A quasi-boundary-value method for a Cauchy problem for elliptic equations with nonhomogeneous Neumann data

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Theoretical aspects

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The Cauchy problem for elliptic equations


Our problem

\[
\begin{cases}
    u_{xx}(x, y) - \mathcal{L}u(x, y) = 0, & x \in (0, 1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\
    u(x, y) = 0, & x \in [0, 1], y \in \partial \Omega, \\
    u(0, y) = g_1(y), & y \in \Omega, \\
    u_x(0, y) = g_2(y), & y \in \Omega,
\end{cases}
\]

where \( \mathcal{L} : D(\mathcal{L}) \subset H \to H \) denotes a linear densely defined self-adjoint and positive definite elliptic operator, and \( \Omega \) is connected bounded domain.

Figure: Three dimensional case
Set \( u = u_1 + u_2, \)
Set $u = u_1 + u_2$, where $u_1$ satisfies
\[
\begin{aligned}
&\begin{cases}
(u_1)_{xx}(x, y) - \mathcal{L}u_1(x, y) = 0, & x \in (0, 1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\
u_1(x, y) = 0, & x \in [0, 1], y \in \partial\Omega, \\
u_1(0, y) = g_1(y), & y \in \Omega, \\
(u_1)_x(0, y) = 0, & y \in \Omega,
\end{cases}
\end{aligned}
\]

and $u_2$ satisfies
\[
\begin{aligned}
&\begin{cases}
(u_2)_{xx}(x, y) - \mathcal{L}u_2(x, y) = 0, & x \in (0, 1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\
u_2(x, y) = 0, & x \in [0, 1], y \in \partial\Omega, \\
u_2(0, y) = 0, & y \in \Omega, \\
(u_2)_x(0, y) = g_2(y), & y \in \Omega,
\end{cases}
\end{aligned}
\]

then according to the linearity, $u = u_1 + u_2$ is the solution of the above problem.
Set \( u = u_1 + u_2 \),
where \( u_1 \) satisfies ILL-POSED PROBLEM

\[
\begin{aligned}
(u_1)_{xx}(x, y) - Lu_1(x, y) &= 0, & x \in (0, 1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\
u_1(x, y) &= 0, & x \in [0, 1], y \in \partial \Omega, \\
u_1(0, y) &= g_1(y), & y \in \Omega, \\
(u_1)_x(0, y) &= 0, & y \in \Omega,
\end{aligned}
\]

and \( u_2 \) satisfies

\[
\begin{aligned}
(u_2)_{xx}(x, y) - Lu_2(x, y) &= 0, & x \in (0, 1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\
u_2(x, y) &= 0, & x \in [0, 1], y \in \partial \Omega, \\
u_2(0, y) &= 0, & y \in \Omega, \\
(u_2)_x(0, y) &= g_2(y), & y \in \Omega,
\end{aligned}
\]

then according to the linearity, \( u = u_1 + u_2 \) is the solution of the above problem.
Set $u = u_1 + u_2$,
where $u_1$ satisfies ILL-POSED PROBLEM

$$\begin{cases}
(u_1)_{xx}(x,y) - \mathcal{L}u_1(x,y) = 0, & x \in (0,1), y \in \Omega \subset \mathbb{R}^n, n \geq 1,
\quad u_1(x,y) = 0, & x \in [0,1], y \in \partial \Omega,
\quad u_1(0,y) = g_1(y), & y \in \Omega,
\quad (u_1)_x(0,y) = 0, & y \in \Omega,
\end{cases}$$

and $u_2$ satisfies ILL-POSED PROBLEM

$$\begin{cases}
(u_2)_{xx}(x,y) - \mathcal{L}u_2(x,y) = 0, & x \in (0,1), y \in \Omega \subset \mathbb{R}^n, n \geq 1,
\quad u_2(x,y) = 0, & x \in [0,1], y \in \partial \Omega,
\quad u_2(0,y) = 0, & y \in \Omega,
\quad (u_2)_x(0,y) = g_2(y), & y \in \Omega,
\end{cases}$$

then according to the linearity, $u = u_1 + u_2$ is the solution of the above problem.
The quasi-boundary value method was introduced by Abdulkerimov. The idea is replacing the boundary value problem with an approximate one which is well-posed, then the latter has a non-local boundary value condition.


\[
\left\{ \begin{array}{ll}
\frac{\partial^2 v^{\alpha,\delta}}{\partial x^2}(x,y) - \mathcal{L}v^{\alpha,\delta}(x,y) = 0, & x \in (0,1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\
v^{\alpha,\delta}(x,y) = 0, & x \in [0,1] \times \partial \Omega, \\
v^{\alpha,\delta}(0,y) = 0, & y \in \Omega, \\
v^{\alpha,\delta}_x(0,y) + \alpha v^{\alpha,\delta}_x(1,y) = g_2^\delta(y), & y \in \Omega,
\end{array} \right.
\]

where the noisy data \( g_2^\delta(y) \) satisfy

\[
\|g_2^\delta - g_2\| \leq \delta.
\]
After using the method of separation of variables, we can get

\[ u_2(x, y) = \sum_{n=1}^{\infty} \kappa_n(x)(g_2, w_n)w_n(y), \]

where \( \kappa_n(x) = \frac{\sinh(\sqrt{\lambda_n}x)}{\sqrt{\lambda_n}}, \) and \( \lambda_n, w_n(y) \) are the corresponding eigenvalues and eigenfunctions of the elliptic operator \( L. \)
After using the method of separation of variables, we can get

\[ u_2(x, y) = \sum_{n=1}^{\infty} \kappa_n(x) (g_2, w_n) w_n(y), \]

\[ v^{\alpha, \delta}(x, y) = \sum_{n=1}^{\infty} \frac{\kappa_n(x)}{1 + \alpha \kappa_n(1)} (g_\delta, w_n) w_n(y). \]
After using the method of separation of variables, we can get

$$u_2(x, y) = \sum_{n=1}^{\infty} \kappa_n(x)(g_2, w_n)w_n(y),$$

$$v^{\alpha, \delta}(x, y) = \sum_{n=1}^{\infty} \frac{\kappa_n(x)}{1 + \alpha \kappa_n(1)}(g_\delta, w_n)w_n(y).$$

$$\kappa_n(x) = \frac{\sinh(\sqrt{\lambda_n}x)}{\sqrt{\lambda_n}},$$

where $\lambda_n, w_n(y)$ are the corresponding eigenvalues and eigenfunctions of elliptic operator $\mathcal{L}$. 
1 Problem and the Quasi-Boundary Value Method

2 Theoretical aspects
   - Preliminaries
   - An a-priori parameter choice
   - An a-posteriori parameter choice
   - Remark

3 Numerical aspects

4 Some phenomena of QuasiBVM

5 Future work
1. For $s > 0$, $0 < \alpha_2 < 1$ and $0 < x < 1$, the following inequalities hold.

(a). $\frac{\sinh(xs)}{s} \leq e^{xs}$;

(b). $\frac{\sinh(xs)}{\sinh(s)} \leq e^{(x-1)s}$;

(c). For $f_1(s) := \frac{\sinh(xs)}{1 + \alpha \frac{\sinh(s)}{s}}$, there holds $f_1(s) \leq \left(\frac{1}{\alpha}\right)^x$;

2. For $s > 0$ and $0 < x < 1$, $f_2(x) := \left(\frac{\sinh(xs)}{s}\right)^{\frac{1}{x}}$ is strict monotonically increasing.
Theorem 1

Let $v^{\alpha,\delta}(x, y)$ be the solution of quasi-boundary-value problem and $u_2(x, y)$ be the exact solution of original problem. If conditions

$$\|g_2^\delta - g_2\| \leq \delta.$$ and $$\|u_2(1, \cdot)\| \leq E.$$

hold and we choose

$$\alpha = \frac{\delta}{E},$$

then there holds error estimate

$$\|v^{\alpha,\delta}(x, \cdot) - u_2(x, \cdot)\| \leq 2\delta^{1-x}E^x, \quad 0 < x < 1.$$
**Theorem 2**

*Suppose there exist the a-priori bound*

\[ \|u_2(1, \cdot)\| \leq E. \]

*and conditions*

\[ \|g_2^\delta - g_2\| \leq \delta \quad \|v_{x,\delta}^0(0, \cdot) - g_2^\delta(\cdot)\| = \tau \delta. \]

*Here \( \tau > 1 \) such that \( 0 < \tau \delta < \|g_2^\delta\| \). Then for \( 0 < x < 1 \), there holds*

\[ \|u_2(x, \cdot) - v_{x,\delta}^\alpha(x, \cdot)\| \leq CE^x \delta^{1-x}, \]

*where \( C = \left(1 + \sqrt{2 \frac{1+(\tau-1)^2}{(\tau-1)^2}}\right)^x (1 + \tau)^{1-x}. \)
Remark

Under the a-priori bound
\[ \|u_2(1, \cdot)\| \leq E, \]
the optimal error bounds for problem talked here is
\[ \omega(\delta, E) = E^x \left( \frac{\delta}{2}\right)^{1-x} \left( \ln \frac{E}{\delta}\right)^{x-1} (1 + o(1)), \quad \text{for } \delta \to 0, 0 < x < 1. \]

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A simple case

\[ \begin{cases} 
\Delta u(x, y) = 0, & (x, y) \in (0, 1) \times (0, 1), \\
u(x, 0) = u(x, 1) = 0, & x \in [0, 1], \\
u(0, y) = 0, & y \in [0, 1], \\
u_x(0, y) = g_2(y), & y \in [0, 1], 
\end{cases} \]

Quasi-boundary-value problem

\[ \begin{cases} 
\Delta v_{\alpha, \delta}(x, y) = 0, & (x, y) \in (0, 1) \times (0, 1), \\
v_{\alpha, \delta}(x, 0) = v_{\alpha, \delta}(x, 1) = 0, & x \in [0, 1], \\
v_{\alpha, \delta}(0, y) = 0, & y \in [0, 1], \\
v_{\alpha, \delta}x(0, y) + \alpha v_{\alpha, \delta}(1, y) = g_2^\delta(y), & y \in [0, 1], 
\end{cases} \] (1)

whose solution is

\[ v_{\alpha, \delta}(x, y) = \sum_{n=1}^{\infty} \frac{\kappa_n(x)}{1 + \alpha \kappa_n(1)} \left( g_2^\delta, \sqrt{2} \sin(n\pi y) \right) \sqrt{2} \sin(n\pi y), \] (2)

with \( \kappa_n(x) = \frac{\sinh(n\pi x)}{n\pi} \).
A simple case

\[
\begin{aligned}
\Delta u(x, y) &= 0, \quad (x, y) \in (0, 1) \times (0, 1), \\
u(x, 0) &= u(x, 1) = 0, \quad x \in [0, 1], \\
u(0, y) &= 0, \quad y \in [0, 1], \\
u_x(0, y) &= g_2(y), \quad y \in [0, 1],
\end{aligned}
\]

Quasi-boundary-value problem The Finite Difference Method (FDM)

\[
\begin{aligned}
\Delta v^{\alpha, \delta}(x, y) &= 0, \quad (x, y) \in (0, 1) \times (0, 1), \\
v^{\alpha, \delta}(x, 0) &= v^{\alpha, \delta}(x, 1) = 0, \quad x \in [0, 1], \\
v^{\alpha, \delta}(0, y) &= 0, \quad y \in [0, 1], \\
v_x^{\alpha, \delta}(0, y) + \alpha v^{\alpha, \delta}(1, y) &= g_2^{\delta}(y), \quad y \in [0, 1],
\end{aligned}
\]

whose solution is

\[
v^{\alpha, \delta}(x, y) = \sum_{n=1}^{\infty} \frac{\kappa_n(x)}{1 + \alpha \kappa_n(1)} (g_2^{\delta}, \sqrt{2} \sin(n\pi y)) \sqrt{2} \sin(n\pi y),
\]

with \(\kappa_n(x) = \frac{\sinh(n\pi x)}{n\pi} \).
A simple case

\[
\begin{cases}
\Delta u(x, y) = 0, & (x, y) \in (0, 1) \times (0, 1), \\
u(x, 0) = u(x, 1) = 0, & x \in [0, 1], \\
u(0, y) = 0, & y \in [0, 1], \\
u_x(0, y) = g_2(y), & y \in [0, 1],
\end{cases}
\]

Quasi-boundary-value problem The Finite Difference Method (FDM)

\[
\begin{cases}
\Delta v^\alpha,\delta(x, y) = 0, & (x, y) \in (0, 1) \times (0, 1), \\
v^\alpha,\delta(x, 0) = v^\alpha,\delta(x, 1) = 0, & x \in [0, 1], \\
v^\alpha,\delta(0, y) = 0, & y \in [0, 1], \\
v_x^\alpha,\delta(0, y) + \alpha v^\alpha,\delta(1, y) = g_2^\delta(y), & y \in [0, 1],
\end{cases}
\]  \tag{1}

whose solution is The Discrete Sine Transform (DST)

\[
v^\alpha,\delta(x, y) = \sum_{n=1}^{\infty} \frac{\kappa_n(x)}{1 + \alpha \kappa_n(1)} (g_2^\delta, \sqrt{2} \sin(n\pi y)) \sqrt{2} \sin(n\pi y),
\]  \tag{2}

with \( \kappa_n(x) = \frac{\sinh(n\pi x)}{n\pi} \).
**Example 1.** We consider a simple example with an analytic solution
\[ u(x, y) = \sin(\pi y) \sinh(\pi x), \] i.e., \[ g_2(y) = \pi \sin(\pi y). \]
Numerical Examples

- **Example 1.** We consider a simple example with an analytic solution
  \[ u(x, y) = \sin(\pi y) \sinh(\pi x), \]
  i.e., \( g_2(y) = \pi \sin(\pi y) \).

- **Example 2.**
  \[
  \begin{aligned}
  \Delta u(x, y) &= 0, & (x, y) &\in (0, 1) \times (0, 1), \\
  u(x, 0) &= u(x, 1) = 0, & x &\in [0, 1], \\
  u(0, y) &= 0, & y &\in [0, 1], \\
  u(1, y) &= h(y), & y &\in [0, 1],
  \end{aligned}
  \]
  where \( h(y) \) is a test function. We give a random signal which starts and ends with the value zero and create \( h(y) \) to have these values used together with a cubic interpolating spline.
Figure: Example 1: The results at $x_0 = 0.5$ with $\epsilon = 10^{-2}$ and $\alpha_1 = 0.0012, \alpha_2 = 4.2943 \times 10^{-4}, \alpha_3 = 1.0928 \times 10^{-4}$. 
Figure: Example 1: The results at $x_0 = 0.85$ with $\epsilon = 10^{-2}$ and $\alpha_1 = 0.0012$, $\alpha_2 = 1.3420 \times 10^{-4}$, $\alpha_3 = 6.1348 \times 10^{-5}$. 
**Figure:** Example 2: Left: The random function $h(y)$; Right: The results at $x_0 = 0.3$ with $\epsilon = 10^{-3}$, $\alpha_1 = 0.0028$, $\alpha_2 = 0.0106$, $\alpha_3 = 0.0029$. 
Figure: Example 2: Left: The random function $h(y)$; Right: The results at $x_0 = 0.8$ with $\epsilon = 10^{-3}$, $\alpha_1 = 0.0021$, $\alpha_2 = 4.6174 \times 10^{-4}$, $\alpha_3 = 0.0020$. 
Figure: The relative errors for different fixed points $x_0$ with the same noisy level $\epsilon = 10^{-3}$ for Example 1 (Left) and Example 2 (Right).
Numerical results

Example 1

Example 2

Figure: The relative errors for different noisy levels at $x_0 = 0.2$ for Example 1 (Left) and Example 2 (Right).
**ALGORITHM 9.4. GMRES with Left Preconditioning**

1. Compute $r_0 = M^{-1}(b - Ax_0)$, $\beta = \|r_0\|_2$, and $v_1 = r_0/\beta$
2. For $j = 1, \ldots, m$, Do
3. Compute $w := M^{-1}Av_j$
4. For $i = 1, \ldots, j$, Do
5. $h_{i,j} := (w, v_i)$
6. $w := w - h_{i,j}v_i$
7. EndDo
8. Compute $h_{j+1,j} = \|w\|_2$ and $v_{j+1} = w/h_{j+1,j}$
9. EndDo
10. Define $V_m := [v_1, \ldots, v_m]$, $H_m = \{h_{i,j}\}_{1 \leq i \leq j+1, 1 \leq j \leq m}$
11. Compute $y_m = \arg\min_y \|\beta e_1 - H_my\|_2$ and $x_m = x_0 + V_my_m$
12. If satisfied Stop, else set $x_0 := x_m$ and go to 1

---

*Yousef Saad, Iterative Methods for Sparse Linear Systems, 2nd ed.*
Two essential requirements for a \textbf{GOOD} preconditioner:

(1) $Mz = b$ can be solved quickly and stably.

(2) $M$ is close to $A$ (the eigenvalues of $M^{-1}A$ are clustered near 1).

\begin{quote}
\end{quote}
\[
A = \begin{pmatrix}
-I & 0 & \ldots & \ldots & -\alpha h_x I \\
T & -I & 0 & \ldots \\
-I & T & -I & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & \ldots & -I & T & -I
\end{pmatrix}
\]

with

\[
T = \begin{pmatrix}
2(1+r) & -r & \ldots & 0 \\
-r & 2(1+r) & -r \\
\vdots & \vdots & \ddots & \ddots \\
0 & \ldots & -r & 2(1+r)
\end{pmatrix}
\]
\[ M = \begin{pmatrix}
-I & 0 & \ldots & \ldots & -\alpha h_x I \\
C & -I & 0 & \ldots \\
-I & C & -I & \ldots \\
\vdots & \vdots & \ddots & \ddots \\
0 & \ldots & -I & C & -I
\end{pmatrix} \]

with

\[ C = \begin{pmatrix}
2(1 + r) & -r & \ldots & -r \\
-r & 2(1 + r) & -r \\
\ddots & \ddots & \ddots \\
-r & \ldots & -r & 2(1 + r)
\end{pmatrix} \]
\[
\begin{pmatrix}
F^* & 0 & \cdots & \cdots & 0 \\
0 & F^* & 0 & \cdots & \\
0 & 0 & F^* & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & F^*
\end{pmatrix}
\begin{pmatrix}
F & 0 & \cdots & \cdots & 0 \\
0 & F & 0 & \cdots & \\
0 & 0 & F & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & F
\end{pmatrix}
M
= \begin{pmatrix}
-I & 0 & \cdots & \cdots & -\alpha h_x I \\
F^* CF & -I & 0 & \cdots & \\
-I & F^* CF & -I & \cdots & \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & -I & F^* CF & -I
\end{pmatrix}
\]
Figure: Arbitrary domain
1 Problem and the Quasi-Boundary Value Method

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I. Problem:

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, & (x,t) \in (0, 1) \times (0, T), \\
u(0, t) = 0, & t \in [0, T], \\
u_x(0, t) = u_x(1, t), & t \in [0, T], \\
u(x, T) = \varphi_1(x), & x \in [0, 1].
\end{cases}
\]

The approximate quasi-boundary value condition:

\[\alpha u(x, 0) + (1 - \alpha)u(x, T) = \varphi_1(x), \quad x \in [0, 1].\]

II. Problem:

\[
\begin{cases}
    u_t(x,t) + \mathcal{L}u(x,t) = 0, & x \in (0,1), (x,t) \in \Omega \times (0,T), \\
    u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\
    u(x,T) = \varphi_2(x), & x \in \Omega,
\end{cases}
\]

The approximate quasi-boundary value conditions:

(1).

\[ u(x,T) - \alpha u_t(x,0) = \varphi_2(x), \quad x \in \Omega. \]

(2).

\[ u(x, T) + \alpha u(x, 0) = \varphi_2(x), \quad x \in \Omega. \]


III. Problem:

\[
\begin{align*}
& u_{xx}(x, y) - \mathcal{L}u(x, y) = 0, \quad x \in (0, x_0), y \in \Omega, \\
& u(x, y) = 0, \quad x \in [0, x_0] \times \partial\Omega, \\
& u(0, y) = \varphi_3(y), \quad y \in \Omega, \\
& u_x(0, y) = 0, \quad y \in \Omega.
\end{align*}
\]

The approximate quasi-boundary value problem:

\[
\begin{align*}
& u_{xx}(x, y) - \mathcal{L}u(x, y) = 0, \quad x \in (0, ax_0), y \in \Omega, a \geq 1, \\
& u(x, y) = 0, \quad x \in [0, ax_0] \times \partial\Omega, \\
& u(0, y) + \alpha u(ax_0, y) = \varphi_3(y), \quad y \in \Omega, \\
& u_x(0, y) = 0, \quad y \in \Omega.
\end{align*}
\]


Figure: The positions of the conditions
Problem.

\[ \begin{align*}
\Delta u(x, y) &= 0, \quad x \in (0, 1), y \in \mathbb{R}^n, n \geq 1, \\
u(0, y) &= \varphi_4(y), \quad y \in \mathbb{R}^n, \\
u_x(0, y) &= 0, \quad y \in \mathbb{R}^n,
\end{align*} \]
2. Problem.

\[
\begin{aligned}
\Delta u(x, y) &= 0, \quad x \in (0, 1), y \in \mathbb{R}^n, n \geq 1, \\
u(0, y) &= \varphi_4(y), \quad y \in \mathbb{R}^n, \\
u_x(0, y) &= 0, \quad y \in \mathbb{R}^n,
\end{aligned}
\]

Failed.

\[
u(0, y) + \alpha u(1, y) = \varphi_4^\delta(y), \quad (3)
\]
\[
\hat{u}_{x}^{\alpha, \delta}(x, \xi) = \varphi_4^\delta(\xi) \frac{|\xi| \sinh(|\xi x|)}{1 + \alpha \cosh(|\xi|)}, \quad (4)
\]
2.

- **Problem.**

\[
\begin{align*}
\Delta u(x, y) &= 0, \quad x \in (0, 1), y \in \mathbb{R}^n, n \geq 1, \\
u(0, y) &= \varphi_4(y), \quad y \in \mathbb{R}^n, \\
u_x(0, y) &= 0, \quad y \in \mathbb{R}^n,
\end{align*}
\]

- **Failed.**

\[
u(0, y) + \alpha u(1, y) = \varphi_4^\delta(y),
\]

\[
\hat{u}_{x}^{\alpha, \delta}(x, \xi) = \hat{\varphi}_4^\delta(\xi) \frac{|\xi| \sinh(|\xi|x)}{1 + \alpha \cosh(|\xi|)},
\]

- **Succeed.**

\[
u(0, y) + \alpha u_x(1, y) = \varphi_4^\delta(y),
\]

\[
\hat{u}_{x}^{\alpha, \delta}(x, \xi) = \hat{\varphi}_4^\delta(\xi) \frac{|\xi| \sinh(|\xi|x)}{1 + \alpha |\xi| \sinh(|\xi|)},
\]
3.

- **Operator equation.**

\[ A(x)X(x) = Y, \quad \text{for} \quad x \in (x_0, x_1], \quad (7) \]
3. 

- **Operator equation.**

\[ A(x)X(x) = Y, \quad \text{for} \quad x \in (x_0, x_1], \quad (7) \]

- **Quasi-boundary-value method.**

\[ (\beta_1(\alpha)A(x) + \beta_2(\alpha)B(x, x_2))X(x) = Y, \quad \text{for} \quad x \in (x_0, x_1], \quad (8) \]

where

\[ \beta_1(\alpha) \to 1 \quad \text{and} \quad \beta_2(\alpha) \to 0, \quad \text{as} \quad \alpha \to 0. \quad (9) \]
3.

- **Operator equation.**

\[
A(x)X(x) = Y, \text{ for } x \in (x_0, x_1],
\]

(7)

- **Quasi-boundary-value method.**

\[
(\beta_1(\alpha)A(x) + \beta_2(\alpha)B(x, x_2))X(x) = Y, \text{ for } x \in (x_0, x_1],
\]

(8)

where

\[
\beta_1(\alpha) \rightarrow 1 \text{ and } \beta_2(\alpha) \rightarrow 0, \text{ as } \alpha \rightarrow 0.
\]

(9)

- **Lavrentiev regularization method.**

\[
(A(x) + \alpha I)X(x) = Y, \text{ for } x \in (x_0, x_1].
\]

(10)

A is self-adjoint.
Problem and the Quasi-Boundary Value Method

Theoretical aspects

Numerical aspects

Some phenomena of QuasiBVM

Future work
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- Theoretical aspects:
  - Improve the present error estimates;
  - The general domain.

- Numerical aspects:
  - Multiple dimensions;
  - Variable coefficients;
  - Arbitrary domain.
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Suggestions & Comments