KAM and Rigidity
(in partially hyperbolic and parabolic dynamics)

ANATOLE KATOK

The Pennsylvania State University

Based on joint work with
Danijela Damjanović and with
Zhenqi Jenny Wang

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Four types of linear maps in $\mathbb{R}^m$. A reminder

1. ELLIPTIC: All eigenvalues of absolute value 1; diagonalizable over $\mathbb{C}$
   DYNAMICS: All orbits bounded.
   STABILITY: Stable among symplectic maps; otherwise unstable.
   EXAMPLES: Rotations in $\mathbb{R}^2$ or $\mathbb{R}^3$.

2. PARABOLIC: All eigenvalues of absolute value 1; Non-diagonalizable.
   DYNAMICS: Most orbits grow polynomially with time; some bounded.
   STABILITY: Unstable. EXAMPLE: Jordan block.

3. HYPERBOLIC: No eigenvalues of absolute value 1.
   DYNAMICS: Every orbit grow exponentially in one time direction.
   STABILITY: Stable. EXAMPLE: $\mathrm{diag}(2, 1/2)$.

4. PARTIALLY HYPERBOLIC: Otherwise.
   DYNAMICS: Most orbits grow exponentially in one time direction.
   STABILITY: Usually become hyperbolic; stable in symplectic case.

Linear group actions

More generally, one considers actions of a (topological) group G by linear transformations; i.e. (continuous) homomorphisms

$$\alpha: \; G \to GL(n, \mathbb{R})$$

For abelian groups above classification extends. It depends only on *absolute values* of the eigenvalues.

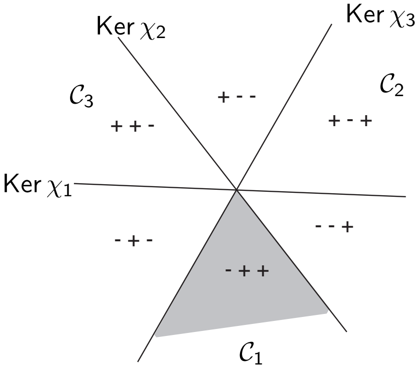
Logarithms of those are *Lyapunov exponents*.

For abelian G Lyapunov exponents are homomorphisms $G \to \mathbb{R}$.
We will only consider abelian groups in this lectures.

- ELLIPTIC and PARABOLIC: All Lyapunov exponents vanish.
- HYPERBOLIC: All Lyapunov exponents are non-zero.
- PARTIALLY HYPERBOLIC: Some vanish and some don't.

Weyl chambers

For $\mathbb{Z}^k$ actions Lyapunov exponents extend to linear functionals of $\mathbb{R}^k$. Kernels of non-zero Lyapunov exponents are *Lyapunov hyperplanes*; those divide $\mathbb{R}^k$ into *Weyl chambers*;



Lyapunov hyperplanes and Weyl chambers for $\mathbb{Z}^2$ or $\mathbb{R}^2$ action in $\mathbb{R}^3$
(compare with Rodriguez Hertz lectures)

Differentiable group actions

- M – differentiable manifold (usually compact).
- G – a group (discrete finitely generated or connected Lie).
- $\mathrm{Diff}^r(M)$ – the group of C^r diffeomorphisms of M;
$$r = 0, 1, 2, \dots; \infty, \omega.$$
- A continuous homomorphism $\alpha : G \to \mathrm{Diff}^r(M)$ is called a *C^r–action* of G on M.

We will discuss the actions of *higher rank* abelian groups: $\mathbb{Z}^k \times \mathbb{R}^l$, ($k + l \geq 2$; commuting maps and vector–fields).

The classical cases of $\mathbb{Z}$ actions (iterates of a single map) and $\mathbb{R}$ actions (flows generated by ODE) will appear occasionally for comparison and contrast.

- $\mathbb{R}^k \subset H$, a connected Lie group.
- Λ a lattice in H.
- $\mathbb{R}^k$ acts on a quotient H/Λ by left translations
- C is a compact subgroup of H which commutes with $\mathbb{R}^k$.

The $\mathbb{R}^k$-action on H/Λ descends to an action on $C \setminus H/\Lambda$.
The general algebraic $\mathbb{R}^k$-action ρ is a finite factor of such an action.

Let $\mathfrak{c}$ be the Lie algebra of C.

The linear part of ρ is the representation of $\mathbb{R}^k$ on $\mathfrak{c} \setminus \mathfrak{h}$ induced by the *adjoint representation* of $\mathbb{R}^k$ on the Lie algebra $\mathfrak{h}$ of H.

Algebraic actions. II: Affine

- H a connected Lie group
- $\Lambda \subset H$ a cocompact lattice.
- Aff (H) the set of diffeomorphisms of H which map right invariant vectorfields on H to right invariant vectorfields.
- Aff (H/Λ) the diffeomorphisms of H/Λ which lift to elements of Aff (H).

An action ρ of a discrete group G on H/Λ is *affine algebraic* if $\rho(g)$ is given by a homomorphism $G \to \text{Aff }(H/\Lambda)$.

Identifying the Lie algebra of H. $\mathfrak{h}$ with the right invariant vectorfields on H, any affine algebraic action determines a homomorphism $\sigma : G \to \text{Aut } \mathfrak{h}$, the *linear part* of this action.

Classification: four types of algebraic actions

According to the properties of the linear part the algebraic actions can be classified into:

1. ELLIPTIC: eigenvalues of modulus one, semisimple (diagonalizable),
2. PARABOLIC: eigenvalues of modulus one with some Jordan blocks),
3. HYPERBOLIC (Anosov): no eigenvalues of absolute value one; if G is discrete H has to be nilpotent,
4. PARTIALLY HYPERBOLIC: otherwise.

Lyapunov exponents for an algebraic action are those of its linear part, i.e logarithms of absolute values of the eigenvalues.

Basic algebraic rank 1 examples; a reminder

1. $\mathbb{Z} \to S^1 : \ n \mapsto \{n\alpha\}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$: irrational rotation of the circle, elliptic;
2. $\mathbb{R}$ embeds into $\mathbb{T}^k$: with dense image: irrational linear flow on the torus, elliptic;
3. $\mathbb{R}$ embeds to $G = SL(2, \mathbb{R})$ as $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$; acts on G/Γ by left translations:
   horocycle flow on a surface of constant negative curvature, parabolic;
4. $\mathbb{R}$ embeds to $G = SL(2, \mathbb{R})$ as $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$,
   geodesic flow on a surface of constant negative curvature, hyperbolic;
   (3) and (4) appeared in Einsiedler's lectures
5. any matrix $A \in SL(n, \mathbb{Z})$ defines an automorphism F_A of the torus $\mathbb{T}^n$.
   For $n = 2$ take $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, hyperbolic.

Some higher rank examples

1. Parabolic $\mathbb{R}^2$ action:

$$U = \{ \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} : (t,s) \in \mathbb{R}^2$$

   Action of U by left translations on $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\Gamma$; Γ an irreducible lattice

2. Hyperbolic $\mathbb{R}^{n-1}$ action (Weyl chamber flow);
   Left translations by positive diagonals
   $\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n}), \sum_{i=1}^{n} t_i = 0$ on $SL(n,\mathbb{R})/\Gamma, \; n \geq 3$
   see Section 2.3 of the Katok-Nitica book for more details.
   ((1) and (2) will appear later)

3. Hyperbolic $\mathbb{Z}^2$ action:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 8 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 8 & 4 \end{pmatrix}.$$

   Automorphisms F_A and F_B of $\mathbb{T}^3$ generate a hyperbolic $\mathbb{Z}^2$ action; prototype example from the class considered in Rodriguez Hertz lectures.

Example: a partially hyperbolic $\mathbb{Z}^2$ action

Matrices $A, B \in SL(6, \mathbb{Z})$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}$$

generate an irreducible partially hyperbolic but not hyperbolic $\mathbb{Z}^2$ action by automorphisms $F_A^n F_B^m$, $m, n \in \mathbb{Z}$ of $\mathbb{T}^6$ (Will appear later).

Dynamical/ergodic properties of algebraic actions: A summary

1. ELLIPTIC. DYNAMICS: Ergodicity depends on arithmetic. Ergodic implies uniquely ergodic, no mixing, discrete spectrum.
2. PARABOLIC. DYNAMICS: Ergodic wrt. Haar measure. Typically uniquely ergodic; other measures algebraic. Einsiedler's course. Mixing of all orders. Zero entropy of course.
3. HYPERBOLIC. DYNAMICS: Ergodic, mixing and more. Dense periodic orbits In rank one enormous variety of invariant sets and invariant measures. Einsiedler's course.

   In higher rank rigidity. Rodriguez Hertz course.
4. PARTIALLY HYPERBOLIC. DYNAMICS: More similar to hyperbolic than to others, except for periodic orbits.

C^r–conjugacy, orbit equivalence, time change; a reminder

- G-actions α and β are *C^r–conjugate* if $\alpha = h \circ \beta \circ h^{-1}$ for some $h \in \mathrm{Diff}^r(M)$.

- G-actions α and β are *C^r–conjugate up to an automorphism* if $\alpha \circ \rho = h \circ \beta \circ h^{-1}$ for some $h \in \mathrm{Diff}^r(M)$ and $\rho \in \mathrm{Aut}(G)$.

For G continuous let $\mathcal{O}_\alpha$ be the orbit foliation of α.

- α and β are *C^r–orbit equivalent* if $O_\alpha = hO_\beta$ for some $h \in \mathrm{Diff}^r(M)$.
- β is a *time change* of α if $O_\alpha = O_\beta$.
- Actions are orbit equivalent iff one is conjugate to a time change of the other.
- Time change is determined by a certain G–valued *1–cocycle* over the action (*old time* expressed as a function of *new time*).

Methods for finding conjugacies; a brief review

- GEOMETRIC or direct: constructing a conjugacy in the phase space
  Including : Fundamental domain method (good for dissipative systems or dissipative parts of more complicated systems),
  coding and closely related shadowing method
  (good for hyperbolic systems; codeword Markov partitions)

- ANALYTIC: finding a right element in a functional space of "candidate" conjugacies. For that usually a (non-linear) operator in the space of candidate conjugacies is constructed such that the conjugacy is a fixed point of that operator.

  There are two basic methods for finding fixed points:
  Contraction mapping principle and
  Newton iteration scheme.

  Both are widely used.

Analytic methods

CONTRACTION MAPPING PRINCIPLE (often in very fancy garb) is used in hyperbolic and partially hyperbolic problems:

- local: Sternberg, Hartman, Grobman, Chen and many others, and
- global Moser, Mather, Hirsch-Pugh-Shub, Franks, Manning and many others.

NEWTON ITERATION SCHEME originally appeared in elliptic problems:

- RANK 1– traditional KAM: Kolmorogov, Moser, Arnol'd, Zehnder, Herman, and many others;
- HIGHER RANK: Moser's scheme for commuting circle maps.

Lately it has been used in

- partially hyperbolic and
- parabolic

HIGHER RANK problems: Damjanović-K. method,

SUBJECT OF THIS COURSE.

Rigidity and its variations: Preliminaries

1. Full rigidity: An action α of a finitely generated discrete group A on a manifold M is $C^{k,r,l}$ *locally rigid* $(k \geq \max(r,l))$ if any sufficiently C^r-small C^k perturbation $\tilde{\alpha}$ is C^l conjugate to α

   For continuous groups such as $\mathbb{R}^k$ one has to allow a "linear time change" i.e an automorphism ρ of the acting group close to *id*.

2. Parametric rigidity: a finite-dimensional family of *standard perturbations* has to be allowed as models.

3. Conditional rigidity involves existence of actions conjugate to the unperturbed one in a generic smooth family of perturbations with a fixed number of parameters. (Example: Traditional KAM)

$C^{k,r,l}$ *orbit rigidity* defined similarly to full rigidity with C^k orbit equivalence instead of conjugacy. $C^{r,r,0}$ orbit rigidity is called C^r *structural stability*.

Review of the classical cases: $G = \mathbb{Z}, \mathbb{R}$; an aside

- C^1 structural stability is completely characterized; closely connected with (uniform) *hyperbolicity*. (D. Anosov, S.Smale, J. Robbin, R.Mañé etc; 1960's–1990's)
- In the structurally stable continuous time case there are infinitely many *moduli* of C^0 conjugacy; related to lengths of periodic orbits.
- *Diophantine* translations on a torus $\mathbb{T}^k$ are conditionally rigid with k parameters (Arnol'd's theorem; prototype of KAM)
- C^r (but not C^1) structural stability for $r > 1$ is unlikely, but the proof is currently beyond reach due to the lack of understanding of *global C^r perturbation constructions* for $r \geq 2$.
- $C^{k,1,1}$–rigidity or orbit rigidity is impossible; proof is based on good understanding of *local C^1 perturbations*.
- $C^{k,r,1}$–rigidity or orbit rigidity for $r > 1$ is highly unlikely and can be ruled out in many cases.

Rigidity in the higher-rank case; a brief history

The differentiable orbit structure of smooth actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k > 1$, is remarkably different from the classical cases $k = 1$.

Differentiable rigidity or parametric rigidity is most likely impossible in the classical cases, but these phenomena do appear in the higher–rank case.

Early indications in the works of N.Kopell and R. Sacksteder (late 1960's–early 1970's).

Differentiable (in particular, $C^{\infty,1,\infty}$) rigidity was proved in the 1990's (A.K– R.Spatzier, M.Guysinsky–A.K.) for most standard *hyperbolic* actions including

- Actions by *hyperbolic automorphisms of the torus*,
- *Weyl Chamber flows*
- Other hyperbolic homogeneous actions such as *twisted Weyl chamber flows*

using the structural stability/ *a prioiri* regularity method.

THE PROGRAM

Establish local differentiable rigidity (of an appropriate flavor) for broad classes of partially hyperbolic and parabolic algebraic actions of $\mathbb{Z}^k \times \mathbb{R}^l$, $k + l \geq 2$, i.e. homogeneous or affine actions on homogeneous and double coset spaces for Lie groups as above.

Partially hyperbolic and parabolic maps are not structurally stable. *Even among linear maps they are not stable*. New methods needed.

Both

- ANALYTIC (Harmonic analysis/KAM) and
- GEOMETRIC (partially hyperbolic dynamics/Algebraic K-theory)

new methods have been developed.

New tools where brought in.

How to decide on correct flavor of rigidity

For an algebraic action look at algebraic perturbations:

1. Maximal action (any algebraic perturbation is a conjugate): full rigidity. This happens for
   - hyperbolic actions,
   - partially hyperbolic actions by automorphisms of tori and nilmanifolds,
   - some very special twisted examples (genuinely partially hyperbolic Weyl chamber flows).
2. Algebraic perturbations are of the same type: parametric rigidity. This happens for
   - partially hyperbolic actions with prevalence of hyperbolic behavior
3. Algebraic perturbations change type: conditional rigidity (at best). This happens in
   - elliptic actions,
   - parabolic actions,
   - partially hyperbolic actions with "large" non-hyperbolic part.

KAM METHOD (general scheme for a single transformation)

- M – compact smooth manifold;
- $g : M \to M$ – a "model" (linear, algebraic, homogeneous, etc); Assume linear structure in a space of diffeomorphisms near g and near id;
- $f = g + u$ – a "perturbation" (nonlinear, general). Want to show that f is (smoothly) conjugate to g via an unknown $h = \mathrm{id} + w$.
- Conjugacy equation as an implicit-function problem:

$$g = \mathcal{F}(f, h) := h^{-1} \circ f \circ h. \tag{1}$$

- The "group property":

$$\mathcal{F}(f, \varphi \circ \psi) = \mathcal{F}(\mathcal{F}(f, \varphi), \psi), \qquad \mathcal{F}(f, \mathrm{id}) = f. \tag{2}$$

Linearized equation.

- $D_1\mathcal{F}$ and $D_2\mathcal{F}$ partial differentials with respect to f.
- First order Taylor expansion of (1) at (g, id):

$$\mathcal{F}(f,h) = \mathcal{F}(g,\mathrm{id}) + D_1\mathcal{F}(g,\mathrm{id})(u) + D_2\mathcal{F}(g,\mathrm{id})(w) + \mathcal{R}(f,h);$$

  $\mathcal{R}(f,h)$ is of second order in (u,w).
- If h solves the linearized equation (obtained by dropping $\mathcal{R}$), then

$$\mathcal{F}(g,\mathrm{id}) + D_1\mathcal{F}(g,\mathrm{id})(f-g) + D_2\mathcal{F}(g,\mathrm{id})w = g. \tag{3}$$

  Since $\mathcal{F}(\cdot,\mathrm{id}) = \mathrm{id}(\cdot)$ by (2), $D_1\mathcal{F}(g,\mathrm{id}) = \mathrm{id}$, and
  $D_2\mathcal{F}(g,\mathrm{id})w = dg(w) - w \circ g$.

$$dg(w) - w \circ g = -u. \tag{4}$$

  This is a twisted cohomological equation.
- If $D_2\mathcal{F}(g,\mathrm{id})$ is invertible, then $w = -\left(D_2\mathcal{F}(g,\mathrm{id})\right)^{-1} u$.

Linearized equation.

Cohomological equations, both untwisted and twisted, have been extensively studied, starting form the fundamental observation of Kolmogorov for Diophantine elliptic systems (1953) and celebrated theorem by Alexander Livshitz for hyperbolic systems (1971), recently extended by Kalinin.

Significant contributions: de la Llave with collaborators, Nitica-Torok, K-Spatzier, K-Kononenko, Flaminio-Forni, and many others.

For elliptic, hyperbolic and partially hyperbolic systems both harmonic analysis methods (for algebraic systems) and geometric methods (for algebraic and non-algebraic systems) are used successfully.

For parabolic systems only harmonic analysis methods are available; hence only algebraic systems have been treated so far.

Four our applications high regularity solutions with tame estimates are needed; this puts premium on harmonic analysis methods.

Quadratic convergence

- In the case of invertibility, w is of the same order as u, and substituting $h = \mathrm{id} + w$ into $\mathcal{F}(f, h)$ we obtain a function $f_1 = h^{-1} \circ f \circ h = \mathcal{F}(f, h) = g + \mathcal{R}(f, h)$, so the size of $u_1 = f_1 - g = \mathcal{R}(f, h)$ is formally of second order in the size of $u = f - g$.
- **Iterative process.** Assuming that $f_1, \ldots, f_n$ have been constructed, solve the equation

$$f_n - g + D_2\mathcal{F}(g, \mathrm{id})w_{n+1} = 0$$

and set

$$h_{n+1} = h_n \circ (\mathrm{id} + w_{n+1}) \text{ and } f_{n+1} = (\mathrm{id} + w_{n+1})^{-1} \circ f_n \circ (\mathrm{id} + w_n).$$

- To justify the iterative process and prove convergence, one needs to estimate the difference between $\mathcal{F}$ and its linearization near (g, id).

Intrinsic subtlety of the conjugacy problem

Notice that at every step the linear part is inverted at (g, id), rather than at the intermediate points as in the elementary Newton method. This is the main reason which makes analytic difficulties manageable (however often still quite formidable):

- There are usually *obstructions* to solvability of the linearized equation: the operator $D_2\mathcal{F}(g, \mathrm{id})$ is only invertible at the kernel of these obstructions which may have finite or infinite codimension.
- Even at the kernel there is usually no bounded inverse in any natural class of regularity (C^r, Sobolev, analytic in a fixed domain).

THE RESULTS. Overview and comparison

D. Damjanovic, A.K., Z.J.Wang, with contributions from
V. Nitica, D. Mieszkowski, L. Flaminio; 2004–

- The program has been mostly completed for partially hyperbolic algebraic actions with the use of the analytic method: [1,2].
- For representative parabolic examples conditional rigidity has been established by the analytic method: [3] ;
  work on other parabolic cases is in progress: [4].
- Analytic method requires very high regularity of the perturbation; i.e. closeness with many derivatives.
- Geometric method is more robust; C^2 closeness is sufficient.
- Geometric method is not applicable in certain cases;
  in others it requires case-by case considerations,
  e.g. going through the list of simple Lie groups and particular resonances in Cartan subgroups
  This involves very heavy algebra and has been done only partially.

Rigidity by the THE ANALYTIC METHOD: examples

1. Full rigidity for commuting partially hyperbolic automorphisms of the torus [1], like

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}$$

2. Parametric rigidity with *algebraic standard perturbations* for restrictions of Weyl chamber flows to most higher rank subgroups [2] like a $\mathbb{Z}^2$ lattice in the positive diagonal subgroup of $SL(4,\mathbb{Z})$:

$$\{\mathrm{diag}(e^{t_1}, e^{t_2}, e^{t_3}, e^{t_4}),\ \sum_{i=1}^{4} t_i = 0\}$$

that does not lie in a Lyapunov plane $t_i = t_j$.

3. Conditional rigidity for unipotent subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ [3]:

$$U = \{ \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} : (t,s) \in \mathbb{R}^2$$

THE ANALYTIC METHOD; General structure of proofs

- Preliminary step.
  Smooth rigidity of neutral foliation by *a priori* regularity method.
  Essential in (2) and similar situations (semisimple and twisted cases),
  can be avoided but is still useful in (1) (actions on the torus),
  vacuous in parabolic cases like (3).

- Main step.
  Solution of the linearized equation and estimation of non-linearity .
  Case sensitive. Uses Fourier analysis in (1)
  and theory of unitary representation of semi-simple Lie groups:
  Decay of matrix coefficients in (2),
  Classification of irreducible representations in (3).

  Classical and functional analysis tools are also used.

- Final step. Iterative scheme.
  General with minor variations.

  Tools: embedding theorems, interpolation inequalities.

THE MAIN STEP

- Conjugacy equation is formally linearized at the target.

  Solution of the linearized conjugacy equation is attempted by formally inverting linearized conjugacy operator.

- *Obstructions* are found:

  There are *infinitely many* obstructions for solving the linearized (cohomological) equations for an individual element.
  Those obstructions are of the type of
  invariant distributions for action elements.

  This is a crucial difference with standard KAM for translations where the only obstruction is Integral w.r.t. Lebegue measure.

  Obstructions can be described by different means;
  for our purposes dual description through harmonic analysis is used *).

*) In the geometric method obstructions are described in a totally different way.

THE MAIN STEP

- All but finitely many obstructions vanish
  due to the commutation relations.
  (the Higher-rank trick: THE HEART OF THE PROOF)
  At this step harmonic analysis is used crucially:
  - Fourier analysis for the torus;
  - unitary group representations for homogeneous actions.

  The "philosophy" of the Higher rank trick is universal,
  its execution is case-sensitive.

- The remaining finitely many parameters are absorbed
  by allowing
  1. automorphism of the group (appears for actions of continuous groups);
  2. a standard perturbation (appears in parametric and conditional rigidity);
  3. adjusting parameters. (appears in conditional rigidity).

THE MAIN STEP. I: Solution of the linearized equation

Linearized equation is solved with tame estimates,
i.e. finite loss of regularity and estimates of lower C^r or Sobolev norms of solutions through those of the data.

- In (2) (semisimple case) this is achieved roughly by prevalence of hyperbolic behavior.
- In (1) (actions on the torus) and (3) (parabolic case) this involves glueing of solutions constructed in certain invariant subspaces of functional spaces:
  - In (1) cyclic spaces of characters.
    Key for tame estimates: Diophantine properties of algebraic numbers.
  - In (3) irreducible representation spaces.
    Key for tame estimates: Spectral gap.

THE MAIN STEP. II: Splitting

- The perturbation can be split into two terms
  due to the commutation relations:
  one for which the linearized equations are satisfied, and
  the other "quadratically small" with tame estimates.
  (builds on Moser's original idea)

- In original proofs for (1) (actions on the torus) and (3) (parabolic) this step uses harmonic analysis by working in invariant spaces as above.

- In the proof for semisimple and twisted cases (2) this step is done using functional analysis and is hence less specific.
  This method can also be applied to other cases.

THE FINAL STEP

- Conjugacy provided by the solution of the linearized equation transforms the (modified in the conditional case) perturbed action into an action quadratically close to the target.

  This is the *Newton's scheme iteration step*.

- Procedure converges providing a smooth conjugacy in the limit.

  In (2) convergence is originally guaranteed in a low regularity norm; high regularity of the conjugacy is then established via Hormander's elliptic regularity theory similarly to the *a priori regularity* method for hyperbolic actions.

- For (1) (actions on the torus; full rigidity) Newton scheme can be substituted by
  Hamilton's Hard Implicit Function Theorem.
  No parametric version of this theorem is available.

Partially hyperbolic automorphisms of a torus

An automorphism F_A of the torus $\mathbb{T}^m$ is determined by an $m \times m$ matrix A with integer entries and determinant ± 1.

The dual (character) group to $\mathbb{T}^m$ is $\mathbb{Z}^m$. The dual to F_A is the automorphism $A^* : \mathbb{Z}^m \to \mathbb{Z}^m$ given by the matrix transposed to A. The following conditions are equivalent:

- None of the eigenvalues of the matrix A is a root of unity.
- The automorphism F_A is ergodic with respect to Lebesgue measure.
- Every orbit of the dual map A^*, except that of zero, is infinite.

An automorphism of a torus satisfying any of the conditions above is called ergodic.

The higher rank condition

For an action α of $\mathbb{Z}^k$ by automorphisms of the torus the following two properties are equivalent descriptions of the genuine higher rank condition:

- α has no non-trivial (positive dimension) rank one factors;
- α contains $\mathbb{Z}^2$ subgroup, all of whose elements except for the identity are ergodic automorphisms.

Main Theorem for the actions on a torus [1]

Theorem

Let $\alpha : \mathbb{Z}^k \times \mathbb{T}^N \to \mathbb{T}^N$ be a C^∞ partially hyperbolic action of $\mathbb{Z}^k$ $(k \geq 2)$ by automorphisms without non-trivial rank-one factors.

There exists l which depends on α such that for any C^l-small C^∞ perturbation of α, $\tilde{\alpha} : \mathbb{Z}^d \times \mathbb{T}^N \to \mathbb{T}^N$ there exists a C^∞ map $H : \mathbb{T}^N \to \mathbb{T}^N$ such that

$$\alpha \circ H = H \circ \tilde{\alpha},$$

i.e α is $C^{\infty,l,\infty}$ locally rigid.

Comments

- Any action which contains an element F_A where the matrix A has an irreducible over $\mathbb{Q}$ characteristic polynomial
  has no nontrivial rank one factors.
- An action of $\mathbb{Z}^k$ by ergodic automorphisms of $\mathbb{T}^N$ is *genuinely partially hyperbolic* if it has a zero Lyapunov exponent , i.e. an invariant root subspace with eigenvalues of absolute value one for *all* elements of the action.
- Multiplicity of the zero exponent for such an action is always even because the eigenvalues corresponding to the exponent are complex and hence come in conjugate pairs.
- If the action is semisimple (diagonalizable over $\mathbb{C}$), the number l of required derivatives can be made depends only on the dimension. This involves using the preliminary step (neutral foliation rigidity) and produces only C^r conjugacy with large r.

Example: a genuinely partially hyperbolic $\mathbb{Z}^2$ action

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}$$

generate an irreducible genuinely partially hyperbolic $\mathbb{Z}^2$ action by automorphisms $F_A^n F_B^m$, $m, n \in \mathbb{Z}$ of $\mathbb{T}^6$.

Similar examples exist in any even dimension ≥ 6; reducible example exist in odd dimensions ≥ 9. No examples in dimension $2, 3, 4, 5, 7$.

Setup of the Proof of the Torus Theorem

α – linear action: $\tilde{\alpha}$ – perturbation

Assume $\mathcal{R} \stackrel{\text{def}}{=} \tilde{\alpha} - \alpha$ is "small", i.e. $\|\mathcal{R}\|_{C^l} \leq \varepsilon$, for a small ε

The goal is to show the existence of $H : \mathbb{T}^N \to \mathbb{T}^N$ such that $\tilde{\alpha} \circ H = H \circ \alpha$, $H = \mathrm{id} + \Delta$ where Δ should be "small" as well.

In terms of Δ we need

$$\alpha\Delta - \Delta \circ \alpha = -\mathcal{R} \circ (\mathrm{id} + \Delta).$$

If Δ is a solution for the corresponding linearized equation:

$$\alpha\Delta - \Delta \circ \alpha = -\mathcal{R},$$

(here we identify α with its linear part), then $\tilde{\alpha}^{(1)} \stackrel{\text{def}}{=} H^{-1} \circ \tilde{\alpha} \circ H$ should be "quadratically" close to α, i.e.

$\mathcal{R}^{(1)} \stackrel{\text{def}}{=} \tilde{\alpha}^{(1)} - \alpha$ should be "quadratically small" with respect to $\mathcal{R}$.

Setup of the Proof of the Torus Theorem

The new error is:

$$\mathcal{R}^{(1)} = \tilde{\alpha}^{(1)} - \alpha = \left[\Delta \circ \tilde{\alpha}^{(1)} - \Delta \circ \alpha + \mathcal{R}(\mathrm{id} + \Delta) - \mathcal{R}\right] +$$

$$\left[\mathcal{R} - [\alpha \circ \Delta - \Delta \circ \alpha]\right].$$

The part of the error in the first parentheses is easy to estimate providing Δ is "small".

Therefore, it is enough to solve the linearized equation approximately (i.e. with an error "quadratically small" w.r.t. $\mathcal{R}$)
in order to run the KAM iteration scheme and produce a solution in the limit.

The Linearized Equation (Main Step I.)

Enough to work with two ergodic elements F_A and F_B that generate $\mathbb{Z}^2$ will all ergodic elements.
Solving the linearized equation

$$\alpha\Delta - \Delta \circ \alpha = -\mathcal{R}$$

reduces to

$$A\Delta - \Delta \circ A = -\mathcal{R}_A, \quad B\Delta - \Delta \circ B = -\mathcal{R}_B.$$

It is possible to solve the linearized equation with *fixed loss of the regularity* in C^∞ case if

$$L(\mathcal{R}_A, \mathcal{R}_B) \stackrel{\text{def}}{=} (\mathcal{R}_A \circ B - B\mathcal{R}_B) - (\mathcal{R}_B \circ A - A\mathcal{R}_B) = 0$$

This is the cocycle condition (vanishing of the second coboundary operator) that is the linearized form of the commutativity condition.

The splitting (Main Step II.)

Even if $\mathcal{R}_A$ and $\mathcal{R}_B$ do not satisfy the solvability condition above exactly, it is still possible to approximate both by maps which satisfy the solvability condition approximately with error bounded by the size of $L(\mathcal{R}_A, \mathcal{R}_B)$. Again with *fixed loss of regularity* in the C^∞ case.

This is closely related to *second C^∞ cohomogy* of the action (first studied by A.K. and S. Katok in 1995).

We show that if $\tilde{\alpha} = \alpha + \mathcal{R}$ is a commutative action then $L(\mathcal{R}_A, \mathcal{R}_B)$ is "quadratically small" w.r.t $\mathcal{R}_A$, $\mathcal{R}_B$.

From these we get an approximate solution to the linearized equation.

The norms

Let f be a C^∞ function on the torus $\mathbb{T}^N$, $f = \sum_n \hat{f}_n e_n$, e_n the characters.
$\|f\|_a \stackrel{\text{def}}{=} \sup_n |\hat{f}_n||n|^a, \qquad a > 0.$

The following relations hold

$$\|f\|_r \le C\|f\|_{C^r} \qquad \text{and} \qquad \|f\|_{C^r} \le C_r \|f\|_{r+\sigma}$$

for any $\sigma > N+1$, and any natural non-negative integer r.
In particular, one may take $\sigma = N + 1 + \delta$ with small $\delta > 0$.

Instead of norms $\|\cdot\|_a$ and one can use Sobolev norms $\|\cdot\|_{H^s}$
Loss of regularity is smaller but calculations are less transparent .

Tame solutions of twisted coboundary equation

- $\theta \in C^\infty(\mathbb{T}^N)$, $\lambda \in \mathbb{C}$, $\lambda \neq 1$.
- $A \in GL(N, \mathbb{Z})$ defines an ergodic automorphism of $\mathbb{T}^N$;
- ρ the minimal expansion/contraction rate for A
- $\mathcal{O}_n(\theta) = \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \hat{\theta}_{A^i n}$

Assume that for all $n \in \mathbb{Z}^N \setminus \{0\}$, $\mathcal{O}_n(\theta) = 0$. (obstructions vanish)

Then the twisted coboubdary equation:

$$\lambda\omega - \omega \circ A = \theta \tag{5}$$

has a C^∞ solution ω and the following estimate:

$$\|\omega\|_{a-\delta} \leq \frac{C_a}{\delta^\nu} \|\theta\|_a \tag{6}$$

holds for $\delta > 0, \nu = aN + 1$ and $a > \frac{|\log|\lambda||}{\log \rho}$. Thus for $r \geq 0$:

$$\|\omega\|_{C^r} \leq C_r \|\theta\|_{C^{r+\sigma}} \tag{7}$$

where σ is an integer greater than $\max\{N + 1, \frac{|\lg|\lambda||}{\lg \rho}\}$.

The Diophantine condition

Lemma (Katznelson)

Let A be an $N \times N$ matrix with integer coefficients. Assume that $\mathbb{R}^N$ splits as $\mathbb{R}^N = V \oplus V'$ with V, V' invariant under A and such that $A|_V$ and $A|_{V'}$ have no common eigenvalues. If $V \cap \mathbb{Z}^N = \{0\}$ then there exists a constant γ such that $d(n, V) \geq \gamma \|n\|^{-N}$ for all $n \in \mathbb{Z}^N$ where $\| \cdot \|$ is Euclidean norm and d is Euclidean distance.

This can be viewed as a version of the Liouville's theorem about rational approximation of algebraic irrationals, i.e.$|\alpha - \frac{m}{n}| \geq Cn^{-N}$ for any non-zero integers m and n, where α is an irrational first order root of an integer polynomial of degree N.

Orbit growth for the dual action

Let α be a $\mathbb{Z}^d$ action by ergodic automorphisms of $\mathbb{T}^N$. Then there exist constants $\tau > 0$ and $C > 0$ depending on the action only, such that: For every integer vector $n \in \mathbb{Z}^N$ and for all $k \in \mathbb{Z}^d$:

$$|\alpha^k n| \geq C \exp\{\tau \|k\|\} |n|^{-N}$$

Tame splitting and loss of regularity

Let θ, ψ, φ be C^∞ functions such that $L(\theta, \psi) = \Delta^\mu\theta - \Delta^\lambda\psi = \varphi$, then it is possible to split θ and ψ as

$$\theta = \mathcal{P}\theta + \mathcal{E}\theta$$
$$\psi = \mathcal{P}\psi + \mathcal{E}\psi$$

so that $L(\mathcal{P}\theta, \mathcal{P}\psi) = 0$, $L(\mathcal{E}\theta, \mathcal{E}\psi) = \varphi$ and the following estimates hold:

$$\|\mathcal{E}\theta, \mathcal{E}\psi\|_{C^r} \leq C\|\varphi\|_{C^{r+\sigma}} \tag{8}$$

for any $r > 0$ and any $\sigma > \tilde{M}_{\lambda,\mu}$ and

$$\|\mathcal{P}\theta, \mathcal{P}\psi\|_{C^r} \leq C\|\theta, \psi\|_{C^{r+\sigma}} \tag{9}$$

for any $r > 0$ and any $\sigma > \dot{M}_{\lambda,\mu}$.
As λ and μ are eigenvalues of A and B, constants $\tilde{M}_{\lambda,\mu}$ and $\dot{M}_{\lambda,\mu}$ depend only on A, B and N.