Changes at each step

MARTA KOSEK

Institute of Mathematics

Jagiellonian University

based on a joint work with Maciej Klimek

The Banach Contraction Principle can be expressed as follows

Theorem 1. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a contraction.*

Then the sequence of iterates (f^n) *converges pointwise to a constant mapping.*

This theorem can be applied to some IFSs.

Let $\{f_1, ..., f_k\}$ be a finite family of contractions of a complete metric space X. Then the mapping

$$A \mapsto \bigcup_{j=1}^{k} f_j(A)$$

is a contraction of a space of nonempty closed bounded sets furnished with Hausdorff metric. The unique fixed point is the attractor of the IFS.

J. E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J., **30** (1981), 713–747.

One can deal also with a countable family of contractions but one has to assume a bound (smaller than 1) of the contraction ratios and take the closure of the union of images. Then the unique fixed point is the closure of the attractor.

L. Baribeau, M. Roy, *Analytic multifunctions, holomorphic motions and Hausdorff dimension in IFSs*, Monatsh. Math., **147** (2006), no. 3, 199–217.

But in real life iterating i.e. repeating the same thing again and again is rare. Wisława Szymborska says even that “Nothing can ever happen twice”.

NOTHING TWICE

Nothing can ever happen twice
In consequence, the sorry fact is
that we arrive here improvised
and leave without the chance to practise.

Even if there is no one dumber,
if you're the planet's biggest dunce,
you can't repeat the class in summer:
this course is only offered once.

No day copies yesterday,
no two nights will teach what bliss is
in the precise the same way,
with exactly the same kisses.

One day, perhaps, some idle tongue
mentions your name by accident:
I feel as if a rose were flung
into the room, all hue ans scent.

The next day, though you're here with me,
I can't help looking at the clock:
A rose? A rose? What could that be?
Is it a flower or a rock?

Why do we treat the fleeting day
with so much needless fear and sorrow?
It's in its nature not to stay:
Today is always gone tomorrow.

With smile and kisses, we prefer
to seek accord beneath our star,
although we're different (we concur)
just as two drops of water are.

Wisława Szymborska
translated by
S. Barańczak and C. Cavanagh

W. Szymborska, *Nothing Twice/Nic dwa razy*, **Wydawnictwo Literackie, Kraków, 1997**

Consider therefore

Theorem 2 (Enhanced version of the Banach Contraction Principle)**.**

Let (Y, ϱ) *be a complete metric space and let* $(H_n)_{n=1}^{\infty}$ *be a sequence of contractions of* Y *with contraction ratios not greater than* $L < 1$*.*

If

$$\forall x \in Y : \quad M_x := \sup_{n \geq 1} \varrho(H_n(x), x) < \infty,$$

then the sequence $(H_1 \circ \cdots \circ H_n)_{n=1}^{\infty}$ *converges pointwise to a constant mapping.*

Here we can change the mapping at each step and still get a generalization of a fixed point.

We can use this to generalize attractors (or Julia sets).

Let $(E, ||\cdot||)$ be a Banach space and $\mathcal{L}(E)$ be the space of bounded linear operators on E furnished with the operator norm. Denote by $\mathcal{A}(E)$ the space of continuous affine operators on E. One can write $\mathcal{A}(E) = \mathcal{L}(E) \oplus E$, since

$$\forall\, T \in \mathcal{A}(E): \quad \widetilde{T} := T - T(0) \in \mathcal{L}(E),$$

and consider it with the natural norm

$$||T|| := ||\widetilde{T}|| + ||T(0)||.$$

Note that $T \in \mathcal{A}(E)$ is a contraction if and only if $||\widetilde{T}|| < 1$.

Take a sequence $(\mathcal{T}_n)_{n=1}^{\infty}$ of countable (finite or not) families of continuous affine operators such that

$$Q_M = \sup_{T \in \bigcup_{n \geq 1} \mathcal{T}_n} ||T|| < \infty$$

and

$$L_M = \sup_{T \in \bigcup_{n \geq 1} \mathcal{T}_n} ||\widetilde{T}|| < 1.$$

Then one can define the attractor of the sequence $(\mathcal{T}_n)_{n=1}^{\infty}$ generalizing the notion of (the closure of) an attractor for an IFS.

M.Klimek and M.Kosek, *Generalized iterated function systems, multifunctions and Cantor sets*, Ann. Polon. Math., **96** (2009), 25-41.

Example

Let $(l_n)_{n=0}^{\infty}$ be a given sequence of positive numbers such that $l_0 = 1$ and $2l_n < l_{n-1}$, $n \geq 2$. Put

$$\mathcal{T}_n := \Big\{ \mathbb{C} \ni z \mapsto \frac{l_n}{l_{n-1}} z \in \mathbb{C}, \quad \mathbb{C} \ni z \mapsto \frac{l_n}{l_{n-1}} z + 1 - \frac{l_n}{l_{n-1}} \in \mathbb{C} \Big\}$$

for $n \geq 1$.

Then the sequence $(\mathcal{T}_n)_{n=1}^{\infty}$ satisfies the assumptions.

The obtained (as attractors) Cantor type sets are important examples in constructive theory of functions.

W. Pleśniak, *A Cantor regular set which does not have Markov's property*, Ann. Polon. Math. 51 (1990), 269–274.

We can choose the sequence $(l_n)_{n\geq 1}$ in such a way that the attractor cannot be obtain by any IFS.

S. Crovisier and M. Rams, *IFS attractors and Cantor sets*, Topology Appl. **153** (2006), no. 11, 1849–1859.

The generalization allows also to construct interesting examples of analytic multifunctions.

M.Klimek and M.Kosek, *Generalized iterated function systems, multifunctions and Cantor sets*, Ann. Polon. Math., **96** (2009), 25-41.