_}^\delta$ converge to x^\dagger as $k \rightarrow \infty$ and $\delta \rightarrow 0$, respectively.*

To obtain convergence rates results, x^\dagger has to satisfy source conditions of the kind

$$x^\dagger - x_0 = (F'(x^\dagger)^* F'(x^\dagger))^\mu v, \quad v \in \mathcal{N}(F'(x^\dagger))^\perp. \quad (8.9)$$

In contrast to Tikhonov regularization, for Landweber iteration we need additional conditions like

$$F'(x) = R_x F'(x^\dagger) \quad \text{and} \quad \|R_x - I\| \leq c \|x - x^\dagger\|, \quad x \in \mathcal{B}_{2\rho}(x_0),$$

where $\{R_x : x \in \mathcal{B}_{2\rho}(x_0)\}$ is a family of bounded linear operators $R_x : \mathcal{Y} \rightarrow \mathcal{Y}$ and c is a positive constant.

Unfortunately, these conditions are not always satisfied. To enlarge the applicability of the results we consider instead of (8.1) the following slightly modified iteration method,

$$x_{k+1}^\delta = x_k^\delta + \omega G^\delta(x_k^\delta)^*(y^\delta - F(x_k^\delta)), \quad k \in \mathbb{N}_0, \quad (8.10)$$

where $G^\delta(x) := G(x, y^\delta)$, and G is a continuous operator mapping $\mathcal{D}(F) \times \mathcal{Y}$ into $L(\mathcal{X}, \mathcal{Y})$. The iteration will again be stopped according to the discrepancy principle (8.2).

To obtain local convergence and convergence rates for the modification (8.10) we need the following assumptions:

Assumption 8.4. Let ρ be a positive number such that $\mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F)$.

(i) The equation $F(x) = y$ has an x_0 -minimum-norm solution x^\dagger in $\mathcal{B}_\rho(x_0)$.

(ii) There exist positive constants c_1, c_2, c_3 and linear operators R_x^δ such that for all $x \in \mathcal{B}_\rho(x^\dagger)$ the following estimates hold:

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| \leq c_1 \|F(x) - F(x^\dagger)\| \|x - x^\dagger\|,$$

$$G^\delta(x) = R_x^\delta G^\delta(x^\dagger),$$

$$\|R_x^\delta - I\| \leq c_2 \|x - x^\dagger\|,$$

$$\|F'(x^\dagger) - G^\delta(x^\dagger)\| \leq c_3 \delta.$$

(iii) The scaling parameter ω in (8.10) satisfies the condition

$$\omega \|F'(x^\dagger)\|^2 \leq 1.$$

If we, in addition to Assumption 8.4, require that x^\dagger satisfies the source condition (8.9), then we even obtain convergence rates which are the same as for linear ill-posed problems if $\mu \leq 1/2$ in (8.9).

Theorem 8.5. *Let Assumption 8.4 hold and let $k_* = k_*(\delta, y^\delta)$ be chosen according to the stopping rule (8.2) with $\tau > 2$. If $x^\dagger - x_0$ satisfies (8.9) with some $0 < \mu \leq 1/2$ and $\|v\|$ sufficiently small, then it holds that*

$$k_* = O\left(\delta^{-\frac{2}{2\mu+1}}\right)$$

and

$$\|x_{k_*}^\delta - x^\dagger\| = \begin{cases} o\left(\delta^{\frac{2\mu}{2\mu+1}}\right), & \mu < \frac{1}{2}, \\ O(\sqrt{\delta}), & \mu = \frac{1}{2}. \end{cases}$$

Under the Assumptions 8.4, and according to Theorem 8.5 the best possible rate of convergence is

$$\|x_{k_*}^\delta - x^\dagger\| = O(\sqrt{\delta})$$

attained when $\mu = 1/2$. Even if $\mu > 1/2$ we cannot improve this rate without an additional restriction of the *nonlinearity* of F .

9. Modified Landweber methods

Sometimes better rates may be obtained for Landweber iteration and solutions that satisfy stronger smoothness conditions if the iteration is performed in a subspace of \mathcal{X} with a stronger norm. This leads us directly to regularization in Hilbert scales. On the other hand for solutions with poor smoothness properties the number of iterations may be reduced if the iteration is performed in a space with a weaker norm.

Other variants are the steepest descent and the minimal error method. These methods can be viewed as Landweber iteration with variable coefficients ω :

$$x_{k+1}^\delta = x_k^\delta + \omega_k^\delta s_k^\delta, \quad s_k^\delta := F'(x_k^\delta)^*(y^\delta - F(x_k^\delta)), \quad k \in \mathbb{N}_0,$$

where the coefficients ω_k^δ are chosen as

$$\omega_k^\delta := \frac{\|s_k^\delta\|^2}{\|F'(x_k^\delta)s_k^\delta\|^2}$$

for the *steepest descent method* and

$$\omega_k^\delta := \frac{\|y^\delta - F(x_k^\delta)\|^2}{\|s_k^\delta\|^2}$$

for the *minimal error method*. Note that the choice $\omega_k^\delta = 1$ corresponds to the classical Landweber iteration.

An interesting alternative is the Landweber-Kaczmarz method, a variant of Landweber iteration for problems where F describes a system of equations, i.e.,

$$F := (F_0, \dots, F_{p-1}) : \bigcap_{i=0}^{p-1} \mathcal{D}(F_i) \subset \mathcal{X} \rightarrow \mathcal{Y}^p$$

and $y = (y_0, \dots, y_{p-1})$ with $p > 1$. For this problem the Landweber iteration reads as follows

$$x_{k+1}^\delta = x_k^\delta + \sum_{i=0}^{p-1} F_i'(x_k^\delta)^*(y_i^\delta - F_i(x_k^\delta)), \quad k \in \mathbb{N}_0.$$

The *Landweber-Kaczmarz method* is defined by

$$x_{k+1}^\delta = x_k^\delta + F_k'(x_k^\delta)^*(y_k^\delta - F_k(x_k^\delta)), \quad k \in \mathbb{N}_0. \quad (9.1)$$

For simplicity of presentation, here we use the notation

$$F_k := F_{r_k} \quad \text{and} \quad y_k^\delta := y_{r_k}^\delta \quad \text{with} \quad k = p \lfloor k/p \rfloor + r_k \quad \text{and} \quad r_k \in \{0, \dots, p-1\}.$$

With $\lfloor x \rfloor$ we denote the smallest integer less or equal x , i.e., $\lfloor 1 \rfloor = 1$, $\lfloor 0.9 \rfloor = 0$.

The Landweber-Kaczmarz method applies the Landweber iteration steps cyclically.

In order to prove convergence and stability of the Landweber-Kaczmarz method one basically requires the same conditions on the operators F_i as they are required for F for proving convergence and stability of Landweber iteration.

As for the Landweber iteration we need a stopping rule. Here it is convenient to modify (9.1) as follows

$$x_{k+1}^\delta = x_k^\delta + \omega_k^\delta F_k'(x_k^\delta)^*(y_k^\delta - F_k(x_k^\delta)), \quad k \in \mathbb{N}_0,$$

where

$$\omega_k^\delta := \begin{cases} 1, & \text{if } \tau\delta < \|y_k^\delta - F_k(x_k^\delta)\|, \\ 0, & \text{else,} \end{cases}$$

with τ as in (8.8). The iteration is terminated if it stagnates over a full cycle of p successive iterates.

10. Newton type methods

The key idea of any Newton type method consists in repeatedly linearizing the operator equation $F(x) = y$ around some approximate solution x_k^δ , and then solving the linearized problem

$$F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) = y^\delta - F(x_k^\delta)$$

for x_{k+1}^δ .

If we apply Tikhonov regularization to the linearized problem, we end up with the Levenberg-Marquardt method:

$$x_{k+1}^\delta = x_k^\delta + (F'(x_k^\delta)^*F'(x_k^\delta) + \alpha_k I)^{-1}F'(x_k^\delta)^*(y^\delta - F(x_k^\delta)).$$

Adding a penalty term yields the well-known iteratively regularized Gauss-Newton method introduced by Bakushinskii.

$$x_{k+1}^\delta = x_k^\delta + (F'(x_k^\delta)^*F'(x_k^\delta) + \alpha_k I)^{-1}(F'(x_k^\delta)^*(y^\delta - F(x_k^\delta)) + \alpha_k(x_0 - x_k^\delta)), \quad (10.1)$$

where, as always, $x_0^\delta = x_0$ is an initial guess for the true solution and α_k is a sequence of positive numbers tending towards zero.

We emphasize that for a fixed number of iterations the process (10.1) is again a stable algorithm if $F'(\cdot)$ is continuous.

This method was introduced by Bakushinskii. He proved local convergence essentially under the source condition (8.9) with $\mu \geq 1$ assuming that F' is Lipschitz continuous.

As for Landweber iteration Lipschitz continuity of F' is not sufficient to obtain rates if $\mu < 1/2$ in (8.9). Similarly to Assumption 8.4 (ii) we then need further conditions on F' that guarantee that the linearization is not too far away from the nonlinear operator F . However, for the case $\mu \geq 1/2$, Lipschitz continuity of F' suffices to prove convergence rates for the iteratively regularized Gauss-Newton method as for Tikhonov regularization.

For our convergence analysis we need an assumption similar to Assumption 8.4.

Assumption 10.1. Let ρ be a positive number such that $\mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F)$.

- (i) The equation $F(x) = y$ has an x_0 -minimum-norm solution x^\dagger in $\mathcal{B}_\rho(x_0)$.
- (ii) x^\dagger satisfies the source condition (8.9) for some $0 \leq \mu \leq 1$, i.e.,

$$x^\dagger - x_0 = (F'(x^\dagger)^*F'(x^\dagger))^\mu v, \quad v \in \mathcal{N}(F'(x^\dagger))^\perp.$$

- (iii) If $\mu < 1/2$, the Fréchet-derivative F' satisfies the following conditions

$$\begin{aligned} F'(\tilde{x}) &= R(\tilde{x}, x)F'(x) + Q(\tilde{x}, x) \\ \|I - R(\tilde{x}, x)\| &\leq c_R \\ \|Q(\tilde{x}, x)\| &\leq c_Q \|F'(x^\dagger)(\tilde{x} - x)\| \end{aligned}$$

for $x, \tilde{x} \in \mathcal{B}_{2\rho}(x_0)$, where c_R and c_Q are nonnegative constants.

If $\mu \geq 1/2$, the Fréchet-derivative F' is Lipschitz continuous in $\mathcal{B}_{2\rho}(x_0)$, i.e.,

$$\|F'(\tilde{x}) - F'(x)\| \leq L \|\tilde{x} - x\|, \quad x, \tilde{x} \in \mathcal{B}_{2\rho}(x_0)$$

for some $L > 0$.

(iv) The sequence $\{\alpha_k\}$ in (10.1) satisfies

$$\alpha_k > 0, \quad 1 \leq \frac{\alpha_k}{\alpha_{k+1}} \leq r, \quad \lim_{k \rightarrow \infty} \alpha_k = 0,$$

for some $r > 1$.

For the following a-priori stopping rule, where the iteration is stopped after $k_* = k_*(\delta)$ steps with

$$\begin{cases} \eta \alpha_{k_*}^{\mu + \frac{1}{2}} \leq \delta < \eta \alpha_k^{\mu + \frac{1}{2}}, & 0 \leq k < k_*, & 0 < \mu \leq 1, \\ k_*(\delta) \rightarrow \infty \quad \text{and} \quad \eta \geq \delta \alpha_{k_*}^{-\frac{1}{2}} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0, & & \mu = 0, \end{cases} \quad (10.2)$$

for some $\eta > 0$, we can prove convergence and convergence rates provided that $\|v\|$, c_R , c_Q , and η are sufficiently small, i.e.:

$$\|x_{k_*}^\delta - x^\dagger\| = \begin{cases} o(1), & \mu = 0, \\ O(\delta^{\frac{2\mu}{2\mu+1}}), & 0 < \mu \leq 1. \end{cases}$$

Unfortunately, the a-priori stopping rule (10.2) has the disadvantage that one needs information on the smoothness of the exact solution, i.e., the parameter μ . A-posteriori stopping rules only need available information. If $k_* = k_*(\delta, y^\delta)$ is chosen according to the discrepancy principle (8.2), one can prove convergence rates at least for $\mu \leq 1/2$.

Of course, one can generalize method (10.1) so that instead of Tikhonov regularization other regularization techniques are used for the linear subproblems in each Newton step.

One disadvantage of Newton type methods is that usually operators involving the derivative of F have to be inverted. For well-posed problems a reduction of the numerical effort is achieved via Quasi-Newton methods, e.g., Broyden's method. Such a modification is also possible for ill-posed problems.

11. Multilevel methods

It is well known from the solution of partial differential equations and optimization problems that multilevel methods have a high numerical efficiency. This is achieved by combining the information on different levels of discretization. Especially multigrid methods play an essential role that use appropriate combinations of smoothing steps on fine grids with coarse grid corrections.

For ill-posed problems the behaviour of MGM is not so well understood, yet. The main problem lies in the definition of smoothing operators. The crucial effect of smoothing out the higher frequency part of the error, that the usual iterative schemes such as Jacobi, Gauss-Seidel, or SOR show in discretizations of PDEs or second kind integral equations, gets lost for first kind integral equations, due to their adverse eigensystem structure: here high frequencies correspond to small singular values of the forward operator and are therefore strongly amplified by inversion.

12. Other iterative regularization approaches

In this chapter we give comments on some other iterative regularization approaches that are not covered in our book.

Recently, there have been many studies devoted to variational regularization in Banach spaces. Our book focuses on solving operator equations in Hilbert spaces with iterative regularization methods in a stable way by *early termination* of the iteration. These methods can be similarly defined in a Banach space setting. To see this, let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be an operator between Banach spaces \mathcal{X} and \mathcal{Y} , and take into account that the steepest descent direction of the functional

$$x \mapsto \frac{1}{2} \|F(x) - y^\delta\|^2$$

is given by $-F'(x)^\#(F(x) - y^\delta)$, where $F'(x)^\# : \mathcal{Y}^* \rightarrow \mathcal{X}^*$ is the dual adjoint operator. Note that the operator is defined on the dual spaces of the Banach spaces \mathcal{X} and \mathcal{Y} , respectively. Then for instance the Landweber iteration reads as follows

$$\mathcal{J}_{\mathcal{X}}(x_{k+1}^\delta - x_k^\delta) = -F'(x_k^\delta)^\#(F(x_k^\delta) - y^\delta),$$

where $\mathcal{J}_{\mathcal{X}}$ is the duality mapping from \mathcal{X} into \mathcal{X}^* .

We emphasize that iterative methods as studied in our book are based on the least squares fit for minimizing $\frac{1}{2} \|F(x) - y^\delta\|^2$. Iterative methods can also be based on other fit-to-data functionals: for instance consider the minimization of

$$x \mapsto G(\|F(x) - y^\delta\|),$$

with a convex function G , then the Landweber iteration iteration reads as follows

$$\mathcal{J}_{\mathcal{X}}(x_{k+1}^\delta - x_k^\delta) = -F'(x_k^\delta)^\#G'(F(x_k^\delta) - y^\delta),$$

where the evaluation of the function G' is understood pointwise.

Another research topic on iterative methods concerns the minimization of *penalized* functionals

$$x \mapsto \|F(x) - y^\delta\|^2 + \alpha \|x\|_{\mathcal{X}}, \quad (12.1)$$

where the penalization term $\|\cdot\|_{\mathcal{X}}$ (with \mathcal{X} a Banach space) is used to take into account additional constraints, such as sparsity. These iteration methods yield approximations for minimizers of the functional above and not of the original problem $F(x) = y$. Typical results on convergence of the iterates are shown for $k \rightarrow \infty$ for fixed $\alpha > 0$. Since the minimization of the functional in (12.1) is a well-posed problem, it is not necessary to terminate the iteration number k as $\delta \rightarrow 0$.

In case of an L^1 -penalization, the following semi-implicit fixed point iteration

$$x_{k+1}^\delta = x_k^\delta - F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) - \alpha \operatorname{sgn}(x_{k+1}^\delta)$$

was considered by Daubechies, Defrise, and De Mol. Here, sgn is a set-valued function, defined as 1 for positive values, -1 for negative values, and $\operatorname{sgn}(0) \in [-1, 1]$. Note that the operator $x + \alpha \operatorname{sgn}(x)$ is invertible.

Part 5: Friday, March 28, 15:00-17:00

Fifth part is taken from my papers:

Estimation of discontinuous solutions of ill-posed problems via adaptive grid regularization, *J. Inv. Ill-Posed Problems* 14 (2006), 705–716.

Computation of discontinuous solutions of 2D linear ill-posed integral equations via adaptive grid regularization, *J. Inv. Ill-Posed Problems* 15 (2007), 99–106.

Solution of ill-posed problems via adaptive grid regularization: convergence analysis, *Numer. Funct. Anal. Optim.* 28 (2007), 405–423.

Identification of a temperature dependent heat conductivity via adaptive grid regularization, *J. Integral Equations Appl.* (2008), to appear

13. Adaptive grid regularization

13.1. Introduction

In this chapter we study (linear or nonlinear) ill-posed problems

$$F(x) = y, \quad F : \mathcal{D}(F) (\subset \mathcal{X}) \rightarrow \mathcal{Y},$$

where \mathcal{Y} is a Hilbert space and \mathcal{X} is a Banach space of functions defined on Ω , an open bounded convex subset of \mathbb{R}^d ($d = 1, 2, 3$) with Lipschitz boundary. Examples are integral equations or parameter identification problems. We are especially interested in problems, where the solution is discontinuous.

Unfortunately, standard regularization methods are not appropriate for such ill-posed problems, since they have a smoothing effect on regularized solutions.

If one expects discontinuous solutions, special care has to be taken in choosing the regularization method. Bounded variation regularization has turned out to be an effective method when dealing with such problems. In this method the penalty term of the Tikhonov functional (see (5.2) and (7.3)) is replaced by the *BV*-seminorm. However, a major drawback of this approach is that this seminorm is not differentiable. Thus, usually a differentiable approximation is used. Moreover, *BV*-regularization shows the so-called staircase effect, i.e., regularized solutions have the tendency to become piecewise constant.

In 1998, Scherzer and Neubauer introduced a new approach for regularizing problems with discontinuous solutions was introduced, *regularization for curve representations*. The essence of this method is to replace a discontinuous function by its continuous graph (see Figure 13.1) and to apply standard regularization methods in Hilbert spaces to this parameterization.

$$x = c(\phi^{-1}), \quad \dot{\phi} \geq 0 \text{ a.e.}$$

The resulting grid in the numerical realization has the property that the mesh size is small wherever the solution has discontinuities. This idea was carried over to two-dimensional problems via a moving grid algorithm: instead of computing the grid via optimization (i.e., minimizing the Tikhonov functional with respect to c and ϕ) the mesh



Figure 13.1: Function and its continuous graph

size is directly adapted to the smoothness of the solution via the so called deformation method. This approach is easy to implement and yields good numerical results.

One disadvantage of the moving grid approach is that in each iteration step the whole grid is changed. As a consequence the system matrices occurring in this method have to be totally recalculated in each iteration step. Moreover, it might happen that very flat grid elements occur which has a negative influence on the stability especially in parameter estimation problems.

To reduce the numerical effort the method of *adaptive grid regularization* was introduced in the papers mentioned above. Numerical results in these papers show that this method is an efficient and fast tool to identify discontinuities of solutions of ill-posed problems.

13.2. The method

The idea of the method is to adjust not only the grid locally but also the regularizing norm after each iteration: Let $w \in L^1(\Omega)$ be a weight function satisfying

$$\int_{\Omega} w(\xi) d\xi = |\Omega|, \quad w > 0 \text{ a.e.}, \quad w^{-1} \in L^{\infty}(\Omega).$$

Then the regularization is carried out in \mathcal{X}_w defined as the Hilbert space $H^1(\Omega)$ or $H_0^1(\Omega)$ equipped with the (semi) norm

$$\|x\|_w := \int_{\Omega} |\nabla x(\xi)|^2 w^{-1}(\xi) d\xi.$$

If one uses Tikhonov regularization, then one looks for a minimizer of

$$\|F(x) - y^{\delta}\|^2 + \alpha \|x - x_*\|_w^2$$

in $x_* + \mathcal{X}_h$, where \mathcal{X}_h is a finite-dimensional subspace of \mathcal{X}_w consisting of finite elements corresponding to a triangulation τ_h of Ω . An appropriate choice are piecewise linear elements. x_* usually plays the role of an initial guess. Since it can always be incorporated into F or y , we assume w.l.o.g. in the following that $x_* = 0$.

Instead of Tikhonov regularization also the iteratively regularized Gauss-Newton method can be used to obtain good results especially for nonlinear problems.

It turns out from regularization for curve representations that a somewhat optimal choice for w would be obtained by minimizing

$$g_{\alpha,\beta}(x, w) := \|F(x) - y^\delta\|^2 + \alpha \int_{\Omega} (\beta^2 + |\nabla x(\xi)|^2) w^{-1}(\xi) d\xi$$

simultaneously with respect to x and w yielding that $w \sim \sqrt{\beta^2 + |\nabla x|^2}$. These considerations led us to the following algorithm:

Algorithm 13.1. (Adaptive grid regularization) Let $\alpha, \beta > 0$.

- (i) Start with a uniform (rather coarse) grid in Ω yielding the triangulation τ_1 . Set $n := 1, x_0^\delta := 0$.
- (ii) Compute a minimizer x_n^δ of

$$\begin{aligned} \tilde{g}_{\alpha,\beta}(x, w_n) &:= \|F(x) - y^\delta\|^2 + \alpha \int_{\Omega} |\nabla x(\xi)|^2 w_n^{-1}(\xi) d\xi \\ w_n(\xi) &:= |\Omega| \sqrt{\beta^2 + |\nabla x_{n-1}^\delta(\xi)|^2} / \int_{\Omega} \sqrt{\beta^2 + |\nabla x_{n-1}^\delta(\xi)|^2} d\xi \end{aligned} \quad (13.1)$$

in the finite element space of piecewise linear functions \mathcal{X}_n corresponding to the triangulation τ_n .

- (iii) If a stopping criterion is satisfied, the iteration is finished; otherwise:
- (iv) Perform a local grid refinement

$$\tau_{n+1} := G(\tau_n, x_n^\delta). \quad (13.2)$$

Set $n := n + 1$ and go to step (ii).

As grid refinement in (13.2) we have chosen the following procedure: triangles $T_i \in \tau_n$ are bisected whenever $|\nabla x_n^\delta|$ is large. This is done according to the following rules: a refinement is only performed if the size of the triangle is larger than a certain threshold, i.e.,

$$\text{diam}(T_i) \geq h_{\min} \quad (13.3)$$

Under all these admissible triangles only those are refined where the corresponding weight $w_{n,i} := w_n|_{T_i}$, which is constant on each triangle T_i (note that the finite elements are linear), is not smaller than the k -th largest weight and close enough to the largest weight, i.e.,

$$w_{n,i} \geq \bar{w}_k := k\text{-th largest element among all } w_{n,j} \text{ with } \text{diam}(T_i) \geq h_{\min}, \quad (13.4)$$

$$w_{n,i} > \kappa * \max\{w_{n,j} : \text{diam}(T_j) \geq h_{\min}\}. \quad (13.5)$$

13.3. Convergence analysis

In this section we want to show that the sequence $\{x_n^\delta\}$ obtained by Algorithm 13.1 is convergent.

First of all, note that the subspaces \mathcal{X}_n are increasing, i.e., $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$. Due to the refinement condition (13.3) no refinement will occur anymore after some iteration step \bar{n} , i.e., $\mathcal{X}_n = \mathcal{X}_{\bar{n}}$ for all $n \geq \bar{n}$. It will turn out that under some conditions x_n^δ will converge towards a minimizer of

$$f_{\alpha,\beta}(x) := \|F(x) - y^\delta\|^2 + \alpha|\Omega|^{-1} \left(\int_{\Omega} \sqrt{\beta^2 + |\nabla x(\xi)|^2} d\xi \right)^2$$

in $\mathcal{X}_{\bar{n}}$ if F is linear and at least towards a stationary point if F is nonlinear.

If the threshold h_{\min} in (13.3) is getting smaller and smaller, then obviously \bar{n} is getting larger and larger. Therefore, we are also interested if such minimizers in $\mathcal{X}_{\bar{n}}$ converge to a minimizer in some infinite-dimensional space if \bar{n} tends to infinity. To be able to show this, the penalty term above has to be replaced by a variational formulation that extends to non-smooth functions, i.e.,

$$f_{\alpha,\beta}(x) := \|F(x) - y^\delta\|^2 + \alpha|\Omega|^{-1} J_\beta(x)^2, \quad (13.6)$$

where

$$\begin{aligned} J_\beta(x) &:= \sup\{Q_\beta(x, v) : v \in [C_c^1(\Omega)]^d, \|v\|_\infty \leq 1\}, \\ Q_\beta(x, v) &:= \int_{\Omega} \left(x(\xi) \operatorname{div} v(\xi) + \beta \sqrt{1 - |v(\xi)|^2} \right) d\xi. \end{aligned}$$

Here, $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^d and $\|v\|_\infty := \sup |v(\xi)|$. It is easy to show that $J_\beta(x)$ coincides with $\int_{\Omega} \sqrt{\beta^2 + |\nabla x(\xi)|^2} d\xi$ if $x \in W^{1,1}(\Omega)$.

As usual, we say that a function $x \in L^1(\Omega)$ is of bounded variation if $J_0(x) < \infty$. It is well known that the space of all functions of bounded variation, $BV(\Omega)$, equipped with the norm

$$\|x\|_{BV} := \|x\|_{L^1} + |x|_{BV}, \quad |x|_{BV} := J_0(x),$$

is a Banach space. Obviously, $J_\beta(x) < \infty$ if and only if $J_0(x) < \infty$.

It was shown by Acar and Vogel that, for any $\beta \geq 0$, $J_\beta(x)$ is convex and weakly lower semicontinuous with respect to the $L^p(\Omega)$ topology for $1 \leq p < \infty$.

To be able to guarantee the existence and stability of minimizers of (13.1) or (13.6) and for our convergence proofs we need some assumptions on the operator F :

Assumption 13.2. Let Ω be an open bounded convex subset of \mathbb{R}^d ($d = 1, 2, 3$) with Lipschitz boundary. The operator $F : \mathcal{D}(F) \subset L^p(\Omega) \rightarrow \mathcal{Y}$ is continuous, with $\mathcal{D}(F)$ convex and closed in $L^p(\Omega)$, for some $p < \bar{p}$ or continuous and weakly sequentially closed for $p = \bar{p}$ in case $d \geq 2$, where

$$\bar{p} := \begin{cases} \infty, & d = 1, \\ \frac{d}{d-1}, & d > 1. \end{cases}$$

- (i) If F is linear, then $\mathcal{D}(F) = L^p(\Omega)$.
- (ii) If F is nonlinear, it holds that $\mathcal{X}_n \cap \mathcal{D}(F) \neq \{\}$ and that F is continuously Fréchet-differentiable from $\mathcal{X}_n \rightarrow L(\mathcal{X}_n, \mathcal{Y})$ for all $n \in \mathbb{N}$, where \mathcal{X}_n is as in Algorithm 13.1. Moreover,

$$\|F(x) - F(\bar{x})\| \leq \rho(\|x - \bar{x}\|_{L^p}), \quad x, \bar{x} \in \mathcal{D}(F),$$

for some continuous monotonically increasing function ρ . Furthermore, it holds for all constant functions κ that $\kappa \in \mathcal{D}(F)$ and that

$$\|F(\kappa)\| \geq \bar{\gamma}|\kappa|$$

for some constant $\bar{\gamma} > 0$.

Proposition 13.3. *Let Assumption 13.2 hold. Then the functional $\tilde{g}_{\alpha,\beta}$ as defined in (13.1) has a minimizer x_n^δ in $\mathcal{X}_n \cap \mathcal{D}(F)$ and the functional $f_{\alpha,\beta}$ as defined in (13.6) has a minimizer $x_{\alpha,\beta,n}^\delta$ in $\mathcal{X}_n \cap \mathcal{D}(F)$ and a minimizer $x_{\alpha,\beta}^\delta$ in $BV(\Omega) \cap \mathcal{D}(F)$, respectively.*

In case that F is linear and injective the minimizer $x_{\alpha,\beta}^\delta$ is unique. If F is linear, the minimizers x_n^δ and $x_{\alpha,\beta,n}^\delta$ are unique in case $1 \notin \mathcal{N}(F)$ and unique up to a constant otherwise.

Results about stability follow similarly as for nonlinear Tikhonov regularization. The next theorem is a convergence result in the finite-dimensional space \mathcal{X}_n .

Theorem 13.4. *Let Assumption 13.2 hold and let x_n^δ be as in Algorithm 13.1 where the grid refinement is done according to (13.3) – (13.5).*

Then $\{x_n^\delta\}$ has a convergent subsequence. The limit of every convergent subsequence is a stationary point of $f_{\alpha,\beta}$, defined by (13.6), in case F is nonlinear.

If F is linear, the limit is even a minimizer $x_{\alpha,\beta,\bar{n}}^\delta$ of $f_{\alpha,\beta}$ in $\mathcal{X}_{\bar{n}}$. If this minimizer is unique, then $x_n^\delta \rightarrow x_{\alpha,\beta,\bar{n}}^\delta$ as $n \rightarrow \infty$.

Since $f_{\alpha,\beta}(x_n^\delta)$ is monotonically decreasing, the stationary point in the theorem above will never be a local maximum.

The next theorem shows that minimizers of (13.6) in \mathcal{X}_n converge towards a minimizer of (13.6) in $BV(\Omega)$ if the spaces \mathcal{X}_n approximate the space $BV(\Omega)$ in an appropriate way: let $P_n : BV(\Omega) \rightarrow \mathcal{X}_n$ be projection operators and let $\mathcal{P}_\beta \subset BV(\Omega)$ be defined as follows

$$\mathcal{P}_\beta := \{x \in BV(\Omega) \cap \mathcal{D}(F) : P_n x \xrightarrow{L^p} x, J_\beta(P_n x) \rightarrow J_\beta(x), P_n x \in \mathcal{D}(F) \text{ for } n \text{ sufficiently large}\} \quad (13.7)$$

with p as in Assumption 13.2.

Theorem 13.5. *Let Assumption 13.2 hold and assume that $\mathcal{P}_\beta \neq \{\}$, where \mathcal{P}_β is defined as in (13.7). Moreover, let $x_{\alpha,\beta,n}^\delta$ be a minimizer of $f_{\alpha,\beta}$, defined as in (13.6), in \mathcal{X}_n .*

Then $\{x_{\alpha,\beta,n}^\delta\}$ has a weakly convergent subsequence in $BV(\Omega)$. The limit \tilde{x} of every weakly* convergent subsequence satisfies the condition*

$$f_{\alpha,\beta}(\tilde{x}) \leq \liminf_{n \rightarrow \infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) \leq \limsup_{n \rightarrow \infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) \leq \inf_{x \in \mathcal{P}_\beta} f_{\alpha,\beta}(x).$$

If

$$\mathcal{D}(F) \cap \bigcup_{n \in \mathbb{N}} \mathcal{X}_n \subset \mathcal{P}_\beta$$

then it even holds that

$$\lim_{n \rightarrow \infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) = \inf_{x \in \mathcal{P}_\beta} f_{\alpha,\beta}(x) = f_{\alpha,\beta}(\tilde{x}).$$

If a minimizer $x_{\alpha,\beta}^\delta$ of $f_{\alpha,\beta}$ in $BV(\Omega)$ is an element of \mathcal{P}_β , then \tilde{x} is also a minimizer of $f_{\alpha,\beta}$ in $BV(\Omega)$ and

$$\lim_{n \rightarrow \infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) = \inf_{x \in BV \cap \mathcal{D}(F)} f_{\alpha,\beta}(x) = f_{\alpha,\beta}(\tilde{x}).$$

If, in addition, $x_{\alpha,\beta}^\delta$ is unique, then $\{x_{\alpha,\beta,n}^\delta\}$ weakly* converges towards $x_{\alpha,\beta}^\delta$ and hence

$$x_{\alpha,\beta,n}^\delta \xrightarrow{L^p} x_{\alpha,\beta}^\delta, \quad \forall 1 \leq p < \bar{p}, \quad \text{and} \quad x_{\alpha,\beta,n}^\delta \xrightarrow{L^{\bar{p}}} x_{\alpha,\beta}^\delta, \quad d \geq 2.$$

In the one-dimensional case one can show that there is a projector P_n such that $P_n x \xrightarrow{L^p} x$ and $J_\beta(P_n x) \rightarrow J_\beta(x)$ holds for all $x \in BV(\Omega)$ if \mathcal{X}_n is the finite element space of piecewise linear functions.

Unfortunately, a similar result will not hold in higher dimensions. Even for piecewise constant functions, where jumps occur along lines of finite perimeter, it does not automatically hold that $J_\beta(P_n x) \rightarrow J_\beta(x)$. This is only the case if the triangulation approximates the line, where the jumps occur, good enough. In all other cases, one can only show that $J_\beta(P_n x)$ remains bounded.

13.4. Applications

The algorithm has been successfully applied to 1D integral and parameter estimation problems, where for the latter the iteratively regularized Gauss-Newton was used instead of Tikhonov regularization, and also to 2D integral equations. 2D parameter estimation problems are treated in a diploma thesis at the moment.

The method turned out to be very fast when applied to identification of a temperature dependent heat conductivity, i.e., we want to identify the parameter a in

$$\begin{aligned} -\operatorname{div}(a(u)\nabla u) &= f \quad \text{in } \Omega \\ a(u)\frac{\partial u}{\partial n} &= h \quad \text{on } \Gamma = \partial\Omega \end{aligned} \tag{13.8}$$

from a single measurement of u at the boundary Γ . Note that this is a mixed 1D-2D parameter estimation problem.

Usually, one assumes that a varies spatially on $x \in \Omega$. However, there are interesting problems such as in the cooling process of a steel strand where the heat conductivity depends merely on the temperature. The stationary case leads to the nonlinear elliptic equation above.

We consider equation (13.8), where Ω is an open bounded convex subset of \mathbb{R}^d ($d = 1, 2, 3$) with Lipschitz boundary Γ , $f \in L^2(\Omega)$, $h \in L^2(\Gamma)$, and the parameter a satisfies the conditions

$$\begin{aligned} 0 < \underline{a} < a < \bar{a} < \infty \quad \text{and} \\ a \text{ is continuous except at most countably many points} \end{aligned} \tag{13.9}$$

for some positive constants \underline{a}, \bar{a} . The direct problem consists in finding a weak solution of (13.8), i.e., in looking for $u \in H^1(\Omega)$ satisfying

$$\langle a(u)\nabla u, \nabla v \rangle = \langle f, v \rangle + \int_{\Gamma} h v d\sigma$$

for all $v \in H^1(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$. Obviously, a solution can only exist if f and h satisfy the condition

$$\int_{\Omega} f dx + \int_{\Gamma} h d\sigma = 0, \quad (13.10)$$

which we assume to hold in the following. We will show in the next proposition that (13.8) has a weak solution and that all solutions may be expressed via the unique solution $w \in \mathcal{V}$ of

$$\langle \nabla w, \nabla v \rangle = \langle f, v \rangle + \int_{\Gamma} h v d\sigma \quad \text{for all } v \in \mathcal{V} \quad (13.11)$$

with $\mathcal{V} := \{v \in H^1(\Omega) : \int_{\Omega} v dx = 0\}$. Note that existence and uniqueness of w follow immediately from the Lax-Milgram Lemma.

Proposition 13.6. *Let a satisfy (13.9) and let $f \in L^2(\Omega)$ and $h \in L^2(\Gamma)$ satisfy (13.10). Then equation (13.8) has a weak solution and all weak solutions u_c may be calculated via*

$$u_c := A^{-1}(w + c),$$

where $w \in \mathcal{V}$ is the unique solution of (13.11), $c \in \mathbb{R}$ is a constant and A is defined via

$$A(s) := \int_0^s a(\xi) d\xi.$$

Let us now turn to the inverse problem, i.e., from a given single measurement of u at the boundary we want to identify a . The first question that arises is of course if this problem has a unique solution. One can show that an identification is possible only on the range of u at Γ .

The inverse problem may be reformulated as a linear problem: let us assume in the following that $u \in H^1(\Omega)$ is a solution of the forward problem for some a satisfying (13.9) and that u is bounded on Γ and not constant, hence,

$$\mathcal{R}(u|_{\Gamma}) \subset I := [u_{min}, u_{max}] \quad \text{with} \quad u_{max} > u_{min}.$$

The inverse problem of identifying a in (13.8) is now equivalent to finding a and $c \in \mathbb{R}$ such that

$$Ta(u(x)) := \int_{u_{min}}^{u(x)} a(\xi) d\xi = w(x) + c, \quad x \in \Gamma,$$

where $w \in \mathcal{V}$ is the solution of (13.11). This problem was solved using adaptive grid regularization.