2
Principal classes of algebraic actions

2.1 Automorphisms of tori and (infra)nilmanifolds

2.1.1 Nilpotency of the ambient group

Let us start with a fundamental result that restricts the array of examples of algebraic actions of interest to us. We show that for an Anosov algebraic action ρ of a discrete group G, $\rho : G \to \text{Aff}\,(H/\Lambda)$, the Lie group H has to be nilpotent. Thus, the most general case of an affine Anosov $\mathbb{Z}^k$ action takes place on an infranilmanifold, that is, a finite quotient of a nilmanifold N/Λ, where N is a connected simply connected nilpotent Lie group and $\Lambda \subset N$ is a co-compact lattice.

The proof of the following theorem is taken from [41, Proposition 3.13].

Theorem 2.1.1 *Let H be a connected, simply connected Lie group. Assume there exists $\Phi \in Aff\,(H)$ such that the linear part of Φ is hyperbolic. Then H is nilpotent.*

Proof Let $\mathfrak{h}$ be the Lie algebra of H. We recall from Section 1.6.2 that any map $\Phi \in \text{Aff}\,(H)$ is a composition of an automorphism ϕ of H with left multiplication L_g by an element of $g \in H$. Let ψ be the automorphism of $\mathfrak{h}$ induced by $Ad(g) \circ \phi_*$. Let ψ_s be the semisimple component of the Jordan decomposition of ψ, which is also an automorphism of $\mathfrak{h}$. Since, with respect to a right invariant metric on H, $Ad(g) \circ \phi_*$ and $(R_{g^{-1}})_* Ad(g) \circ \phi_* = \phi_*$ have the same norm for any $v \in \mathfrak{h}$, Φ Anosov implies that ψ, and hence ψ_s, cannot have eigenvalues of modulus one.

If $\mathfrak{s}$ is the (solv)radical of $\mathfrak{h}$, that is, the maximal solvable ideal of $\mathfrak{h}$, then ψ_s induces an automorphism of $\mathfrak{h}/\mathfrak{s}$, which is also denoted by ψ. Since $\mathfrak{h}/\mathfrak{s}$ is a semisimple Lie algebra, some finite power ψ_s^q of ψ_s

coincides to $Ad(h+\mathfrak{s})$ for some $h \in \mathfrak{h}$. Moreover, $Ad(h+\mathfrak{s})$ must contain eigenvalues of modulus one. Since the eigenvalues of ψ_s^q are powers of those of ψ, this gives a contradiction unless $\mathfrak{h}/\mathfrak{s}$ is trivial. So one can assume that H is solvable.

To show that $\mathfrak{h}$ is nilpotent one shows that it coincides with its nil-radical (the maximal nilpotent ideal in $\mathfrak{h}$). It is enough to do this for the complexification of $\mathfrak{h}$, which it is also denoted by $\mathfrak{h}$. Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{h}$. Then $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{n}$. If $\mathfrak{h} \neq \mathfrak{n}$, then there exists $X \in \mathfrak{h}$ such that $X \notin \mathfrak{n}$ and X is an eigenvector for ψ_s with eigenvalue λ, $|\lambda| \neq 1$. To finish the proof it is enough now to show that $\mathbb{R}X + \mathfrak{n}$ is a nilpotent ideal.

The nilpotent algebra $\mathfrak{n}$ has two natural filtrations. The first one is given by the descending central series, which is finite. Let $\mathcal{C}^0\mathfrak{n} = \mathfrak{n}$, $\mathcal{C}^i\mathfrak{n} = [\mathfrak{n}, \mathcal{C}^{i-1}\mathfrak{n}]$, and k be the first integer such that $\mathcal{C}^k\mathfrak{n} = 0$. Then

$$\mathfrak{n} = \mathcal{C}^0\mathfrak{n} \supset \mathcal{C}^1\mathfrak{n} \supset \cdots \supset \mathcal{C}^k\mathfrak{n} = 0$$

and $[X, \mathcal{C}^i\mathfrak{n}] \subset \mathcal{C}^i\mathfrak{n}$.

To define the second filtration, order the eigenvalues $\{\lambda_1, \ldots, \lambda_r\}$ of ψ_s on $\mathfrak{n}$ in increasing order if $|\lambda| > 1$, and in decreasing order if $|\lambda| < 1$. If V_i is the eigenspace of the eigenvalue λ_i, and $W_I = \oplus_{j=i+1}^r V_i$, then

$$\mathfrak{n} = W_0 \supset W_1 \supset W_1 \supset \cdots \supset W_r = 0.$$

We show now that $\mathbb{R}X+\mathfrak{n}$ has nilpotency degree kr. Let $Y \in \mathcal{C}^l(\mathbb{R}X+\mathfrak{n})$ with $l > kr$, that is:

$$Y = [a_l X + N_l, [\cdots [a_2 X + N_2, a_1 X + N_1] \cdots],$$

where $a_i \in \mathbb{R}$ and $N_i \in \mathfrak{n}$. An easy computation shows that Y can be written as a linear combination of terms of the form

$$y = ad(Y_l)ad(Y_{l-1}) \cdots ad(Y_2)(Y_1)$$

where either $Y_i = X$ or $Y_i \in \mathfrak{n}$. Since $l > kr$, either k of Y_i's lie in $\mathfrak{n}$, or there exists a string of r consecutive Y_i's all equal to X. In the first situation use $[X, \mathcal{C}^i\mathfrak{n}] \subset \mathcal{C}^i\mathfrak{n}$ to conclude that $y = 0$. In the second case use that $[X, W_i] \subset W_{i+1}$ to conclude again that $y = 0$. In either case $Y = 0$. □

2.1.2 Ergodic automorphisms of the torus

An automorphism of the torus $\mathbb{T}^m$ is determined by an $m \times m$ matrix A with integer entries and determinant ± 1. Our standard notation for this automorphism is F_A. The group of all such matrices, which is isomorphic to the group of automorphisms of the torus $\mathbb{T}^m$, is denoted by $GL(m, \mathbb{Z})$.

Recall that the transformation F_A is *ergodic* with respect to Lebesgue measure μ on $\mathbb{T}^m$ if and only if any L^2 function that is F_A-invariant is constant. The dual (character) group of $\mathbb{T}^m$ is $\mathbb{Z}^m$. By looking at the dual action of A on the characters one can easily characterize ergodic automorphisms of a torus. The dual to F_A is the automorphisms $A^* : \mathbb{Z}^m \to \mathbb{Z}^m$ given by $A^* = (A^t)^{-1}$.

Proposition 2.1.2 *An automorphism F_A is ergodic with respect to the Lebesgue measure if and only if none of the eigenvalues of the matrix A is a root of unity.*

Proof Let $\phi \in L^2(\mu, \mathbb{T}^m)$ that is F_A-invariant. The ϕ can be written as a Fourier series

$$\phi = \sum a_n e_n, \qquad e_n(x) = e^{2\pi i <n,x>}.$$

From the F_A-invariance of ϕ it follows that $a_{A^*n} = a_n$ for all $n \in \mathbb{Z}^m$. But $\sum_{n\in\mathbb{Z}^m} |a_n|^2 < \infty$, so $a_n \neq 0$ is equivalent to the fact that the set $\{(A^t)^k n\}_{k\in\mathbb{Z}}$ is finite. Now, the last set is finite if and only if $1 \in \text{spec}(A^k)$ for any k, or if $n = 0$. So ϕ has to be constant. □

Remark 2.1.3 *It is an immediate corollary of the previous proof that ergodicity of F_A is equivalent to one of the following: the periodic points of F_A are exactly points all of whose coordinates are rational; every orbit of the dual map A^*, except that of zero, is infinite.*

One can show that ergodicity of F_A is equivalent to F_A being Bernoulli with respect to Lebesgue measure. This equivalence is proved in [83]. See [3] for a simple proof.

Proposition 2.1.4 *Any ergodic automorphism of a torus is partially hyperbolic.*

Proof Assume first that the matrix A of the linear part is semisimple (no non-trivial Jordan blocks). If all eigenvalues have absolute value one then $A^{n_k} \to \text{Id}$ for a certain sequence $n_k \to \infty$. Since all powers of A are integer matrices this implies that for a large enough k $A^{n_k} = \text{Id}$, so F_A cannot be ergodic.

If there are Jordan blocks, then there is an invariant rational subspace L such that A restricted to L is semisimple. Since L is rational its intersection with the integer lattice is a lattice in L. Hence restriction of A to L is an integer matrix expressed in that basis. Now the previous argument applies. □

The classification of ergodic automorphisms from a measure theory point of view is given by their entropy which is equal to the sum of positive Lyapunov characteristic exponents. This follows from Ornstein Isomorphism Theorem [133] and the fact that every ergodic automorphism of a torus is Bernoulli with respect to Lebesgue measure [83].

Since for $A \in \mathrm{GL}(n, \mathbb{R})$ Lyapunov exponents are equal to the logarithms of the absolute values of eigenvalues of the matrix A, entropy is determined by the conjugacy class of A over $\mathbb{Q}$ (or over $\mathbb{C}$). As a consequence, all ergodic automorphisms of a torus which are conjugate over $\mathbb{Q}$ are measurably conjugate with respect to Lebesgue measure.

2.1.3 Anosov diffeomorphisms on nilmanifolds

All known examples of Anosov diffeomorphisms are topologically conjugate to affine Anosov diffeomorphisms of infranilmanifolds (including nilmanifolds and tori as special cases). Moreover, it was proved by Franks and Manning [39, 109] that an arbitrary Anosov diffeomorphism of an infranilmanifold is topologically conjugate to one of this type. It is a well known conjecture that, up to topological conjugacy, affine Anosov diffeomorphisms are the only Anosov diffeomorphisms on infranilmanifolds. It has been verified in dimensions two and three where the only possible ambient manifold is a torus. †

Now we present several examples of affine Anosov diffeomorphisms on nilmanifolds of higher dimension.

2.1.3.1 Preliminaries on nilpotent Lie groups and lattices

Let G be a connected simply connected Lie group. A subgroup $\Gamma \subset G$ is called *lattice* if the orbit space G/Γ has finite volume, and it is called *uniform* or *co-compact* if in addition G/Γ is compact. If Γ is torsion free uniform lattice, then the action of Γ by left translations on G is properly discontinuous, i.e. for every compact set $C \subset G$ the set $\{g \in \Gamma | gC \cap C \neq \emptyset\}$ is finite, and the orbit space G/Γ is a well defined compact manifold.

† Nothing like that holds for Anosov flows where already in dimension three there is a variety of essentially non-algebraic possibilities. This is a manifestation of somewhat less rigid nature of continuous group actions mentioned before.

Let N be a connected simply connected nilpotent Lie group, and $\mathfrak{n}$ its Lie algebra. It is known that N and $\mathfrak{n}$ are diffeomorphic via the exponential map, and the groups of automorphisms of N and $\mathfrak{n}$ are can be identified.

Recall that a Lie algebra $\mathfrak{g}$ is a (real) vector space endowed in addition with a bilinear operation, called *bracket*, $\mathfrak{g} \times \mathfrak{g} \ni (x, y) \to [x, y] \in \mathfrak{g}$, that satisfies Jacobi identity, $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ and is anti-symmetric, $[x, y] = -[y, x]$. If a basis for $\mathfrak{g}$ is fixed, say $x_1, \ldots, x_n$, then the bracket is uniquely determined by the *structure constants* defined by

$$[x_i, x_j] = \sum_k c_{ij}^k x_k.$$

Let $\Gamma \subset N$ be a lattice. It follows from results of Malcev [107] that N admits a lattice if and only if there exists a basis in its Lie algebra $\mathfrak{n}$ for which the structure constants are rational. The following two properties are specific for lattices in nilpotent Lie groups: First, any lattice Γ in a connected nilpotent Lie group is uniform. Second, Γ is a finitely generated torsion free nilpotent subgroup of N. The manifold N/Γ is called *nilmanifold.*

Conversely, for any discrete finitely generated torsion free nilpotent group Γ there exists a unique connected simply connected nilpotent Lie group $\Gamma_{\mathbb{R}}$, called the *Malcev completion* of Γ, in which Γ is a uniform lattice. We briefly recall the construction of the Malcev completion.

The lower central series of Γ is defined inductively via $c_1(\Gamma) = \Gamma$, $c_{i+1}(\Gamma) = [c_i(\Gamma), \Gamma]$, where $[\cdot, \cdot]$ is the notation for the centralizer. The group Γ is called *r-step nilpotent* if $c_r(\Gamma) \neq 1$, but $c_{r+1}(\Gamma) = 1$. The *isolator* of a subgroup $S \subset \Gamma$ is

$$\sqrt[\Gamma]{S} := \{\gamma \in \Gamma | \gamma^k \in S \text{ for some } k \in \mathbb{N}\}.$$

If Γ is r-step nilpotent, the sequence

$$\begin{aligned} \Gamma_{r+1} = 1 \subset \Gamma_r = \sqrt[\Gamma]{c_r(\Gamma)} \subset \Gamma_{r-1} = \sqrt[\Gamma]{c_{r-1}(\Gamma)} \subset \ldots \\ \subset \Gamma_2 = \sqrt[\Gamma]{c_2(\Gamma)} \subset \Gamma_1 = \sqrt[\Gamma]{c_1(\Gamma)} = \Gamma \end{aligned}$$

forms a central series with $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}^{k_i}$ for some $k_i \in \mathbb{N}$.

Fix now

$$a_{r,1}, \ldots, a_{r,k_r}, a_{r-1,1}, \ldots, a_{r-1,k_{r-1}}, \ldots, a_{2,1}, \ldots, a_{2,k_2}, a_{1,1}, \ldots, a_{1,k_1}$$

a set of generators for Γ such that for any integer $1 \leq i \leq r$, the classes $\bar{a}_{i,1}, \ldots, \bar{a}_{i,k_i} \in \Gamma_i/\Gamma_{i+1}$ freely generate the free abelian group Γ_i/Γ_{i+1}.

Any $\gamma \in \Gamma$ can be written as a product

$$\gamma = a_{r,1}^{v_{r,1}} \dots a_{r,k_r}^{v_{r,k_r}} a_{r-1,1}^{v_{r-1,1}} \dots a_{r-1,k_{r-1}}^{v_{r-1,k_{r-1}}} \dots a_{2,1}^{v_{2,1}} \dots a_{2,k_2}^{v_{2,k_2}} a_{1,1}^{v_{1,1}} \dots, a_{1,k_1}^{v_{1,k_1}}$$

with

$$\mathbf{v} = (v_{r,1}, \dots, v_{r,k_r}, v_{r-1,1}, \dots, v_{r-1,k_{r-1}}, \dots, v_{2,1}, \dots, v_{2,k_2}, v_{1,1}, \dots, v_{1,k_1})$$

a vector in $\mathbb{Z}^{k_1+\dots+k_r}$. The notation $\gamma(\mathbf{v})$ shows the dependence of γ on $\mathbf{v}$. By [107] it is known that the product in Γ is given by polynomial functions in $\mathbf{v}$, that is, there exists a polynomial $P : \mathbb{Z}^{2(k_1+\dots+k_r)} \to \mathbb{Z}^{k_1+\dots+k_r}$ such that $\gamma(\mathbf{v}_1)\gamma(\mathbf{v}_2) = \gamma(P(\mathbf{v}_1, \mathbf{v}_2))$, for all $\mathbf{v}_1, \mathbf{v}_2 \in Z^{k_1+\dots+k_r}$. The Malcev completion $\Gamma_{\mathbb{R}}$ is defined to be the set of all formal products

$$n(\mathbf{w}) = a_{r,1}^{w_{r,1}} \dots a_{r,k_r}^{w_{r,k_r}} a_{r-1,1}^{w_{r-1,1}} \dots a_{r-1,k_{r-1}}^{w_{r-1,k_{r-1}}} \dots a_{2,1}^{w_{2,1}} \dots a_{2,k_2}^{w_{2,k_2}} a_{1,1}^{w_{1,1}} \dots, a_{1,k_1}^{w_{1,k_1}}$$

with

$$\mathbf{w} = (w_{r,1}, \dots, w_{r,k_r}, w_{r-1,1}, \dots, w_{r-1,k_{r-1}}, \dots, w_{2,1}, \dots, w_{2,k_2}, w_{1,1}, \dots, w_{1,k_1})$$

a vector in $\mathbb{R}^{k_1+\dots+k_r}$. The product in $\Gamma_{\mathbb{R}}$ is given by

$$n(\mathbf{w}_1)n(\mathbf{w}_2) = n(P(\mathbf{w}_1, \mathbf{w}_2)),$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^{k_1+\dots+k_r}$.

A similar completion can be done over $\mathbb{Q}$ and is denoted by $\Gamma_{\mathbb{Q}}$. A group is called *radicable* if any element of it has roots of any order. In particular, $\Gamma_{\mathbb{Q}}$ is the torsion free radicable nilpotent group containing Γ as a subgroup and such that each element in $\Gamma_{\mathbb{Q}}$ has some positive power lying in Γ.

2.1.3.2 Hyperbolic automorphisms

In order to exhibit concrete examples of Anosov diffeomorphisms on nilmanifolds, we restrict the discussion to the class of nilpotent Lie algebras with integer structure constants. Let $\mathfrak{n}$ be a Lie algebra as above. A $\mathbb{Z}$-*subalgebra* in it is the set of all $\mathbb{Z}$-linear combinations of an integer basis of $\mathfrak{n}$.

To find an Anosov diffeomorphism on an nilmanifold N/Γ one finds a hyperbolic automorphism A of $\mathfrak{n}$ for which there exists a basis of $\mathfrak{n}$ in which the matrix of A is hyperbolic, has integer entries and determinant ± 1. Since $\mathfrak{n}$ has integer structure constants, there exists a $\mathbb{Z}$-Lie subalgebra $\mathfrak{n}_1$ (of finite index) in $\mathfrak{n}$ such that $\Gamma = \exp(\mathfrak{n}_1)$ is a lattice in N. If $\{u_1, \dots, u_n\}$ is a basis in $\mathfrak{n}$ with integer structure constants, take $\mathfrak{n}_1 = \mathbb{Z}mu_1 + \dots + \mathbb{Z}mu_n$, where m is chosen so that the denominators in the Campbell-Hausdorff formula divide the products of the constant structures of $\{mu_1, \dots, mu_n\}$. Now there is an integer k such that $A^k(\mathfrak{n}_1) = \mathfrak{n}_1$. Then the lift of A^k via the exponential map is a hyperbolic automorphism of N that induces an Anosov diffeomorphisms on N/Γ.

Note that the existence of a hyperbolic automorphism A of $\mathfrak{n}$ with integer matrix in a basis of $\mathfrak{n}$ is a necessary condition for the existence of a hyperbolic automorphism of N that preserves a lattice Γ. Indeed, let ϕ be a hyperbolic automorphism of N that induces an automorphism of Γ. Then it follows from [118, Theorem 2] that there is a subgroup $\Gamma_1 \subset \Gamma$ of finite index such that $\log(\Gamma_1)$ is a $\mathbb{Z}$-subalgebra of $\mathfrak{n}$. Then there is an integer k such that $\phi^k(\Gamma_1) = \Gamma_1$ and the pull back of ϕ^k via the exponential map is a hyperbolic automorphism of $\mathfrak{n}$ with integer matrix.

In order to simplify the verification that certain maps on nilmanifolds are hyperbolic, we need the following lemma. Note that for N connected simply connected Lie group the lower central series is defined inductively via $c_1(N) = N, c_{i+1}(N) = [c_i(N), N]$. Then $c_i(N)/c_{i+1}(N) \cong \mathbb{R}^{k_i}$ and any $\sigma \in Aut(N)$ induces automorphisms $\sigma_i \in Aut\,(c_i(N)/c_{i+1}(N))$.

Lemma 2.1.5 *Let N be a connected simply connected Lie group. Let $\sigma \in Aut(N)$. Let $d\sigma$ be the derivative of σ. Then the set of eigenvalues of $d\sigma$ is equal to the set of all eigenvalues of the automorphisms σ_i.*

Proof Let the nilpotency degree of N be c. Choose a basis

$$x_{c,1}, x_{c,2}, \dots, x_{c,k_c}, x_{c-1,1}, \dots, x_{1,k_1} \tag{2.1.1}$$

in $\mathfrak{n}$ such that the elements $x_{c,1}, x_{c,2}, \dots, x_{c,k_c}, \dots, x_{i,k_i}$ form a basis of $c_i(\mathfrak{n})$. Define $X_{i,j} = \exp(x_{i,j})$. The set $X_{i,1}c_{i+1}(N), X_{i,2}c_{i+1}(N), \dots, X_{i,k_i}c_{i+1}(N)$ form a basis of the vector space $c_i(N)/c_{i+1}(N)$. The matrix of $d\sigma$ with respect to the basis (2.1.1) is

$$\begin{pmatrix} A_c & * & \dots & * \\ 0 & A_{c-1} & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & * \\ 0 & 0 & \dots & A_1 \end{pmatrix}$$

with $A_i \in GL(k_i, \mathbb{R})$. Note that

$$d\sigma(x_{i,p}) + c_{i+1}(\mathfrak{n}) = \sum_{q=1}^{k_1} (A_i)_{q,p}(x_{i,p}) + c_{i+1}(\mathfrak{n}). \tag{2.1.2}$$

Using now Campbell-Baker-Hausdorff formula one has:

$$\begin{aligned}\sigma(X_{i,p})c_{i+1}(N) &= \sigma(\exp(x_{i,p}))c_{i+1}(N) = \exp(d\sigma(x_{i,p}))c_{i+1}(N) \\ &= x_{i,1}^{(A_i)_{1,p}} x_{i,2}^{(A_i)_{2,p}} \dots x_{i,k_1}^{(A_i)_{k_i,p}} c_{i+1}(N),\end{aligned}$$

so σ_i is also represented by the matrix A_i. □

2.1.3.3 First examples

The first example of Anosov diffeomorphism t hat is not an automorphism of a torus, which we present below, was found by A. Borel, answering a question of Smale [162].

Example 2.1.6 *A familiar example of connected simply connected nilpotent Lie group is the Heisenberg group:*

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} | x, y, z \in \mathbb{R} \right\}.$$

The group H can be identified to its Lie algebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} | x, y, z \in \mathbb{R} \right\},$$

which is generated by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

subject to the relation $[X, Y] = Z$. The Campbell-Baker-Hausdorff formula is

$$X \cdot Y = X + Y + \frac{1}{2}[X, Y].$$

Consider now the direct sum Lie algebra $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{h}$, which is the Lie *algebra of* $N = H \times H$ *and has a basis* $\mathfrak{b}_1$ *consisting of the following* 6×6 *matrices*

$$X_1 = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}$$

subject to the relations $[X_1, X_2] = X_3, [Y_1, Y_2] = Y_3$.

If $\lambda \in \mathbb{R}$, *define an automorphism* ϕ_λ *of* $\mathfrak{n}$ *by*

$$\phi_\lambda(X_i) = \lambda^i X_i$$
$$\phi_\lambda(Y_i) = \lambda^{-i} Y_i$$

for $i = 1, 2, 3$.

If $a \in \mathbb{Z}, a \geq 2$, *then the roots* λ, λ^{-1} *of the polynomial* $x^2 - 2ax + 1$ *are*

$$\lambda = a + (a^2 - 1)^{1/2}, \quad \lambda^{-1} = a - (a^2 - 1)^{1/2},$$

$0 < \lambda^{-1} < 1 < \lambda$ *and* ϕ_λ *is a hyperbolic automorphism.*

Consider now a new basis $\mathfrak{b}_2$ *in* $\mathfrak{n}$ *given by*

$$X_1 + Y_1, (a^2 - 1)^{1/2}(X_1 - Y_1),$$
$$X_2 + Y_2, (a^2 - 1)^{1/2}(X_2 - Y_2),$$
$$X_3 + Y_3, (a^2 - 1)^{1/2}(X_3 - Y_3).$$

The only non-trivial relations for $\mathfrak{b}_2$ *are*

$$[X_1 + Y_1, X_2 + Y_2] = X_3 + Y_3,$$
$$[X_1 + Y_1, (a^2 - 1)^{1/2}(X_2 - Y_2)] = (a^2 - 1)^{1/2}(X_3 - Y_3),$$
$$[(a^2 - 1)^{1/2}(X_1 - Y_1), X_2 + Y_2] = (a^2 - 1)^{1/2}(X_3 - Y_3),$$
$$[(a^2 - 1)^{1/2}(X_1 - Y_1), (a^2 - 1)^{1/2}(X_2 - Y_2) = (a^2 - 1)(X_3 - Y_3),$$

which implies that $\mathfrak{b}_2$ *is a* $\mathbb{Z}$*-basis.*

Observe now that if $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ *is the matrix of a linear transformation in the basis* $\{X_i, Y_i\}$, *then* $\begin{pmatrix} a & a^2 - 1 \\ 0 & a \end{pmatrix} \in SL(2, \mathbb{Z})$ *is the matrix associated to the same linear transformation in the basis* $\{X_i + Y_i, (a^2 -$

$1)^{1/2}(X_i - Y_i)\}$. *Hence the matrix associated to* ϕ_λ *in the basis* $\mathfrak{b}_2$ *is*

$$\begin{pmatrix} a & a^2-1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & a^2-1 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & a^2-1 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \in SL(6,\mathbb{Z}).$$

Thus the automorphism ϕ_λ *induces an Anosov diffeomorphism on* N/Γ, $\Gamma = \exp(C)$ *where* C *is the* $\mathbb{Z}$*-subalgebra generated by* $\mathfrak{b}_2$.

2.1.3.4 More examples

Borel examples were generalized in [92] to a larger class of nilmanifolds. We present this construction here.

Example 2.1.7 *Let* $\mathfrak{n}$ *be a Lie algebra over* $\mathbb{R}$ *that is graded, that is there exist subspaces* $\mathfrak{n}_i$ *of* $\mathfrak{n}$ *such that* $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k$ *and* $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$. *Any graded Lie algebra is nilpotent. Assume that there is a* $\mathbb{Z}$*-basis* $\{X_1, \dots, X_d\}$ *of* $\mathfrak{n}$ *that is compatible with the graduation, i.e., each* $X_i \in \mathfrak{n}_j$ *for some* j. *Then the direct sum Lie algebra* $\mathfrak{n} \oplus \mathfrak{n}$ *has an Anosov automorphism.*

Let $\{Y_i\}_i$ *be a copy of the basis and consider*

$$\mathfrak{b}_1 = \{X_1, \dots, X_d, Y_1, \dots, Y_d\}$$

a basis in $\mathfrak{n} \oplus \mathfrak{n}$. *Then the relations in* $\mathfrak{n} \oplus \mathfrak{n}$ *are*

$$[X_i, X_j] = \sum_{k=1}^{d} p_{ij}^k X_k, \quad [Y_i, Y_j] = \sum_{k=1}^{d} p_{ij}^k Y_k, \quad p_{ij}^k \in \mathbb{Z}. \tag{2.1.3}$$

For any $\lambda \in \mathbb{R}$ *define an automorphism* ϕ_λ *of* $\mathfrak{n} \oplus \mathfrak{n}$ *given by* $\phi_\lambda(X_i) = \lambda^j X_i$ *and* $\phi_\lambda(Y_i) = \lambda^{-j} Y_i$, *where* $X_i \in \mathfrak{n}_j$. *Take* $a \in \mathbb{Z}$ *and* λ *as in the previous example and consider a new basis* $\mathfrak{b}_2$ *for* $\mathfrak{n} \oplus \mathfrak{n}$ *given by*

$$\{X_1 + Y_1, (a^2-1)^{1/2}(X_1 - Y_1), \dots, X_d + Y_d, (a^2-1)^{1/2}(X_d - Y_d)\}.$$

Using (2.1.3) *it follows that* $\mathfrak{b}_2$ *is a* $\mathbb{Z}$*-basis. Moreover, the computations done in the previous example shows that the matrix of* ϕ_λ *with respect to the basis* $\mathfrak{b}_2$ *is hyperbolic and block diagonal, with* 2×2 *blocks on the diagonal from* $SL(2,\mathbb{Z})$.

Note that Smale's example is supported by a 2*-step nilpotent Lie algebra. An example of* k*-step nilpotent Lie algebra supporting Anosov* *automorphisms can be constructed in the following way: let* $\mathfrak{n}$ *be the Lie algebra generated by* $\mathfrak{b} = \{X_1, \ldots, X_{k+1}\}$ *subject to the relations:*

$$[X_1, X_2] = X_3, \; [X_1, X_3] = X_4, \; \ldots, \; [X_1, X_k] = X_{k+1}.$$

It is clear that $\mathfrak{n}$ *is graded by* $\mathfrak{n} = \mathbb{R}X_1 \oplus \ldots \mathbb{R}X_{k+1}$ *and the basis* $\mathfrak{b}$ *is a* $\mathbb{Z}$*-basis. So the construction above gives an Anosov automorphism of* $\mathfrak{n}$*.*

A different class of examples of Anosov automorphisms on nilmanifolds was found by Auslander and Scheuneman in [4], using free k-step nilpotent Lie algebras. This class of examples actually allows for higher rank abelian groups actions on nilmanifolds. We present this class of examples later in Section 2.2.9.

Finding Anosov diffeomorphisms on nilmanifolds is a field of active research. More examples and references can be found in [25] where it is shown that for every $n \geq 17$ there exist a n-dimensional 2-step connected simply connected nilpotent Lie group N which is indecomposable, that is not a direct product of lower dimensional nilpotent Lie groups), and a lattice Γ in N such that N/Γ admits an Anosov diffeomorphism.

2.1.4 Anosov diffeomorphisms on infratori and infranilmanifolds

Slightly more generally, one can introduce infranilmanifolds, which are finitely covered by nilmanifolds. Let N be a connected simply connected Lie group and let $\mathrm{Aff}(N)$ be the group of affine transformations of N. An *almost crystallographic group* is a subgroup E of $\mathrm{Aff}(N)$ such that its subgroup $E \cap N$ of pure translations is a uniform lattice in N, and moreover $E \cap N$ is of finite index in E. The finite quotient group $E/(E \cap N)$ is isomorphic to the image of E under the projection $\mathrm{Aff}(N) \to \mathrm{Aut}(N)$, and so it can be viewed as a subgroup of $\mathrm{Aut}(N) \subset \mathrm{Aff}(N)$. Any almost crystallographic group acts properly discontinuously on N and the orbit space N/E is compact. When E is torsion free the orbit space is a compact manifold. Such a manifold is called *infranilmanifold*.

Note that if $N = \mathbb{R}^n$ then infra-nilmanifolds and almost crystallographic groups become tori, flat Riemannian manifolds, and crystallographic groups. A flat Riemannian manifold is sometimes called *infratorus*.

Example 2.1.8 *We describe an example from [63] of an Anosov auto-*

morphism of an orientable finite factor of the four-dimensional torus $\mathbb{T}^4$ *which is an infratorus, but not a torus.*

One considers a group Γ *of isometries of* $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ *generated by the integral translations* $\mathbb{Z}^4 = \mathbb{Z}^2 \times \mathbb{Z}^2$ *and an element* γ *such that for* $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\gamma(x, y) = (x + v, -y)$, *where* $v = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$. *Note that* $\gamma^2 \in \mathbb{Z}^4$, *and* $\mathbb{Z}^4$ *is a normal subgroup of index 2 in* Γ.

It is easy to see that the group Γ *acts on* $\mathbb{R}^4$ *without fixed points. Hence* $M = \mathbb{R}^4/\Gamma$ *is a flat manifold whose double cover is* $\mathbb{T}^4$. *Note that* M *is orientable since both* $\mathbb{Z}^4$ *and* γ *preserve the orientation of* $\mathbb{R}^4$. *Note also that* M *is not a torus since* Γ *is not abelian. Indeed, if* $\beta(x, y) = (x, y + y')$, *where* $y' \neq (0, 0)$, *then* $\beta \circ \gamma \neq \gamma \circ \beta$.

Let $\mathbf{A}$ *be the direct product of an Anosov automorphism* A *of* $\mathbb{R}^2$ *with itself:*

$$\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : \mathbb{R}^4 \to \mathbb{R}^4, \qquad \textit{where} \qquad A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

To show that the action of $\mathbf{A}$ *on* $\mathbb{R}^4$ *projects to* M *we verify that for any* $(x, y) \in \mathbb{R}^4$, $\mathbf{A}(\Gamma(x, y)) = \Gamma(\mathbf{A}(x, y))$. *Since* $\det \mathbf{A} = 1$, $\mathbf{A}(\mathbb{Z}^4) = \mathbb{Z}^4$. *Thus it suffices to check that* $\mathbf{A}(\gamma(x, y)) \in \mathbb{Z}^4(\gamma(\mathbf{A}(x, y))$ *and hence* $\mathbf{A}(\mathbb{Z}^4(\gamma(x, y))) = \mathbb{Z}^4(\gamma(\mathbf{A}(x, y))$. *This can be seen as follows.*

$$\begin{aligned} &\mathbf{A}(\gamma(x, y)) - \gamma(\mathbf{A}(x, y)) = \mathbf{A}(x + v,\, -y) - \gamma(Ax, Ay) = \\ &(Ax + Av,\, -Ay) - (Ax + v, -Ay) = (Av - v,\, 0) = \\ &\left(\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, 0 \right) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0 \right) \in \mathbb{Z}^2 \times \mathbb{Z}^2. \end{aligned}$$

If E is a torsion free almost crystallographic group, i.e. a Bieberbach group, then an equivalent description for it is given in [94]: E is a discrete group that has a finitely generated torsion free nilpotent normal subgroup Γ such that Γ is of finite index in E and Γ is maximal nilpotent amongst all subgroups of E. One can associate to the pair $\Gamma \subset E$ an extension $1 \to \Gamma \to E \to F \to 1$. The finite group F is called the *holonomy* of E, Γ is called the *translational subgroup* of E, and $\Gamma_{\mathbb{R}}/\Gamma$ is the *covering nilmanifold* of the infranilmanifold $\Gamma_{\mathbb{R}}/E$. There always exists a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 1 \\ & & \big\downarrow j & & \big\downarrow i & & \big\downarrow \phi & & \\ 1 & \longrightarrow & \Gamma_{\mathbb{R}} & \longrightarrow & Aff(\Gamma_{\mathbb{R}}) & \longrightarrow & Aut(\Gamma_{\mathbb{R}}) & \longrightarrow & 1 \end{array}$$

where $j : \Gamma \to \Gamma_{\mathbb{R}}$ is the canonical inclusion of Γ into its Malcev completion, $i : E \to \mathrm{Aff}(\Gamma_{\mathbb{R}})$ realizes E as a genuinely almost crystallographic group, and $\phi : F \to \mathrm{Aut}(\Gamma_{\mathbb{R}})$ is the lift of the homomorphism $\psi : F \to \mathrm{Out}(\Gamma_{\mathbb{R}})$ induced by a normalized section $s : F \to E$.

In [140] Porteous completely characterized flat Riemannian manifolds supporting Anosov diffeomorphisms. Each such manifold M defines an effective action $T : F \to GL(n, \mathbb{Z})$ of the holonomy group on the translational subgroup $\mathbb{Z}^n$ of M. The manifold M supports an Anosov diffeomorphism if and only if each $\mathbb{Q}$-irreducible component of T which is of multiplicity one is irreducible over $\mathbb{R}$.

We are ready to show several explicit examples of Anosov diffeomorphisms on infranilmanifolds. We follow closely [108], where the reader can find more details. The first example is inspired by an example of Shub [160]. It is shown in [108] that the original example of Shub defines only an orbifold rather than an infranilmanifold.

Example 2.1.9 *Consider the group E presented by*

$$E := < a, b, c, d, e, f, \alpha | [c, a] = e^2, [d, a] = f^2, [c, b] = f^2, [d, b] = e^6,$$
$$\alpha a = a^{-1}\alpha, \alpha b = b^{-1}\alpha, \alpha c = c^{-1}\alpha, \alpha d = d^{-1}\alpha$$
$$\alpha e = e\alpha, \alpha f = f\alpha, \alpha^2 = ef >$$

Note that the commutators between the generators that do not appear in the presentation are zero.

E is part of the extension $1 \to \Gamma \to E \to F \to 1$, where Γ is a finitely generated torsion free nilpotent normal subgroup, of finite index in E, maximal in E, and $F = \mathbb{Z}_2$. The element of E with nontrivial image in F is α. A presentation for Γ is given by:

$$\Gamma :< a, b, c, d, e, f | [c, a] = e^2, [d, a] = f^2, [c, b] = f^2, [d, b] = e^6 > .$$

Let $\Gamma_{\mathbb{R}}$ be the Malcev completion of Γ. $\Gamma_{\mathbb{R}}$ is a 6-dimensional connected simply connected Lie group. An integer $\mathbb{Z}$-basis in its Lie algebra is given by the logarithms of a, b, c, d, e, f. So E is an almost-Bieberbach group that defines an infranilmanifold $M = \Gamma_{\mathbb{R}}/E$.

We check that E is torsion free. Using the commutation relations, any element $g \in E$ can be written in the form $g = \alpha^\epsilon e^s f^t a^x b^y c^z d^r$, where s, t, x, y, z, r are integers and $\epsilon \in \{0, 1\}$. If $\epsilon = 0$ then g belongs to the nilpotent subgroup N and cannot be torsion unless is trivial. If $\alpha = 1$, then due to the multiplication formula

$$g^2 = e^{1+2s+2x+6y-2z-6r} f^{1+2s+2x-2r+2y-z}, \tag{2.1.4}$$

g^2 belongs to N and is nontrivial, so again g is not torsion.

The group E embeds into $Aff(\Gamma_{\mathbb{R}}) \cong \Gamma_{\mathbb{R}} \rtimes Aut(\Gamma_{\mathbb{R}})$ by sending $a \to (a,1)$, $b \to (b,1)$, $c \to (c,1)$, $d \to (d,1)$, $e \to (e,1)$, $f \to (f,1)$, $\alpha \to (e^{1/2}f^{1/2}, \phi^\alpha)$, where $\phi^\alpha \in Aut(\Gamma_{\mathbb{R}})$ is given by

$$\phi^\alpha(a) = a^{-1}, \phi^\alpha(b) = b^{-1}, \phi^\alpha(c) = c^{-1},$$
$$\phi^\alpha(d) = d^{-1}, \phi^\alpha(e) = e, \phi^\alpha(f) = f.$$

Consider now the automorphism of Γ, which lift uniquely to an automorphism of $\Gamma_{\mathbb{R}}$:

$$\nu : a \to a^2b^{-1}, b \to a^{-3}b^2, c \to c^7d^4, d \to c^{12}d^7, e \to e^2f, f \to e^3f^2.$$

To check that ν is an automorphisms, observe that ν preserves the relations, and that the restriction of ν to each of the abelian subgroups $< a,b >$, $< c,d >$, $< e,f >$ is an automorphism. Note that ν and ϕ^α commute on $\Gamma_{\mathbb{R}}$. Moreover, it is easy to check that $(1,\nu)$ normalizes E.

In the block diagonal representation required in Lemma 2.1.5 ν is given by

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & 4 & 7 \end{pmatrix}.$$

Since the eigenvalues of A are $2-\sqrt{3}, 2+\sqrt{3}, 7-4\sqrt{3}, 7+4\sqrt{3}$, the first two with multiplicity two, ν defines an Anosov diffeomorphism on M.

Example 2.1.10 *Let E be the group presented by*

$$E := < a,b,c,d,e,f,\alpha | [b,a] = d^2, [c,a] = e^2, [c,b] = f^2,$$
$$\alpha a = a^{-1}\alpha, \alpha b = b^{-1}\alpha, \alpha c = c^{-1}\alpha, \alpha d = d\alpha$$
$$\alpha e = e\alpha, \alpha f = f\alpha, \alpha^2 = d > .$$

Note that the commutators between the generators that do not appear in the presentation are zero.

E is part of the extension $1 \to \Gamma \to E \to F \to 1$, where Γ is a finitely generated torsion free nilpotent normal subgroup, of finite index in E,

maximal in E, and $F = \mathbb{Z}_2$. The element of E with nontrivial image in F is α. A presentation for Γ is given by:

$$\Gamma :< a, b, c, d, e, f | [b, a] = d^2, [c, a] = e^2, [c, b] = f^2 > .$$

Let $\Gamma_{\mathbb{R}}$ be the Malcev completion of Γ. $\Gamma_{\mathbb{R}}$ is a 6-dimensional connected simply connected Lie group. An integer $\mathbb{Z}$-basis in its Lie algebra is given by the logarithms of a, b, c, d, e, f. So E is an almost-Bieberbach group that defines an infranilmanifold $M = \Gamma_{\mathbb{R}}/E$.

The group E embeds into Aff$(\Gamma_{\mathbb{R}}) \cong \Gamma_{\mathbb{R}} \rtimes Aut(\Gamma_{\mathbb{R}})$ by sending $a \to (a, 1)$, $b \to (b, 1)$, $c \to (c, 1)$, $d \to (d, 1)$, $e \to (e, 1)$, $f \to (f, 1)$, $\alpha \to (d^{1/2}, \phi^\alpha)$, where $\phi^\alpha \in Aut(\Gamma_{\mathbb{R}})$ is given by

$$\phi^\alpha(a) = a^{-1}, \phi^\alpha(b) = b^{-1}, \phi^\alpha(c) = c^{-1}, \phi^\alpha(d) = d, \phi^\alpha(e) = e, \phi^\alpha(f) = f.$$

Consider now the automorphism of Γ, which lift uniquely to an automorphism of $\Gamma_{\mathbb{R}}$:

$$\nu : a \to abc^2, b \to ab^2c^2, c \to abc^3, d \to df^{-2}, e \to def^{-1}, f \to ef^2.$$

To check that ν is an automorphisms, observe that ν preserves the relations, and that the restriction of ν to each of the abelian subgroups $< a, b, c >$, $< d, e, f >$ is an automorphism. Note that ν and ϕ^α commute on $\Gamma_{\mathbb{R}}$. Moreover, it is easy to check that $(1, \nu)$ normalizes E.

In the block diagonal representation required in Lemma 2.1.5 ν is given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 3 \end{pmatrix}.$$

Since the eigenvalues of A are nonzero and not of absolute value 1, ν defines an Anosov diffeomorphism on M.

2.2 Actions of $\mathbb{Z}^k$, $k \geq 2$ on tori and nilmanifolds

2.2.1 On algebraic conjugacy

Definition 2.2.1 *Let α and α' be actions of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^m$ and $\mathbb{T}^{m'}$ correspondingly.*

The actions α and α' are algebraically isomorphic *if $m = m'$ and there is a group automorphism $h : \mathbb{T}^m \to \mathbb{T}^m$ such that*

$$\alpha'(g) \circ h = h \circ \alpha(g), \text{ for all } g \in \mathbb{Z}^k.$$

The action α' is called an algebraic factor of α *if there exists a surjective homomorphism $h : \mathbb{T}^m \to \mathbb{T}^{m'}$ such that*

$$\alpha'(g) \circ h = h \circ \alpha(g), \text{ for all } g \in \mathbb{Z}^k.$$

In particular, if h is everywhere finite to one, then α' is called a finite factor *or a* factor with finite fibers *of α.*

The factor action α' is called a rank–one factor of α if $\alpha'(\mathbb{Z}^k)$ has a subgroup of finite index which consists of powers of a single map.

The actions α and α' are weakly algebraically isomorphic *if each one is an algebraic factor of the other. In this case $m = m'$ and each factor map has finite fibers.*

These algebraic notions have natural measure-theoretic counterparts. We will discuss mainly the algebraic setting, but will give from time to time references for the measurable properties.

We show now the relationships between the conjugacy over $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}$ for algebraic actions by automorphisms of a torus. Any $\mathbb{Z}^k$ action α by automorphisms of $\mathbb{T}^m$ generated by $F_{A_1}, \ldots, F_{A_k}$ where $A_1, \ldots, A_k$ are integral matrices of determinant ± 1, defines an embedding $\rho_\alpha : \mathbb{Z}^k \to GL(n, \mathbb{Z})$ by

$$\rho_\alpha^{\mathbf{n}} = A_1^{n_1} \ldots A_k^{n_k},$$

where $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$. Conversely, any embedding $\rho : \mathbb{Z}^d \to GL(n, \mathbb{Z})$ defines an action by automorphisms. Similarly, one can consider actions by endomorphisms induced by general embeddings $\rho : (\mathbb{Z}^+)^d$ into the semi-group of invertible matrices with integer entries.

Two actions α and α' are conjugate via an automorphism (algebraically isomorphic) if and only if the corresponding embeddings ρ_α and $\rho_{\alpha'}$ are conjugated over $\mathbb{Z}$. This implies conjugacy over $\mathbb{Q}$, which is equivalent to conjugacy over $\mathbb{C}$. Note that conjugacy over $\mathbb{C}$ is determined by the eigenvalue structure.

In general, the opposite is not true. The conjugacy over $\mathbb{Z}$ is determined not just by the linear algebra, as is the case for the conjugacy over $\mathbb{Q}$, but also by the algebraic number theory data. Classification, up to conjugacy over $\mathbb{Z}$, of matrices in $SL(n, \mathbb{Z})$ which are irreducible and conjugate over $\mathbb{Q}$, has to do with the *class numbers of the algebraic* *fields*. The simplest case of this situation appears for $n = 2$. In this case the trace determines conjugacy over $\mathbb{Q}$, and in particular the entropy. However, if the class number of the corresponding number field is greater than one, there are matrices with the given trace that are not conjugate over $\mathbb{Z}$. For example, one can consider

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}.$$

This is in contrast with the case of measurable isomorphism, which is completely determined by the entropy. For the case of automorphisms of the torus, concrete metric isomorphisms in the case of equal entropies are constructed using Markov partitions by Adler and Weiss [1]. These are more specific then those produced by the general Ornstein isomorphism theory, and yet not algebraic.

One should note that every action α by automorphisms of a torus has many algebraic factors with finite fibers. These factors are in one-to-one correspondence with lattices $\Gamma \subset \mathbb{R}^k$ which contain the standard lattice $\Gamma_0 = \mathbb{Z}^k$, and which satisfy $\rho_\alpha^{\mathbf{n}}(\Gamma) \subset \Gamma$ for all $\mathbf{n} \in \mathbb{Z}^{\mathbf{k}}$.

We denote the factor action associated to a particular lattice $\Gamma_0 \subset \Gamma$ by α_Γ. In the case of actions by automorphisms the embeddings ρ_α are invertible. Hence $\rho_\alpha^{\mathbf{n}}(\Gamma) = \Gamma$ for all $\mathbf{n} \in \mathbb{Z}^{\mathbf{k}}$.

Let $\Gamma_0 \subset \Gamma$. Take any basis in Γ and let $S \in GL(n, \mathbb{Q})$ be the matrix which maps the standard basis in Γ_0 to this basis. Then obviously the factor-action α_Γ is equal to the action $\alpha_{S\rho_\alpha S^{-1}}$. In particular, ρ_α and ρ_{α_Γ} are conjugate over $\mathbb{Q}$, although not necessarily over $\mathbb{Z}$.

For any positive integer q, the lattice $\frac{1}{q}\Gamma_0$ is invariant under any automorphism in $GL(n, \mathbb{Z})$ and gives rise to a factor which is conjugate to the initial action: one can set $S = \frac{1}{q}Id$ and obtains that $\rho_\alpha = \rho_{\alpha_{\frac{1}{q}\Gamma_0}}$. On the other hand, one can find, for any lattice $\Gamma_0 \subset \Gamma$ a positive integer q such that $\Gamma \subset \frac{1}{q}\Gamma_0$ (take q to be the least common multiple of the denominators of all coordinates for a basis of Γ). Thus $\alpha_{\frac{1}{q}\Gamma_0}$ appears as a factor of α_γ.

We summarize these considerations in the following proposition:

Proposition 2.2.2 *Let α and α' be $\mathbb{Z}^d$-actions by automorphisms of the torus $\mathbb{T}^n$. Then the following are equivalent:*

(i) *ρ_α and $\rho_{\alpha'}$ are conjugate over $\mathbb{Q}$;*
(ii) *there exists an action α'' such that both α and α' are isomorphic to finite algebraic factors of α'';*

(iii) *α and α' are weakly algebraic isomorphic, that is, each of them is isomorphic to a finite algebraic factor of the other.*

2.2.2 The genuine higher rank condition

As we will see later, the absence of rank-one algebraic factors is one of the most general situation in which certain rigidity phenomena appear. The following equivalent characterization of this condition can be found in [164].

Proposition 2.2.3 *For an action α of $\mathbb{Z}^k$ by automorphisms of a torus the following two conditions are equivalent:*

(i) *The action α contains a subgroup isomorphic to $\mathbb{Z}^2$, for which all nontrivial elements are ergodic automorphisms of the torus.*
(ii) *The action α does not possess any non–trivial rank one algebraic factor.*

Proof Let $\Sigma = \alpha(\mathbb{Z}^k) \subset SL(n, \mathbb{Z})$. Every element $g \in \Sigma$ admits a Jordan decomposition $g = g_s \times g_u$ into semisimple and unipotent part, and both g_s, g_u are in $SL(n, \mathbb{Q})$. Moreover, there exists $m \in \mathbb{N}$ such that $g^m = g_s^m \times g_u^m$ is a decomposition inside $SL(n, \mathbb{Z})$. So there exists $\Sigma' \subset \Sigma$ a subgroup of index finite such that any element in Σ' has a decomposition inside $SL(n, \mathbb{Z})$. We denote by Σ_s the semisimple part of Σ, and by Σ_u the unipotent part. Note that $\Sigma = \Sigma_s \times \Sigma_u$ and $\Sigma_s \cap \Sigma_u$ is finite.

If $g = g_s \times g_u$ is a Jordan decomposition and $g, g_s, g_u \in SL(n, \mathbb{Z})$, then it follows from Proposition 2.1.2 that g induces an ergodic automorphism of a torus if and only if g_s does. Consequently Σ contains a subgroup isomorphic to $\mathbb{Z}^2$ consisting of ergodic automorphisms if and only if Σ_s does.

We show now that if Σ_s admits a rank one factor then Σ does. This will reduce the proof of the proposition to the case when Σ is semisimple.

Let $\mathbb{T} \subset \mathbb{T}^n$ be a Σ_s-invariant subtorus such that $\Sigma_s|_{\mathbb{T}^n/\mathbb{T}}$ is a finite extension of $\mathbb{Z}$. Then Σ_s contains a subgroup H such that Σ_s/H is a finite extension of $\mathbb{Z}$ and H acts trivially on $\mathbb{T}^n/\mathbb{T}$. Let $V = \mathrm{Fix}(H)$. It follows from the fact that H is semisimple that V has an H-invariant complement V'. Moreover both V and V' are defined over $\mathbb{Q}$ and both are invariant under the centralizer of H in $SL(n, \mathbb{R})$. This implies that both V and V' are Σ and Σ_s invariant. If $\mathbb{T}' = V'/\mathbb{Z}^n \cap V'$, then the action of Σ_s on $\mathbb{T}^n/\mathbb{T}'$is of rank one. Note now that the unipotent component of $\Sigma|_{\mathbb{T}^n/\mathbb{T}}$ has an invariant proper subspace W defined over $\mathbb{Q}$ such that the induced action on $\mathbb{R}^n/W$ is trivial, and conclude that Σ has a rank one factor.

If Σ has an ergodic element, one can show also that if Σ admits a rank one factor then Σ_s does. Indeed, let $\mathbb{T} \subset \mathbb{T}^n$, $\Sigma(\mathbb{T}) = \mathbb{T}$ and $\Sigma|_{\mathbb{T}^n/\mathbb{T}}$ is a finite extension of $\mathbb{Z}$. Let H be the stabilizer of $\mathbb{T}^n/\mathbb{T}$ in Σ. Then Σ/H is a finite extension of $\mathbb{Z}$. Let T' be the maximal H_s-fixed subtorus. Clearly T' is non-trivial, and has an H_s-invariant complement T''. Then T'' is Σ_s-invariant and $\Sigma_s|_{\mathbb{T}^n/T''}$ contains an ergodic element and is included in a finite extension of $\mathbb{Z}$, so it has to be a finite extension of $\mathbb{Z}$.

Note now that if Σ is semisimple, as we will assume from now on, then the implication (1) $\Rightarrow$ (2) is obvious, so we need to show only (2) $\Rightarrow$ (1).

Observe that in any subgroup of $SL(n, \mathbb{Z})$ isomorphic to $\mathbb{Z}^k$ which has no proper invariant tori in $\mathbb{T}^n$, every (nontrivial) element is ergodic. Let $A \subset \Sigma$ ergodic, and let $\mathbb{T}^n = T_1 \times \cdots \times T_s$ be an almost(i.e. up to a finite extension) direct product of Σ-invariant irreducible subtori. From the observation above, for each i there exists $B_i \in \Sigma$ such that the restrictions of A and B_i to T_i generates a $\mathbb{Z}^2$-action with every (nontrivial) element ergodic. We want to find $B \in \Sigma$ such that A and B generate a $\mathbb{Z}^2$-action with every (nontrivial) element ergodic.

We will use the following fact that can be easily proved by induction:

Let $A, B, C \in SL(n, \mathbb{Z})$ semisimple commuting elements such that A, B generate a $\mathbb{Z}^2$ action on $\mathbb{T}^n$ with every (nontrivial) element ergodic. Then there exist at most a finite number of relatively prime triples (k, l, m) such that $A^k B^l C^m$ is not ergodic.

Suppose now that $A, B, C \in \Sigma$ such that $\mathbb{T}^n = T \times T'$, T, T' Σ-invariant, A, B generate a $\mathbb{Z}^2$ action on T, A, C generate a $\mathbb{Z}^2$ action on T', and both actions have each nontrivial element ergodic. Then the previous fact implies that there are only finitely many rational directions (k, l, m) in $\mathbb{R}^3$ with non-ergodic element $A^K B^l C^m$. So we can find a rational plane in $\mathbb{R}^3$ which consists of ergodic triples only, and the theorem follows now by induction. □

Either one of the conditions in Proposition 2.2.3 describes the most general "genuine higher rank" situation. Accordingly we will call such actions *genuinely higher rank*.

2.2.3 Rigidity of genuinely higher rank actions

Genuinely higher rank actions by automorphisms of a torus possess a number of strong rigidity properties.

Local differentiable rigidity: Any smooth action whose generators are sufficiently close to those of a genuinely higher rank action α is differentiably conjugate to α [81, 20, 22].

Isomorphism rigidity: Any action by automorphisms of a torus measurably isomorphic to a genuinely higher rank action by automorphisms is algebraically isomorphic to it [60, 70].

Measure rigidity (Anosov case): The only ergodic invariant measures for α such that some element has positive entropy are the Lebesgue measures on closed invariant subgroups [31].

Rigidity of measurable centralizer: The centralizer of a genuinely higher rank action by automorphisms of a torus in the group of Lebesgue measure preserving transformations consists of affine transformations [60, 70].

Now we will consider various interesting classes and examples of genuinely higher rank actions by automorphisms of a torus. We begin with some preliminaries from algebraic number theory.

2.2.4 Irreducible actions and units in number fields

An important class of genuinely higher rank abelian actions are those irreducible over $\mathbb{Q}$.

Definition 2.2.4 *A genuine higher rank $\mathbb{Z}^k$-action α on $\mathbb{T}^n$ is called irreducible if any nontrivial algebraic factor of α has finite fibers.*

From Proposition 2.1.2 follows that:

Proposition 2.2.5 *Any irreducible action α over $\mathbb{Q}$ by automorphisms of a torus has all nontrivial elements acting ergodically.*

The following equivalent conditions for irreducibility can be found in [7].

Proposition 2.2.6 *Let α be a $\mathbb{Z}^k$-action on $\mathbb{T}^n$ by automorphisms of a torus. The following conditions are equivalent:*

(i) *α is irreducible;*

(ii) *the image of* α *in* $GL(n,\mathbb{Z})$ *contains a matrix with characteristic polynomial irreducible over* $\mathbb{Q}$*;*

(iii) α *does not have a nontrivial invariant rational subspace or, any* α*–invariant closed subgroup of* $\mathbb{T}^n$ *is finite.*

There are close connections between irreducible actions on $\mathbb{T}^n$ and groups of units in number fields of degree n. In fact, algebraic number theory provides several important technical tools for the study of $\mathbb{Z}^k$ actions by automorphisms of a torus. We review below Section 3.3 in [70], which contains a discussion of these issues.

Let $A \in GL(n,\mathbb{Z})$ be a matrix with an irreducible characteristic polynomial f, hence with distinct eigenvalues. The centralizer of A in the group of matrices $M(n,\mathbb{Q})$ can be identified with the ring of all polynomials in A with rational coefficients modulo the principal ideal generated by the polynomial $f(A)$, and hence with the field $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of A. The identification is given by the map

$$\mathcal{G} : p(A) \to p(\lambda) \tag{2.2.1}$$

with $p \in \mathbb{Q}[x]$. Notice that if $B = p(A)$ is an integer matrix then $\mathcal{G}(B)$ is an algebraic integer, and if $B \in GL(n,\mathbb{Z})$ then $\mathcal{G}(B)$ is an algebraic unit. The converse is not necessarily true.

Lemma 2.2.7 *The map* $\mathcal{G}$ *in* (2.2.1) *is injective.*

Proof If $\mathcal{G}(p(A)) = 1$ for $p(A) \neq Id$ then $p(A)$ has 1 an an eigenvalue and hence has a rational subspace consisting of all invariant vectors. This subspace must be invariant under A which contradicts its irreducibility. □

Denote by $\mathcal{O}_K$ the ring of integers of K, by $\mathcal{U}_K$ the group of units of $\mathcal{O}_k$, by $C(A)$ the centralizer of A in $M(n,\mathbb{Z})$ and by $Z(A)$ the centralizer of A in the group $GL(n,\mathbb{Z})$.

Lemma 2.2.8 $\mathcal{G}(C(A))$ *is a ring in* K *such that* $\mathbb{Z}[\lambda] \subset \mathcal{G}(C(A)) \subset \mathcal{O}_K$*, and* $\mathcal{G}(Z(A)) = \mathcal{U}_K \cap \mathcal{G}(C(A))$*.*

Proof $\mathcal{G}(C(A))$ is a ring because $C(A)$ is a ring. As observed above images of integer matrices are algebraic integers and images of matrices with determinants ± 1 are algebraic units. Hence $\mathcal{G}(C(A)) \subset \mathcal{O}_K$. Finally, for every polynomial p with integer coefficients, $p(A)$ is an integer matrix, hence $\mathbb{Z}[\lambda] \subset \mathcal{G}(C(A))$. □

Notice that $\mathbb{Z}[\lambda]$ is a finite index subring of $\mathcal{O}_K$. Hence $\mathcal{G}(C(A))$ has the same property.

Remark 2.2.9 *The groups of units in two different rings, say $\mathcal{O}_1 \subset \mathcal{O}_2$, may coincide. Examples can be found in the table of totally real cubic fields [16].*

Proposition 2.2.10 *$Z(A)$ is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times F$ where r_1 is the number of real embeddings, r_2 is the number of pairs of complex conjugate embeddings of the field K into $\mathcal{C}$ and is a finite cyclic group.*

Proof By Lemma 2.2.8 $Z(A)$ is isomorphic to the group of units in the ring $\mathcal{G}(C(A))$, so the statement follows from the classical Dirichlet Unit Theorem ([8],Ch. 2, §4.3). □

Note that since $r_1 + 2r_2 = n$, Proposition 2.2.10 gives an upper bound on the rank of an irreducible $\mathbb{Z}^k$ action on $\mathbb{T}^n$.

2.2.5 Cartan actions

Of particular interest are abelian groups of ergodic automorphisms of $\mathbb{T}^n$ of maximal possible rank $n-1$ in agreement with the real rank of the Lie group $SL(n,\mathbb{R})$. These are a particular class of irreducible actions.

Definition 2.2.11 *An action of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$ for $n \geq 3$ by ergodic automorphisms is called a* Cartan action.

The following facts proved in [70, Proposition 4.1] are, essentially, consequences of the Dirichlet Unit Theorem.

Proposition 2.2.12 *Let α be a Cartan action on $\mathbb{T}^n$. Then:*

(i) *Any element of the action other than identity has real eigenvalues and is hyperbolic and thus Bernoulli.*
(ii) *α is irreducible.*
(iii) *The centralizer of α is a finite extension of α.*

Lemma 2.2.13 *Let A be a hyperbolic matrix in $SL(n,\mathbb{Z})$ with irreducible characteristic polynomial and distinct real eigenvalues. Then every element of the centralizer $Z(A)$ other than $\{\pm\mathrm{Id}\}$ is hyperbolic.*

Proof Assume that $B \in Z(A)$ is not hyperbolic. As B is simultaneously diagonalizable with A and has real eigenvalues, it has an eigenvalue $+1$ or -1. The corresponding eigenspace is rational and A–invariant. Since A is irreducible, this eigenspace has to coincide with the whole space and hence $B = \pm\mathrm{Id}$. □

Corollary 2.2.14 *Cartan actions are exactly the maximal rank irreducible actions corresponding to totally real number fields. The centralizer $Z(\alpha)$ for a Cartan action α is isomorphic to $\mathbb{Z}^{n-1} \times \{\pm\mathrm{Id}\}$. Lyapunov exponents for a Cartan action are simple and Lyapunov hyperplanes are in general position and are* completely irrational, *i.e. none of them contains an integer point.*

In addition to rigidity properties enumerated in Section 2.2.3 Cartan actions are *globally rigid*: Any Anosov action homotopic to a Cartan action is differentiable conjugate to it [151]. There is also strong isomorphism rigidity for actions with *Cartan homotopy data*, i.e. actions whose elements are homotopic to those of a Cartan action [74].

The following example demonstrates highly non-trivial consequences of isomorphism rigidity in the case of Cartan actions:

Example 2.2.15 *[70, Section 6.3] Consider two Cartan actions of $\mathbb{Z}^2$ on $\mathbb{T}^3$ generated by automorphisms F_A, F_B and $F_{A'}$, $F_{B'}$ correspondingly, where*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 8 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 8 & 4 \end{pmatrix},$$

and

$$A' = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ -5 & 9 & 2 \end{pmatrix} \quad B' = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ -5 & 9 & 5 \end{pmatrix}.$$

These two actions are isomorphic over $\mathbb{Q}$ and hence are algebraic factors of each other with finite fibers. They are however not isomorphic over $\mathbb{Z}$; this remains true even if one is allowed to change generators within *each group. By the isomorphism rigidity property (see Section 2.2.3), the actions are not measurably isomorphic. However, every element of the actions is Bernoulli, and hence has a huge measurable centralizer. This gives a remarkable example of rigid construction from soft elements.*

2.2.6 Symplectic actions on $\mathbb{T}^4$

The $\mathbb{R}$-rank (see Section 2.3.3.2) of the group $Sp(4, \mathbb{R})$ of invertible symplectic 4×4 matrices is two. Accordingly, any maximal split Cartan subgroup of $Sp(4, \mathbb{R})$ may intersect the integer lattice $Sp(4, \mathbb{Z})$ by a group of rank at most two. In fact, such an intersection may have rank two and be irreducible over $\mathbb{Q}$. Using it as the linear part one obtains an irreducible $\mathbb{Z}^2$ Anosov action on $\mathbb{T}^4$ by symplectic automorphisms. Let F_A and F_B be the generators of such an action. Each of the matrices A and B has two pairs of mutually inverse real eigenvalues. Hence, the four Lyapunov exponents of the action split into two pairs with exponents in each pair differing by sign. Thus there are only two Lyapunov hyperplanes (lines in this case). Geometrically the picture of exponents and Weyl chambers is the same as for the product action generated by $C \times \mathrm{Id}$ and $\mathrm{Id} \times D$ where $C,\ D \in SL(2, \mathbb{Z})$ are hyperbolic matrices.

The difference between the product case and the irreducible case is in that the latter contains a subgroup isomorphic to $\mathbb{R}^2$ consisting of ergodic elements, while the former does not. Alternatively, one can explain this as follows. The Lyapunov lines in the irreducible case are irrational and in the product case they are simply coordinate axes. If one consider the suspension of the action in the irreducible case *every* one-parameter subgroup of $\mathbb{R}^2$ acts ergodically including those represented by the Lyapunov line. Each of those subgroups acts by isometries along one of the invariant one-dimensional Lyapunov foliations thus providing an essential geometric ingredient for rigidity properties.

Notice that since A is irreducible with real eigenvalues its centralizer has rank three by Proposition 2.2.10. Thus the $\mathbb{Z}^2$ symplectic action is embedded into an Anosov action of $\mathbb{Z}^3$. A third generator of this action may be chosen that is not a symplectic matrix.

Example 2.2.16 *We show now an explicit example of $\mathbb{Z}^2$ symplectic action on $\mathbb{T}^4$. Let*

$$A = \begin{pmatrix} 2 & 2 & 2 & -1 \\ -1 & -2 & -1 & 1 \\ 0 & 5 & 1 & -3 \\ -7 & -4 & -6 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -41 & 100 & 40 & -25 \\ 25 & -61 & -25 & 15 \\ 40 & -95 & -36 & 25 \\ 55 & -130 & -50 & 34 \end{pmatrix}.$$

Both A and B are symplectic and hyperbolic. The eigenvalues of A are (approximately) $\lambda_1 = 5.2828, \lambda_2 = -3.1552, \lambda_3 = -0.3169, \lambda_4 = 0.1893$ *and the eigenvalues of B are* $\lambda_1 = -107.8924, \lambda_2 = 3.6259, \lambda_3 = 0.2758, \lambda_4 = -0.0093$*. It is clear that A and B are not powers of the same element in $Sp(4, \mathbb{R})$. The characteristic polynomial of A is* $P_A = x^4 - 2x^3 - 17x^2 - 2x + 1$ *and the characteristic polynomial of B is* $P_B = x^4 + 104x^3 - 419x^2 + 104x + 1$*. Both P_A and P_B are irreducible over $\mathbb{Q}$ and reciprocal. One can easily check that $AB = BA$. Thus F_A, F_B determine a genuine $\mathbb{Z}^2$-action on $\mathbb{T}^4$. The $\mathbb{Z}^2$ action can be embedded into a $\mathbb{Z}^3$ action. A possible choice for the extra generator is*

$$C = \begin{pmatrix} -6 & 13 & 4 & -4 \\ 1 & -5 & -4 & 0 \\ 16 & -23 & 2 & 13 \\ 19 & -28 & 1 & 15 \end{pmatrix}. \tag{2.2.2}$$

The eigenvalues of C are $\lambda_1 = 18.5110$, $\lambda_2 = -12.6222$, $\lambda_3 = 0.1414$, $\lambda_4 = -0.0303$*. The characteristic polynomial of C is* $P_C = x^4 - 6x^3 - 233x^2 + 26x + 1$*. P_C is irreducible over $\mathbb{Q}$. The matrix C is hyperbolic, but not symplectic because its characteristic polynomial is not reciprocal. So C does not belong to the abelian group generated by A and B. Nevertheless, C commutes to A and B.*

2.2.7 Genuinely partially hyperbolic actions

We proceed to construct examples of genuinely higher rank partially hyperbolic actions on tori that are irreducible and not hyperbolic.

The first dimension in which there are partially hyperbolic ergodic automorphisms of a tori that are not hyperbolic is 4. If $N = 2$, since the purely hyperbolic case is excluded, the two commuting matrices have both eigenvalues real and of absolute value 1. Then they are roots of unity and the matrices cannot induce ergodic automorphisms. If $N = 3$, the matrix has two complex conjugate eigenvalues, and an additional one that is real. The real one has to be of 1. This is excluded by the ergodicity assumption.

The simplest reducible examples for $N = 4$ are given by products. For irreducible examples, one needs to find an irreducible matrix in $\mathrm{SL}(n, \mathbb{Z})$ that has two complex conjugate eigenvalues of absolute value one, and two real eigenvalues λ and λ^{-1}, $0 < |\lambda| < 1$. To produce such a matrix start with an irreducible quadratic polynomial that has a root larger than two and one less than two in absolute value. Then the substitution $x \to x + \frac{1}{x}$ gives a quadric polynomial that is the characteristic polynomial of a matrix with the desired properties.

Example 2.2.17 *For example, start with the quadratic polynomial* $P = x^2 - 3x + 1$, *which after substitution becomes* $Q = x^4 - 3x^3 + 3x^2 - 3x + 1$, *which is irreducible over* $\mathbb{Q}$. *Then the companion matrix of* Q*:*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -3 & 3 \end{pmatrix}$$

has the desired properties. The eigenvalues of A *are:* $\lambda_1 = .4643, \lambda_2 = 2.1532, \lambda_3 = 0.1909 - 0.9816i, \lambda_4 = 0.1909 + 0.9816i$.

There is no reducible or irreducible matrix with the desired properties if $N = 5$. Since the matrix is not hyperbolic, there are two cases to consider: either there are two complex conjugate eigenvalues of absolute value one and three real eigenvalues, or there are two pairs of complex conjugate eigenvalues, exactly one of these of absolute value one, and one remaining real eigenvalue.

In the first case, let the characteristic polynomial of the matrix be:

$$\begin{aligned} f(x) &= (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \mu)(x - \bar{\mu}) \\ &= x^5 + \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x - 1, \end{aligned}$$

with $\lambda_1, \lambda_2, \lambda_3$ real, and $\mu, \bar{\mu}$ complex conjugate with $|\mu| = 1$. Then

$$f(x) = (x^3 - ax^2 + bx - 1)(x^2 + cx + 1)$$

where $a = \lambda_1 + \lambda_2 + \lambda_3, b = \lambda_1\lambda_2\lambda_3$ and $c = 2\mathrm{Re}\mu$. The coefficients of f are $\alpha = c - a, \beta = b + 1 - ac, \gamma = -a + bc - 1, \delta = b - c$. Since all are rational numbers, $\alpha + \delta = b - a$ and $\beta + \gamma = (b - a)(1 + c)$ are rational. If $a \neq b$ this implies c rational, which is in contradiction to ergodicity. If $a = b$, then $f(x)$ has $x - 1$ as a factor, so again we have a contradiction to ergodicity.

In the second case, let the characteristic polynomial of the matrix be:

$$\begin{aligned} f(x) &= (x-\lambda)(x-\nu)(x-\bar{\nu})(x-\mu)(x-\bar{\mu}) \\ &= (x-\lambda)(x^2-ax+1)(x^2-bx+\rho^2) \\ &= x^5+\alpha x^4+\beta x^3+\gamma x^2+\delta x-1, \end{aligned}$$

where $|\mu| = 1, |\nu| = \rho \neq 1, a = 2\text{Re}\mu, b = 2\text{Re}\nu = 2\rho\cos\theta$. If we denote $A = \lambda + b$ and $B = \lambda b + \rho^2$, then the coefficients of f are $\alpha = -A-a, \beta = Aa+1+B, \gamma = -A-1-aB, \delta = B+a$. Since all are rational, $\alpha+\delta = B-A$ and $\beta+\delta = (B-A)(1-a)$ are rational. So either $B = A$, or a is rational. If $A = B$, then $\lambda\rho^2 = 1$ implies $|\rho| = 1$ or $1+\rho^2-2\rho\cos\theta = 0$. The first situation implies $\lambda = 1$, which contradicts ergodicity. The second situation is equivalent to $\sin^2\theta+(\cos\theta-\rho)^2 = 0$, which implies $|\rho| = 1$, contradicting ergodicity. If a is rational, then μ has to be a root of unity, giving again a contradiction to ergodicity.

More general, one can show that there are no irreducible partially hyperbolic and not hyperbolic automorphisms of a torus in any odd dimensions.

Let N be an odd integer, and let $f(x)$ be the irreducible characteristic polynomial of degree N of an integer matrix A. Let λ be an eigenvalue of A of absolute value 1, and let $\mu = \lambda + \frac{1}{\lambda}$. The for the number fields $L = \mathbb{Q}(\lambda)$ and $K = \mathbb{Q}(\mu)$ one has:

$$|L:K| \cdot |K:\mathbb{Q}| = N.$$

Observe now that the field K is real, since $\mu = \lambda + \frac{1}{\lambda} = \lambda + \bar{\lambda} \in \mathbb{R}$, so $|L:K > 2$. Since $\lambda^2 - \lambda\mu + 1 = 0$, we also have $|L:K| < 2$. This implies $|L:K| = 2$, so N is even.

We start investigating higher rank abelian partially hyperbolic automorphisms.

Definition 2.2.18 *A genuinely higher rank action of $\mathbb{Z}^k$ is called* genuinely partially hyperbolic *if it has at least one zero Lyapunov exponent. In fact, multiplicity of the zero exponent for such an action is always even because the eigenvalues corresponding to the exponent are complex and hence come in conjugate pairs.*

The following existence results appear in [22, Theorem 3].

Theorem 2.2.19 *Irreducible genuinely partially hyperbolic actions by automorphisms of a torus exist in any even dimension starting from six and not in any other dimension.*

Reducible genuinely partially hyperbolic actions exist in any odd dimension starting from nine.

No genuinely partially hyperbolic actions exist in dimensions strictly less than up to six.

Proof One shows first that there are no genuinely partially hyperbolic $\mathbb{Z}^2$ actions on a torus $\mathbb{T}^N$ if $N < 6$, or irreducible genuinely partially hyperbolic actions if N is odd. The case of an odd dimension and the cases $N = 2, 3, 5$ are discussed above. The only case remaining is $N = 4$.

If $N = 4$ it is possible to find a partially hyperbolic matrix which is genuinely partially hyperbolic and acts ergodically on $\mathbb{T}^4$. See Example 2.2.17. Nevertheless, it is impossible to find a $\mathbb{Z}^2$-action genuinely partially hyperbolic. Indeed, assume that such an action exists and let A, B be its generators. Then A and B have a common neutral space. Otherwise a product of a power of A and a power of B is hyperbolic. Since the roots are either real or complex conjugate, the common neutral space has to be two dimensional, corresponding to a pair of complex conjugate roots. Without loss, one can assume that A has real roots $0 < \alpha < 1 < \alpha^{-1}$, and B has real roots $0 < \beta < 1 < \beta^{-1}$. Both α and β, as well as the complex conjugate roots of both A and B, have to be irrational. By choosing k, l large enough, one can assume that the eigenvalues of $A^l B^k$ are arbitrarily close to identity. So the matrix $A^l B^k$ is arbitrarily close to identity. But this is impossible because $A^l B^k$ is an integer matrix not equal to identity.

We continue by showing the existence of irreducible genuinely partially hyperbolic $\mathbb{Z}^2$ actions in any even dimension $N \geq 6$. We use here Dirichlet units theorem.

Let Q be an irreducible integer polynomial of degree n with all roots real and leading coefficient one. Assume that Q has $s, s \geq 2$, roots greater than two and at least one root less that two in absolute value. Let $R(x) = Q(x+\frac{1}{x})x^n$ be the reciprocal polynomial of degree $2n$ determined by Q. Let ψ be a root of R and $\theta = \psi + \frac{1}{\psi}$ a root of Q. Let $L = \mathbb{Q}(\theta)$ and $K = \mathbb{Q}(\psi)$ be the corresponding number fields. Let $\sigma_i, i = 1, \dots, 2n$, be the embeddings of K into $\mathbf{C}$. Since all roots of R come in pairs $\{\lambda, \frac{1}{\lambda}\}$ we can assume that $\sigma_i(\psi)\sigma_{i+n}(\psi) = 1, i = 1, \dots, n$. Let $\alpha = P(\psi) \in K$, where P is a polynomial of degree less than n, an arbitrary element in K. Then

$$\sigma_i(P(\psi^{-1})) = P(\sigma_i(\psi^{-1})) = P(\sigma_i(\psi)^{-1}) = P(\sigma_{n+1}(\psi)) = \sigma_{n+1}(P(\psi)).$$

So $P(\psi)$ and $P(\psi^{-1})$ have the same norm:

$$N(P(\psi)) = N(P(\psi^{-1})) = \sigma_1(\psi)\ldots\sigma_{2n}(\psi).$$

So $P(\psi)$ is a unit if and only if $P(\psi^{-1})$ is a unit.

Let U_L, U_K be the group of units of L, K respectively. Define a homomorphism $f : U_K \to U_L$ by $f(P(\psi)) = P(\psi)P(\psi^{-1})$. We show that $f(U_K) \subset U_L$. For any integer polynomial $P(x)$ one has $P(x)P(\frac{1}{x}) = S(x+\frac{1}{x})$, where S is a rational polynomial. This implies $f(P(\psi)) = S(\theta)$, so $f(U_K) \subset L$. Moreover, $P(\psi)$ unit implies $P(\psi^{-1})$ unit, so $S(\theta)$ is a unit in K. Since $\theta = \psi + \frac{1}{\psi}$ we have $|K : L| \le 2$. Since $|K : K \cup \mathbb{R}| = 2$ and due to the fact that K is real, one has $K = K \cup \mathbb{R}$. But this implies that any unit in U_K must lie in U_L, so the image of f is in U_L.

Let A be an integer matrix in $\mathrm{SL}(2n, \mathbb{R})$ with characteristic polynomial R. For example, A can be the companion matrix of R. In order to obtain matrices that commute to A it is enough to show that the kernel of f contains at least two independent units. Indeed, let $v = P(\psi)$ be a unit such that $B = P(A)$ commutes to A, and A and B are independent under multiplication. If v is in the kernel of f then $\sigma_i(f(v)) = 1$ for all i, that is:

$$\begin{aligned} 1 &= \sigma_i(P(\psi)P(\psi^{-1})) = \sigma_i(P(\psi))\sigma_i(P(\psi)^{-1}) \\ &= P(\sigma_i(\psi))P(\sigma_i(\psi^{-1})) = P(\psi_i)P(\psi_i^{-1}). \end{aligned}$$

If ψ_i is a root of R of absolute value 1, then $\psi_i^{-1} = \bar{\psi}_i$, therefore

$$1 = P(\psi_i)P(\bar{\psi}_i) = P(\psi_i)\overline{P(\psi_i)} = |P(\psi_i)|.$$

Thus B has eigenvalues of absolute value one in the same directions as A, and the $\mathbb{Z}^2$-action generated by A and B is genuinely partially hyperbolic.

It remains to show that the kernel of f contains at least two independent units. We show that the kernel contains s independent units. The polynomial R has $2s$ real roots, and $n - s$ pairs of complex conjugate roots of absolute value 1. The structure of the groups U_L, U_K is given by Dirichlet units theorem. Every unit in U_L can be written as $\rho u_1^{k_1} u_2^{k_2} \ldots u_{s+n-1}^{k_{s+n-1}}$, where u_i are independent units. The images $f(u_i)$ are units in U_K. By Dirichlet units theorem, U_K can have at most $n-1$ independent units. Therefore in $\{f(u_i)\}_i$ there are at most $n-1$ independent units. Without loss, assume that these are $f(u_i), i = 1, \ldots, n-1$. This implies the existence of at least s relations of type:

$$1 = f(u_1)^{l_{1,k}} f(u_2)^{l_{2,k}} \ldots f(u_{n-1})^{l_{n-1,k}} f(u_k)^{l_k},$$

where $k = n, n+1, \ldots, n+s$. These relations implies the existence of s units in the kernel of f:

$$v_k = u_1^{l_{1,k}} u_2^{l_{2,k}} \ldots u_{n-1}^{l_{n-1,k}} u_k^{l_k}.$$

Moreover, u_i independent implies that v_k are independent.

Finally, the existence of reducible genuinely partially hyperbolic $\mathbb{Z}^2$-actions on tori of any odd dimension greater that 9 can be shown by taking products of partially hyperbolic $\mathbb{Z}^2$-actions on tori of even dimension with hyperbolic $\mathbb{Z}^2$-actions on tori of dimension 3. Examples of the latest are shown in Examples 2.2.16 and 2.2.15. □

Example 2.2.20 *The following example of irreducible genuine partially hyperbolic $\mathbb{Z}^2$ action on $\mathbb{T}^6$ appears in [22, Section 6.2]. It was found by S. Katok using the program PARI [134].*

We start with the irreducible degree three polynomial $Q = x^3 - 2x^2 - 8x + 1$, which has two real roots of absolute value larger than two, and a real root of absolute value less than two. Its reciprocal polynomial is

$$f(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1,$$

which is irreducible over $\mathbb{Q}$, has four real roots and a pair of complex conjugate roots of absolute value one. A system of fundamental units for $\mathbb{Q}(f)$ is given by $u_1 = x, u_2 = x+1, u_3 = x^4 - 2x^3 - 6x^2 - x + 1, u_4 = 2x^5 - 6x^4 - 3x^3 - 6x^2 - 6x$.

The $\mathbb{Z}^2$ partially hyperbolic action is generated by the companion matrix A of f and by $B = 2A^5 - 6A^4 - 3A^3 - 6A^2 - 6A$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}.$$

The eigenvalues of A are $3.6863, -1.3236, 0.06076+0.9981i, 0.06076-0.9981i, -0.7555, 0.2712$. *The eigenvalues of B are* -22.1542, -2.1586, $0.9105+0.4133i$, $0.9105-0.4133i$, $-0.4632, -0.04513$. *Both A and B have four real eigenvalues and a pair of complex conjugate eigenvalues of absolute value one. The matrices A and B commute and have a common two dimensional neutral space.*

Example 2.2.21 *Here is an example of irreducible genuinely partially hyperbolic action of* $\mathbb{Z}^2$ *on* $\mathbb{T}^8$. *We start with the irreducible degree four polynomial* $Q = x^4-x^3-7x^2-x+1$. *Q has two real roots of absolute value greater than two, and two real roots of absolute value less that* 2. *The reciprocal polynomial of Q is* $R = x^8-x^7-4x^6-7x^4-4x^3-3x^2-x+1$, *which has four real roots and two pairs of complex conjugate roots of absolute value* 1. *A set of fundamental units for R, obtained using the program PARI [134], is given by:* $u_1 = x, u_2 = x^7-x^6-4x^5-3x^4-3x^3-x^2, u_3 = x^7-2x^6-x^5-3x^4-4x^3-3x+1, u_4 = 8x^7-17x^6-3x^5-32x^4-22x^3-15x^2-15x+4$. *The generators of the* $\mathbb{Z}^2$ *action are the companion matrix A of R and* $B = 8A^7-17A^6-3A^5-32A^4-22A^3-15A^2-15A+4$*:*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 3 & 4 & 7 & 4 & 3 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & -15 & -15 & -22 & -32 & -3 & -17 & 8 \\ -8 & 12 & 9 & 17 & 34 & 0 & 21 & -9 \\ 9 & -17 & -15 & -27 & -46 & -2 & -27 & 12 \\ -12 & 21 & 19 & 33 & 57 & 2 & 34 & -15 \\ 15 & -27 & -24 & -41 & -72 & -3 & -43 & 19 \\ -19 & 34 & 30 & 52 & 92 & 4 & 54 & -24 \\ 24 & -43 & -38 & -66 & -116 & -4 & -68 & 30 \\ -30 & 54 & 47 & 82 & 144 & 4 & 86 & -38 \end{pmatrix}.$$

The eigenvalues of A are 2.8854, -1.2628, $-0.2433+0.9699i$, $-0.2433-0.9699i$, $0.1547+0.9879i$, $0.1547-0.9879i$, -0.7918, 0.3465. *The eigenvalues of B are* -138.319, -4.4174, $0.7076+0.7065i$, $0.7076-0.7065i$,

$0.7770+0.6294i, 0.7770-0.6294i, -0.2263, -0.0072$. *Both A and B have four real eigenvalues and two pairs of complex conjugate eigenvalues of absolute value* 1. *The matrices A and B commute and have a common four dimensional neutral space; eight is the lowest dimension when the latter property is compatible with partial hyperbolicity for a higher rank action.*

2.2.8 Affine actions of the torus without fixed points

Any affine map of the torus whose linear part A has no roots of unity has a fixed point and hence is isomorphic to the ergodic automorphism F_A via a translation which takes zero into a fixed point.

For several commuting affine maps the set of fixed points of any of them is invariant under the others. Thus any abelian group of affine maps of a torus which contains an element with linear part without roots of unity has a finite orbit and contains a subgroup of finite index which has a fixed point and is hence isomorphic to an action by automorphisms. However, the whole group may not have a fixed point even if all non-zero elements of the action are hyperbolic [54]. We describe such an example on the torus and the relevant theoretical background below.

Let Γ be a discrete finitely generated group. Let $\phi_0 : \Gamma \to \mathrm{Aut}(\mathbb{T}^n) \equiv GL(n, \mathbb{Z})$ be a representation. We recall in this particular setting some notions from Section 1.4. A map $\tau : \Gamma \to \mathbb{T}^n$ is a 1-cocycle over the action ϕ if for each $\gamma_1, \gamma_2 \in \Gamma$

$$\tau(\gamma_1\gamma_2) = \tau(\gamma_1) \cdot \phi_0(\gamma_1)(\tau(\gamma_2)). \tag{2.2.3}$$

A 1-cocycle is *trivial* if there exists a point $x_0 \in \mathbb{T}^n$ so that for all $\gamma \in \Gamma$

$$\tau(\gamma) = \phi_0(\gamma)(x_0)^{-1} \cdot x_0. \tag{2.2.4}$$

The cohomology group $H^1(\Gamma; \mathbb{T}^n_{\phi_0})$ is the quotient space of the 1-cocycles modulo the trivial 1-cocycles.

The following proposition relates the basic structure of affine structures to 1-cohomology.

Proposition 2.2.22 *Let Γ be a finitely generated group and $\phi : \Gamma \to GL(n, \mathbb{Z})$ a representation.*

(i) *There is a one-to-one correspondence between the affine actions ϕ of Γ on $\mathbb{T}^n$ with linear part ϕ_0 and 1-cocycles τ over the action ϕ_0. We will denote by τ_ϕ the 1-cocycle over ϕ_0 associated to an action*

α. The translational class associated to ϕ is the cohomology class $[\tau_\phi] \in H^1(\Gamma; \mathbb{T}^n_{\phi_0})$.

(ii) *An affine action with linear part ϕ_0 has a fixed point if and only if $[\tau_\phi]$ is trivial.*

(iii) *An affine action with linear part ϕ_0 has a dense set of periodic points if and only if $[\tau_\phi]$ is torsion.*

Proof (1) Let $\phi(\gamma)$ be an affine transformation. Define the translational part $\tau_\phi(\gamma) \in \mathbb{T}^n$ to be its action on the identity element, $\tau(\gamma)(0) = \phi(\gamma)(0)$. The group law for the action ϕ becomes the cocycle law (2.2.3). Vice-versa, given a 1-cocycle $\tau : \Gamma \to \mathbb{T}^n$ over ϕ_0, define an affine action by the rule $\phi(\gamma)(x) = \tau(\gamma) \cdot \phi_0(\gamma)(x)$ for $x \in \mathbb{T}^n, \gamma \in \Gamma$.

(2) Suppose that x_0 is a fixed-point for the action. For each $\gamma \in \Gamma, \tau(\gamma) \cdot \phi_0(\gamma)(x_0) = x_0$, so $\tau(\gamma) = \phi_0(\gamma)(x_0)^{-1} \cdot x_0$, and τ is a trivial cocycle. Vice-versa, if x_0 satisfies equation (2.2.4)

$$\phi_\tau(\gamma)(x_0) = \tau(\gamma) \cdot \phi_0(\gamma)(x_0) = \tau(\gamma) \cdot \tau(\gamma)^{-1}(x_0) = x_0.$$

(3) Let ϕ be an affine action with a periodic orbit $x_0 \in \mathbb{T}^n$. The stabilizer of the orbit of x_0 is a normal subgroup $\Gamma' \subset \Gamma$ with finite index. Let $R : H^1(\Gamma; \mathbb{T}^n_{\phi_0}) \to H^1(\Gamma'; \mathbb{T}^n_{\phi_0})$ be the restriction map and $T : H^1(\Gamma'; \mathbb{T}^n_{\phi_0}) \to H^1(\Gamma; \mathbb{T}^n_{\phi_0})$ the transfer. The restriction class $R[\tau_\phi] \in H^1(\Gamma'; \mathbb{T}^n_{\phi_0})$ is 0 by statement (2). The composition $T \circ R = [\gamma : \Gamma'] \cdot Id$ (by Proposition 10.1, [13]), so $0 = T \circ R[\tau_\phi] = [\Gamma : \Gamma'][\tau_\phi]$ which implies that $[\tau_\phi]$ is a torsion class.

Conversely, let $p > 0$ so that $p \cdot \tau(\gamma) = \phi_0(\gamma)(x_0)^{-1} \cdot x_0$ for all $\gamma \in \Gamma$. Choose $y_o \in \mathbb{T}^n$ so that $p \cdot y_0 = x_0$ and calculate

$$p \cdot \left(\tau(\gamma) \cdot \phi + 0(\gamma)(y_0) \cdot y_0^{-1}\right) = e \in \mathbb{T}^n$$

hence τ is cohomologous to a cocycle taking values in the finite group $(\frac{1}{p}) \cdot \mathbb{Z}/\mathbb{Z} \subset \mathbb{T}$. The statement from proposition follows now if we show that an affine action ϕ, defined by a translational cocycle τ'_ϕ with values in a finite subgroup $G \subset \mathbb{T}$, has a dense set of periodic points.

The isotropy subgroup $\Gamma_G \subset \Gamma$ of G for the linear action ϕ_0 is a normal subgroup of finite index. The restriction of τ'_ϕ to Γ_G is a homomorphisms $\tau'_\phi : \Gamma_G \to \Gamma$ with kernel Γ' of finite index. It follows that every rational point of $\mathbb{T}^n$ is a periodic point for the action ϕ. □

The following result appears in [54].

Theorem 2.2.23 *Let Γ be a free abelian group of rank $r \geq 2$, and $\phi_0 : \Gamma \to SL(n, \mathbb{Z})$ a representation such that $\phi_0(\gamma_h)$ is hyperbolic for some $\gamma_h \in \Gamma$. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index and an affine Anosov action ϕ of Γ' on $\mathbb{T}^n$ with linear part $\phi_0|\Gamma'$ and no fixed points.*

Proof The fixed-point set $\text{Fix}(\phi(\gamma_h)) \subset \mathbb{T}^n$ for $\phi(\gamma_h))$ is finite, and as Γ is abelian, is invariant under the action of $\phi(\Gamma)$. Thus, the existence of a fixed point for the action ϕ is equivalent to the existence of a fixed point for the restricted action of Γ on $\text{Fix}(\phi(\gamma_h))$. Introduce now the relative cohomology group $H^1(\Gamma, \gamma_h; \text{Fix}(\phi_0(\gamma_h))_{\phi_0})$ spanned by the 1-cocycles $\tau : \Gamma \to \text{Fix}(\phi_0(\gamma_h))$ over ϕ_0 which vanish on γ_h. Note that this defines a subcomplex as $\tau(\gamma_h^m) = \tau(\gamma_h)^m$ using that $\phi_0(\gamma_h)$ acts trivially on $\text{Fix}(\phi_0(\gamma_h))$. So there is a natural map

$$H^1(\Gamma, \gamma_h; \text{Fix}(\phi_0(\gamma_h))_{\phi_0}) \to H^1(\Gamma; \mathbb{T}^n_{\phi_0}). \tag{2.2.5}$$

We show now that this map is injective and that each affine action ϕ with the linear part ϕ_0 as in the theorem yields a 1-cohomology class in the image of this map. Together with Proposition 2.2.22 this implies that the set of affine actions of Γ with linear part ϕ_0 is indexed by the cohomology group $H^1(\Gamma, \gamma_h; \text{Fix}(\phi_0(\gamma_h))_{\phi_0})$, and that each class in the previous group gives rise to an Anosov action without fixed points.

Assume that $\tau : \Gamma \to \text{Fix}(\phi_0(\gamma_h))$ is a 1-cocycle over ϕ_0 which vanishes on γ_h and is a coboundary as a map into $\mathbb{T}^n$. Then the corresponding affine action ϕ_τ admits a fixed point, so is conjugate to the linear action ϕ_0 via translation by some $x_0 \in \mathbb{T}^n$. Translation by x_0 maps the fixed point set of ϕ to that of ϕ_0. The hypothesis $\tau(\gamma_h) = 0$ implies that $\text{Fix}(\phi(\gamma_h))$ contains 0, so $x_0 \in \text{Fix}(\phi_0(\gamma_h))_{\phi_0}$ and thus τ also defines the zero class in $H^1(\gamma; \text{Fix}(\phi_0(\gamma_h))_{\phi_0})$. A corollary of this argument is that $\text{Fix}(\phi(\gamma_h)) = \text{Fix}(\phi_0(\gamma_h))$ whenever $0 \in \text{Fix}(\phi(\gamma_h))$.

Assume now that ϕ is an affine action as in the theorem. We can conjugate the action by a translation so that $0 \in \text{Fix}(\phi(\gamma_h))$. Let $\tau_\phi : \Gamma \to \mathbb{T}^n$ denote the corresponding 1-cocycle. The set $\text{Fix}(\phi(\gamma_h))$ is invariant under the action $\phi(\Gamma)$, so $\tau_\phi(\gamma) = \phi(\gamma_i)(0) \in \text{Fix}(\phi(\gamma_h)) = \text{Fix}(\phi_0(\gamma_h))$, and so the class $[\tau_h]$ is in the image of the homomorphisms (2.2.5).

If $\Gamma' \subset \Gamma$ is the subgroup of finite index consisting of the elements which act trivially when restricted to $\text{Fix}(\phi(\gamma_h))$, and $\mathcal{A}$ an abelian group, let $\text{Hom}(\Gamma', \gamma_h; \mathcal{A})$ denote the group homomorphisms that maps

γ_h into the trivial element. The theorem now follows by observing that

$$\mathrm{Hom}(\Gamma', \gamma_h; \mathrm{Fix}(\phi_0(\gamma_h))) \subset H^1(\Gamma', \gamma_0; \mathrm{Fix}(\phi_0(\gamma_h))_{\phi_0}), \tag{2.2.6}$$

and using Proposition 2.2.22. Note that for the cohomology that appear on the right hand side of (2.2.6), 1-cocycles are representations and there is only one trivial 1-cocycle, the trivial one. □

We show now a concrete example of higher rank abelian action on tori without any periodic point.

Example 2.2.24 *Consider the action* $\phi_0 : \mathbb{Z}^2 \times \mathbb{T}^3 \to \mathbb{T}^3$ *on the three dimensional torus induced by* F_A, F_B*, where*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 8 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 8 & 4 \end{pmatrix}.$$

One can easily check that both A *and* B *are hyperbolic, not powers of each other, and* $AB = BA$*. We take* $A = \gamma_h$*, where* γ_h *is as above. Let*

$$\mathit{Fix}(A) = \left\{ \left(\frac{i}{10}, \frac{i}{10}, \frac{i}{10} \right) \middle| 0 \leq 1 \leq 1 \right\}$$

be the set of fixed elements of F_A*.* $\mathit{Fix}(A)$ *is an abelian group under addition isomorphic to* $\mathbb{Z}_{10}$*. The action of* B *on* $\mathit{Fix}(A)$ *is given by multiplication by* 3*, so* B^{10} *is the first power of* B *that acts trivially on* $\mathit{Fix}(A)$*. The subgroup* $\Gamma' \subset \mathbb{Z}^2$ *that acts trivially on* $\mathit{Fix}(A)$ *is the subgroup generated by* A *and* B^{10}*. Let* $\tau \in \mathit{Hom}(\Gamma', \gamma_h; \mathit{Fix}(\phi_0(\gamma_h)))$ *be defined by* $\tau(A) = \hat{0}$ *and* $\tau(B^{10}) = \hat{1}$*. Since the action of* Γ' *on* $\mathit{Fix}(A)$*,* τ *is also a non-trivial* 1*-cocycle, and hence induces a nontrivial element in* $H^1(\Gamma', \gamma_0; \mathit{Fix}(\phi_0(\gamma_h))_{\phi_0})$*. Moreover,* $\tau : \Gamma' \to \mathbb{T}^3$ *is a* 1*-cocycle that is not trivial.*

One defines now the affine action $\phi : \Gamma' \times \mathbb{T}^3 \to \mathbb{T}^3$ *by* $\phi(\gamma)(x) = \tau(\gamma)\phi_0(\gamma)(x)$ *for all* $\gamma \in \Gamma', x \in \mathbb{T}^3$*. The action* ϕ *does not have any fixed point.*

2.2.9 Higher rank abelian actions on nilmanifolds

Examples of higher rank abelian actions on nilmanifold are constructed by Qian [145], developing on the work of Auslander and Scheuneman [4].

Let V be an n-dimensional vector space over $\mathbb{R}$. Let $\{x_1, x_2, \ldots, x_n\}$

be a basis. We denote by $N(V)$ the free Lie algebra associated to V, that is, the Lie algebra generated by all non-associative words

$$x_1, x_2, \dots, x_n, [x_1, x_2], \dots, [[x_1, x_2], x_3], \dots, \text{ etc.} \tag{2.2.7}$$

subject only to the Jacobi and skew-symmetry relations. The free k-step nilpotent Lie algebra, denoted by $N_k(V)$, is obtained by imposing the condition that every word of length $k+1$ is zero.

To find a lattice in $N_k(V)$, consider C_0 to be the basis in $N_k(V)$ consisting of all words in (2.2.7) of length at most $k-1$, and let $C = \mathbb{Z}$-span of C_0. Then there exists an integer $m \in \mathbb{Z}$ such that mC is a uniform lattice in $N_K(V)$. For example, using Campbell-Baker-Hausdorff formula it is easy to see that for $k = 1, 2, 3, 4$ one may take $m = 1, 2, 12, 48$ respectively.

We denote the lattice $\Lambda = \exp(mC)$, the corresponding nilmanifold $N(n,k) := N_k(V)/\Lambda$, and the group of automorphisms of $N(n,k)$ by $\mathrm{Aut}(n,k)$. Recall that an element $A \in \mathrm{Aut}(n,k)$ is a linear automorphism $A : N(n,k) \to N(n,k)$ that preserves the bracket operation and satisfies $A(\Lambda) = \Lambda$.

The standard basis in $N_k(V)$, consisting of words of length at most k in (2.2.7), determines a decomposition $N_k(V) = V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(k)}$, where $V^{(k)}$ is the subspace generated by the words of length exactly k. The decomposition is kept invariant by any automorphism $A \in N(n,k)$. One can check that the matrix of A with respect to the standard basis is of type:

$$A = \begin{pmatrix} A_0^{(0)} & 0 & 0 & \dots & 0 \\ A_1^{(0)} & A_0^{(1)} & 0 & \dots & 0 \\ A_2^{(0)} & A_1^{(1)} & A_0^{(2)} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_{k-1}^{(0)} & A_{k-2}^{(1)} & A_{k-3}^{(2)} & \dots & A_0^{(k-1)} \end{pmatrix}, \tag{2.2.8}$$

where $A_0^{(0)} \in SL(n,\mathbb{Z}), A_i^{(0)} \in Mat(n_i \times n, \mathbb{Z}), n_i = \dim(V^{(i)}, A_i^{(0)}$ are arbitrary, and the matrices $A_i^{(l)}$ are determined inductively by $A_j^{(0)}, j < i$, using the fact that A preserves the brackets.

Example 2.2.25 *Consider the free Lie algebra $N_2(2)$, which is generated by $\{x_1, x_2, [x_1, x_2]\}$. An automorphism $A \in Aut(2,2)$ can be represented* *as a matrix:*

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ b_1 & b_2 & c \end{pmatrix} \tag{2.2.9}$$

where $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2,\mathbb{Z}), b_1, b_2 \in \mathbb{Z}$ *arbitrary, and* $c = a_{11}a_{22} - a_{12}a_{21} = 1$. *In particular,* $N_2(2)$ *does not have any Anosov automorphism.*

Note that by choosing in (2.2.8) the matrices $A_1^{(0)}, A_2^{(0)}, \ldots, A_{k-1}^{(0)}$ to be trivial, one obtains a representation of $SL(n,\mathbb{Z})$ into $Aut(n,k)$. We will refer to this representation as the *standard representation* of $SL(n,\mathbb{Z})$ into $Aut(n,k)$. Note that the standard representation is block diagonal. In this case the only not-trivial entries in (2.2.8) are the diagonal blocks $A_0^{(i)}, 0 \le i \le k$.

One can generalize the notion of Cartan action, introduced in Section 2.2.5, to arbitrary manifolds and, in particular, to (infra)-nilmanifolds. The definition below was introduced by Hurder in [55]. It was frequently used in the last years in the rigidity theory of higher rank lattice actions on compact manifolds.

Definition 2.2.26 *Let* $\phi : \mathbb{Z}^k \times M \to M$ *be a* C^∞ *action on a compact manifold* M*. The action* ϕ *is called Cartan if there exists a set of generators* $\{\gamma_1, \ldots, \gamma_n\}$ *of* $\mathbb{Z}^k$ *such that*

(i) *each* $\phi(\gamma_i)$ *is hyperbolic and has a* 1*-dimensional, strongest stable foliation* $\mathcal{F}^{ss}$*;*

(ii) *the tangential distributions are pair-wise transverse with their direct sum* $E_1^{ss} \oplus \cdots \oplus E_n^{ss} \cong TM$*.*

More general, if Γ *is a discrete group and* $\phi : \Gamma \times M \to M$ *is a* C^∞ *action, then* ϕ *is called Cartan if there exists a set of commuting elements* $\{\gamma_1, \ldots, \gamma_n\}$*, which generate an abelian group* $\mathcal{A}$*, such that the restriction of* ϕ *to* $\mathcal{A}$ *is a Cartan action on* M*.*

Remark 2.2.27 *Note that from a result of Franks [39] it follows that if a manifold admits an Anosov diffeomorphism with a* 1*-dimensional stable foliation, then the manifold is a torus. Some of the following examples are on nilmanifolds. In these cases, the* 1*-dimensional foliations* $\mathcal{F}^{ss}$ *that appear in the Cartan actions are always strict subfoliations of some stable foliations.*

Remark 2.2.28 *It is immediate that a $\mathbb{Z}^k$ Cartan action generated by hyperbolic automorphisms of an infranilmanifold is a genuine $\mathbb{Z}^k$ action.*

The following lemma shows the existence of many Cartan actions on tori. It allows to build hyperbolic matrices in $SL(n, \mathbb{Z})$ with specific eigenvalue structure, and is used to build examples of higher rank abelian actions on nilmanifolds. Its proof is based on more general results from [142].

Lemma 2.2.29 *Let $\Gamma = SL(n, \mathbb{Z}), n \geq 2$, or more generally, a subgroup of finite index in $SL(n, \mathbb{Z})$. Then there exists a Cartan subgroup H of $SL(n, \mathbb{Z})$ such that the quotient $H/(H \cup \Gamma)$ is compact. In particular, there exists a subgroup $\mathcal{A} \subset \Gamma$ such that*

(i) *the elements of $\mathcal{A}$ are simultaneously diagonalizable over $\mathbb{R}$;*
(ii) *$\mathcal{A}$ is isomorphic to a free abelian group of rank $n-1$.*

Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be a basis of simultaneously eigenvectors for the group $\mathcal{A}$, and $\lambda_i : \mathcal{A} \to \mathbb{R} - \{0\}$ the character of $\mathcal{A}$ defined by $Av_i = \lambda_i(A)v_i, A \in \mathcal{A}$. One can pass to a subgroup of finite index and assume that each λ_i takes values in $\mathbb{R}^+$. If H^0 is the connected component of the identity in the the Cartan group in Lemma 2.2.29 then λ_i's extend to H^0:

$$\lambda_i : H^0 \to \{(x_1 \ldots x_n) \in (\mathbb{R}^+)^n | x_1 \ldots x_n = 1\} \tag{2.2.10}$$

as an isomorphism of analytic groups. Note that $H^0/\mathcal{A}$ is compact. As a consequence of the compactness of $\mathcal{A}$ in H^0, one has that for each $i, 1 \leq i \leq n$, there exists $A_i \in \mathcal{A}$ such that $\lambda_i(A_i) < 1$ and $\lambda_j(A_i) > 1$ for each $j \neq i$. Moreover, for each $1 \leq i < j \leq n$ there exists $B_{ij} \in \mathcal{A}$ such that $\lambda_i(B_{ij}) < 1, \lambda_j(B_{ij}) < 1$, and $\lambda_k(B_{ij}) > 1$ for all $k \neq i, k \neq j$. Note that by standard results the torus H^0 is $\mathbb{Q}$-anisotropic, that is, none of the eigenspaces $\mathbb{R}v_i$ is a rational line and its image under the standard projection is dense in $\mathbb{T}^n$. As a consequence, if $A \in \mathcal{A}, A \neq 1$, then $\lambda_i(A) \neq 1$ for each $1 \leq i \leq n$.

Theorem 2.2.30 *If $n \geq 3$, then the action induced on $N(n, 2)$ by the standard representation of $SL(n, \mathbb{Z})$ is a Cartan action.*

Proof Let $\mathcal{A} \cong \mathbb{Z}^{n-1}$ be the diagonalizable subgroup of $SL(n, \mathbb{Z})$ which existence is guaranteed by Lemma 2.2.29. Let $\{v_1, \ldots, v_n\}$ be the basis in $\mathbb{R}^n$ in which $\mathcal{A}$ is diagonalizable. Then there exist hyperbolic matrices

$A_k \in \mathcal{A}, 1 \leq k \leq n$ such that A_k has all eigenvalues positive numbers, with only one eigenvalue $\lambda_k^{(k)}$ in position k greater than one, and all the other eigenvalues less than one, and there exist hyperbolic matrices $B_{ij}, 1 \leq i < j \leq n$ such that B_{ij} has all eigenvalues positive numbers, with only two eigenvalues, $\lambda_i^{(ij)}, \lambda_j^{(ij)}$, in positions i, j, greater than one, and all the other eigenvalues less than one.

Then the set

$$\mathcal{B} = \{v_k | 1 \leq k \leq n\} \cup \{[v_i, v_j] | 1 \leq i < j \leq n\} \tag{2.2.11}$$

is a basis for $N(2, n)$ in which the linear action induced by $\mathcal{A}$ on $N(2, n)$ is diagonalizable. It is clear now that the action induced by the matrix A_k on $N(2, n)$ is hyperbolic and has a strong stable one dimensional foliation parallel to the vector v_k, and the action induced by the matrix B_{ij} is hyperbolic and has a strong stable one dimensional foliation parallel to the vector $[v_i, v_j]$. □

Remark 2.2.31 *One can show that for any $k > 2$ the standard representation of $SL(n, \mathbb{Z})$ into $Aut(n, k)$ is not a Cartan action. In the case $n = 3$ this follows from the fact that two different eigenvectors v_i, v_j, v_k belonging to a basis in $\mathbb{R}^n$ generate two different words of length 3 in $N(n, 3)$, namely $[v_i, [v_j, v_k]]$ and $[v_k, [v_i, v_j]]$.*

Nevertheless, the standard action still has many Anosov diffeomorphisms, and, as follows from the theorem below, they actually generate the action.

First we prove an auxiliary lemma.

Lemma 2.2.32 *Let $A \in SL(n, \mathbb{R})$. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+, \lambda_i \neq 1$ for all $1 \leq i \leq n$. Then there exists k positive integer such that the matrix $ADiag\{\lambda_1^k, \ldots, \lambda_n^k\}$ is hyperbolic.*

Proof It is enough to show that for any nonzero unit vector v and for k positive integer sufficiently large one has $\|ADiag\{\lambda_1^k, \ldots, \lambda_n^k\}v\| \neq \|v\|$. We proceed by contradiction. Let $v = (v_1, v_2, \ldots, v_n)$ be a unit vector such that

$$\|ADiag\{\lambda_1^k, \ldots, \lambda_n^k\}v\| = \|v\|. \tag{2.2.12}$$

Since all λ_i's are positive we can assume without loss that

$$\lambda_1 = \cdots = \lambda_l > \lambda_{l+1} \geq \cdots \geq \lambda_n$$

for some $2 \leq l < n$. We rewrite (2.2.12) as:

$$\lambda_1^k \left\| ADiag\{v_1, \ldots, v_l, \left(\frac{\lambda_{l+1}}{\lambda_1}\right)^k v_{l+1}, \ldots, \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n\} \right\| = \|v\|. \quad (2.2.13)$$

Observe now that $ADiag\{v_1, \ldots, v_l, 0, 0, \ldots, 0\} \neq 0$ and k large enough gives a contradiction in (2.2.13). Thus the first l components of v are zero. In a similar way one can show that all components of v are zero, so $v = 0$. But this is in contradiction with v of length 1. □

Theorem 2.2.33 *Consider the standard representation of $SL(n, \mathbb{Z})$ into $Aut(n, k)$. Then its image is generated by Anosov diffeomorphisms if $n \geq k + 1$.*

Proof It follows from Lemma 2.2.29 that $SL(n, \mathbb{Z})$ contains a hyperbolic element A that has all eigenvalues $\lambda_i, 1 \leq i \leq n$, real eigenvalues, and moreover, any product of λ_i's of length strictly less than n is different from 1. Choose now a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ of V consisting of eigenvectors of A. Then the set of words of length less then $k + 1 (\leq n)$ in letters from $\mathcal{B}$ gives a basis in $N(n, k)$ which consists of eigenvectors for the action of A on $N(n, k)$. The corresponding eigenvalues are products of distinct eigenvalues $\lambda_i, 1 \leq i \leq n$, of length less than n, so are different from 1 and positive. This implies that A, viewed as an element in $Aut(n, k)$, has all eigenvalues different from 1. For any $M > 0$, by taking higher powers of A we can assume that the eigenvalues of A on $N(n, k)$ are such that $\lambda > M$ or $\lambda^{-1} > M$.

It is well known that $SL(n, \mathbb{Z})$ is generated by the elementary matrices $E_{i,j} := Id + e_{i,j}$, where $e_{i,j}$ is the matrix with 1 in the (i, j) position and 0's elsewhere. Since each $E_{i,j}$ can be written as

$$E_{i,j} = (E_{i,j} A^{-1} E_{i,j}^{-1})(E_{i,j} A),$$

to finish the proof of the theorem it is enough to show that the action of the matrix $E_{i,j} A$ on $N(n, k)$ is hyperbolic. But this follows from Lemma 2.2.32. □

Remark 2.2.34 *Using Theorem 2.2.33 one can show that the whole group $Aut(n, k)$ is generated by Anosov diffeomorphisms.*

Example 2.2.35 *We describe now another example of a $\mathbb{Z}^3$-action on a nilmanifold. Let $\mathfrak{n}$ be the 2-step nilpotent Lie algebra generated by $\{e_i; 1 \le i \le 10\}$, with the following relations:*

$$[e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_1, e_4] = e_7,$$
$$[e_2, e_3] = e_8, [e_2, e_4] = e_9, [e_3, e_4] = e_{10},$$

and all the other brackets between the generators zero. Define $C = span_{\mathbb{Z}}\{e_i\}$. Denote $N = \exp(\mathfrak{n})$ and $\Gamma = \exp(2C)$. Then N is a connected, simply connected nilpotent Lie group, and Γ is a co-compact lattice in N.

Consider the standard representation of $SL(4, \mathbb{Z})$ on $span\{e_i; 1 \le i \le 4\}$. Then, using the relations between e_i's, we find a representation of $SL(4, \mathbb{Z})$ on $span\{e_i; 5 \le i \le 10\}$. So we have a representation of $SL(4, \mathbb{Z})$ on $\mathfrak{n}$, and therefore an action on N, which preserves Γ. One can find an abelian subgroup generated by three hyperbolic matrices in $SL(4, \mathbb{Z})$ that gives a $\mathbb{Z}^3$ action on the nilmanifold N/Γ.

2.3 Higher rank $\mathbb{R}^k$-actions

2.3.1 Summary

Starting with Anosov flows and doing standard constructions as products, quotients and covers, or starting with Anosov diffeomorphisms and doing suspensions, provide a large class of hyperbolic $\mathbb{R}^k$-actions. These examples do not exhibit many rigidity properties.

An interesting class of examples of higher rank $\mathbb{R}^k$-actions, that exhibit many rigidity properties, comes from the following unified algebraic construction. None of the examples below have a finite cover with a smooth factor on which the action is not faithful, not transitive or is generated by a rank one group.

Let G be a connected Lie group, $A \subset G$ a closed abelian subgroup which is isomorphic with $\mathbb{R}^k$, M a compact subgroup of the centralizer $Z(A)$ of A, and Γ a co-compact lattice in G. Then A acts by left translation on the compact space $M \setminus G/\Gamma$.

We will discuss in this section the following specific types corresponding to the general construction:

(i) For *suspensions of actions by automorphisms of tori and nilmanifolds* take $G = \mathbb{R}^k \ltimes \mathbb{R}^m$ or $G = \mathbb{R}^k \ltimes N$, the semi-direct product of $\mathbb{R}^k$ with $\mathbb{R}^m$ or a simply connected nilpotent Lie group N.

(ii) For the *symmetric space examples* take G a semisimple Lie group of the non-compact type.

(iii) For the *twisted symmetric space examples* take $G = H \ltimes \mathbb{R}^m$ or $G = H \ltimes N$, a semi-direct product of a reductive Lie group H with semisimple factor of the non-compact type with $\mathbb{R}^m$ or a simply connected nilpotent group N.

Those examples are partially hyperbolic and many among them are Anosov. Further interesting partially hyperbolic examples are obtained by taking restrictions of those actions to higher-rank subgroups of A, i.e. subgroups which contain a discrete $\mathbb{Z}^2$ subgroup.

2.3.2 Suspensions of automorphisms of tori and nilmanifolds

Consider a genuine higher rank $\mathbb{Z}^k$-action on $\mathbb{T}^n$ by automorphisms of a torus. Recall that such an action contains a $\mathbb{Z}^2$-action such that every nontrivial element of $\mathbb{Z}^2$ acts ergodically on $\mathbb{T}^n$ with respect to the Haar measure. Embed $\mathbb{Z}^k$ as a lattice in $\mathbb{R}^k$. Let $\mathbb{Z}^k$ acts on $\mathbb{R}^k \times \mathbb{T}^n$ by $z(x, m) = (x - z, zm)$, and let $M = (\mathbb{R}^k \times \mathbb{T}^n)/\mathbb{Z}^k$ be the orbit space of the action.

Observe now that the group $\mathbb{R}^k$ acts naturally on $\mathbb{R}^k \times \mathbb{T}^n$ by $x(y, m) = (x + y, m)$ and the $\mathbb{R}^k$ action commutes with the previous $\mathbb{Z}^k$-action. Hence the $\mathbb{R}^k$-action descends to M. The induced $\mathbb{R}^k$-action is called the *suspension* of the $\mathbb{Z}^k$-action.

These examples generalize to irreducible Anosov $\mathbb{Z}^k$-actions by automorphisms of nilmanifolds. See Section 2.2.9 for examples of such actions.

2.3.3 Symmetric spaces and Weyl chamber flows

Now we consider the leading class of algebraic Anosov and partially hyperbolic higher rank $\mathbb{R}^k$ actions. We start with a review of relevant facts from Lie group theory. A good reference for this material is the book of Helgason [48].

2.3.3.1 Summary of Lie group theory

All our Lie groups are considered to be real, that is Lie groups over $\mathbb{R}$. Let G be a Lie group. The maximal connected solvable normal subgroup $R \subset G$ is said to be the *radical* of G. A connected Lie group is said to be *semisimple* if its radical is trivial. G is said to be *simple* if it has no nontrivial proper normal connected subgroups. Every connected semisimple Lie group G can be uniquely decomposed into an *almost direct product* $G = G_1 \dots G_n$ of its normal simple subgroups, called the *simple factors* of G. In the previous product G_i, G_j commute and the intersection $G_i \cap G_j$ is discrete if $i \neq j$. If, in addition, G is simply connected and center free, then G is the direct product $G_1 \times \cdots \times G_n$. Note that if G is semisimple, its center $Z(G)$ is discrete, and moreover, it coincides with the center of the adjoint representation if G is connected.

Let G be a connected semisimple Lie group. Then G is equal to the almost direct product $G = KS$ of its compact and totally noncompact parts, where $K \subset G$ is the product of all compact simple components of G, and S is the product of all noncompact simple components. G is called *totally noncompact* if K is trivial. A subgroup $H \subset G$ is said to be *Cartan* if H is a maximal connected abelian subgroup consisting of semisimple elements. Any Cartan subgroup has a unique decomposition $H = T \times A$ into a direct product of a compact torus T and an $\mathbb{R}$-diagonalizable subgroup A.

All maximal connected $\mathbb{R}$ diagonalizable subgroups of G are conjugate and their common dimension is called the $\mathbb{R}$-*rank* of G. If maximal $\mathbb{R}$-diagonalizable subgroups are Cartan subgroups then G is said to be $\mathbb{R}$-*split* group.

If $\mathfrak{g}$ is a finite dimensional Lie algebra, then there exists a uniquely solvable ideal in $\mathfrak{g}$ containing all solvable ideals of $\mathfrak{g}$. This ideal is called the *radical* of $\mathfrak{g}$ and is denoted *rad* $\mathfrak{g}$. A finite dimensional Lie algebra $\mathfrak{g}$ is called *simple* if it is nonabelian and has no proper nonzero ideals. It is called *semisimple* if *rad* $\mathfrak{g} = 0$, that is $\mathfrak{g}$ has no nontrivial solvable ideals. The Lie algebra of a semisimple Lie group is semisimple. If $\mathfrak{g}$ is semisimple then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and if $\mathfrak{g}$ is an arbitrary finite dimensional Lie algebra, then $\mathfrak{g}/rad\ \mathfrak{g}$ is semisimple.

Let $\mathfrak{g}$ be a Lie algebra. The bilinear form $B(X,Y) = Tr(adX adY)$ on $\mathfrak{g} \times \mathfrak{g}$ is called the *Killing form* of $\mathfrak{g}$.

An automorphism Θ of a Lie algebra is called *involutive* if $\Theta^2 = Id_{\mathfrak{g}}$. An involutive automorphism Θ of a semisimple Lie algebra $\mathfrak{g}$ is called *Cartan involution* if the bilinear form $B_\Theta(X,Y) = -B(X, \Theta Y)$ is strictly positive definite.

Let G be a semisimple Lie group of noncompact type with $\mathfrak{g}$ its semisimple Lie algebra. Let B be the Killing form and let Θ be any Cartan involution of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding *Cartan decomposition*, that is the decomposition in eigenspaces corresponding to eigenvalues 1 and -1 of Θ. Note that $\mathfrak{k}$ is the fixed point set of Θ. It also coincides with the Lie algebra of a maximal compact subgroup $K \subset G$. Let $\mathfrak{a} \subset \mathfrak{p}$ be any maximal abelian subspace. All such subspaces have the same dimension. Let $A = \exp \mathfrak{a} \subset G$ the corresponding subgroup. Then A is the connected component of identity of a split Cartan subgroup of G. We denote by $\log : A \to \mathfrak{a}$ the inverse of the exponential map.

The centralizer $Z(A)$ of A splits as a product $Z(A) = M\,A$ where M is compact. M coincides with the centralizer of $\mathfrak{a}$ in K. If $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{t}$, then $\mathfrak{m}$ is the Lie algebra of M.

For each λ in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ let

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | [H, X] = \lambda(H)X \text{ for } H \in a\}.$$

Then λ is called a *restricted root* if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$.

The simultaneous diagonalization of $ad_{\mathfrak{g}}(\mathfrak{a})$ gives the decomposition

$$\mathfrak{g} = \mathfrak{g}_o + \sum_{\lambda \in \Lambda} \mathfrak{g}_\lambda, \ \mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m},$$

where Λ is the set of restricted roots. The spaces $\mathfrak{g}_\lambda$ are called *root spaces.* A point $H \in \mathfrak{a}$ is called *regular* if $\lambda(H) \neq 0$ for all $\lambda \in \Lambda$. Otherwise it is called *singular.* The set of regular elements consists of the complement of finitely many hyperplanes, and its components are called *Weyl chambers*.

2.3.3.2 Definition of Weyl chamber flow and hyperbolicity

Let G be a semisimple connected real Lie group of the noncompact type, with Lie algebra $\mathfrak{g}$. Let $K \subset G$ be a maximal compact subgroup that gives a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{g}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra and $A = \exp \mathfrak{a}$ be the corresponding subgroup. Let M be the centralizer of A in K. Suppose Γ is an irreducible torsion-free co-compact lattice in G. Since A commutes with M, the action of A by left translations on G/Γ descends to an A-action on $N \stackrel{\text{def}}{=} M \setminus G/\Gamma$. We call this action the *Weyl chamber flow* of A. Notice that the rank of the acting group is equal to the $\mathbb{R}$-*rank* of the group G. From the dynamical point of view there is a great difference between the cases or rank one and higher rank. Usually the name Weyl chamber flow is applied only to the higher rank case. The rank one case corresponds to the geodesic flow on the corresponding locally symmetric space $C \setminus G/\Gamma$ whew C is a maximal compact subgroup of G. It is an Anosov flow and does not have any rigidity properties beyond structural stability.

The following result appears in [57].

Proposition 2.3.1 *Any Weyl chamber flow $\alpha : A \times N \to N$,*

$$\alpha(a, \widehat{Mg}) = \alpha_a(\widehat{Mg}) = \widehat{Mag},$$

where $\widehat{Mg}$ is the class of Mg in $M \setminus (G/\Gamma)$, is an Anosov action. If the real rank of G is higher than 2, then the action α is a higher rank hyperbolic $\mathbb{R}^k$-action.

Proof We shall prove that all regular elements of A are Anosov elements for α. It is enough to prove this for the lifted action on $M \setminus G$, which for simplicity we denote by α as well. We need to compute the differential of $\alpha_a : M \setminus G \to M \setminus G$. Since the tangent spaces to $M \setminus G$ can be canonically identified with $\mathfrak{m} \setminus \mathfrak{g}$ using right translations with elements in G, it is enough to compute the differential $d\alpha_a : \mathfrak{m} \setminus \mathfrak{g} \to \mathfrak{m} \setminus \mathfrak{g}$. It is clear that $d\alpha_a = Ad(a)$, where $Ad(a)$ is the projection on $\mathfrak{m} \setminus \mathfrak{g}$ of $Ad(a) : \mathfrak{m} \setminus \mathfrak{g}$.

Let Λ denote the restricted root system of G. Then the Lie algebra $\mathfrak{g}$ of G has the root space decomposition

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_{\alpha \in \Lambda} \mathfrak{g}^{\alpha}$$

where $\mathfrak{g}^{\alpha}$ is the root space of α and $\mathfrak{m}$ and $\mathfrak{a}$ are the Lie algebras of M and A. Then $\mathfrak{m} \setminus \mathfrak{g} \cong \mathfrak{a} + \sum_{\alpha \in \Lambda} \mathfrak{g}^{\alpha}$ and any element $\xi \in \mathfrak{m} \setminus \mathfrak{g}$ can be written as $\xi = \xi_0 + \sum_{\lambda \in \Lambda} \xi_{\lambda}$, where $\xi_0 \in \mathfrak{a}$ and $\xi_{\lambda} \in \mathfrak{g}^{\alpha}$.

The identity $Ad(a) = \exp(ad(\log a))$ implies that for ξ as above one has

$$Ad(a)(\xi) = \xi_0 + \sum_{\lambda \in \Lambda} e^{\lambda(\log a)} \xi_{\lambda}. \tag{2.3.1}$$

Assume now that $a \in A$ is a regular element. The set Λ splits in two subsets:

$$\Lambda^+ = \{\lambda \in \Lambda | \lambda(\log a) > 0\}, \quad \Lambda^- = \{\lambda \in \Lambda | \lambda(\log a) < 0\}$$

and $\mathfrak{m} \setminus \mathfrak{g}$ splits into a direct sum invariant under $d\alpha_a$:

$$\mathfrak{m} \setminus \mathfrak{g} = \mathfrak{a} + \mathfrak{n}^+ + \mathfrak{n}^-,$$

where

$$\mathfrak{n}^+ = \sum_{\alpha \in \Lambda^+} \mathfrak{g}^{\alpha}, \qquad \mathfrak{n}^- = \sum_{\alpha \in \Lambda^-} \mathfrak{g}^{\alpha}.$$

One has

$$d\alpha_a(\xi) = \xi, \text{ if } \xi \in \mathfrak{a},$$
$$d\alpha_a(\xi) = \sum_{\lambda \in \Lambda^+} e^{\lambda(\log a)} \xi_\lambda, \text{ if } \xi \in \mathfrak{n}^+,$$
$$d\alpha_a(\xi) = \sum_{\lambda \in \Lambda^-} e^{\lambda(\log a)} \xi_\lambda, \text{ if } \xi \in \mathfrak{n}^-.$$

This decomposition, extended to the tangent space $T(M \setminus G)$, gives a hyperbolic decomposition. Indeed, inverting the sign of the Killing form on $\mathfrak{k}$ and the extending the scalar product on $\mathfrak{g}$ by left translations to the $M \setminus G$, one obtains a Riemannian metric on $M \setminus G$. The root spaces are orthogonal in this metric, and moreover one has:

$$\|d\alpha_a(\xi)\| = \|\xi\|, \text{ for } \xi \in \mathfrak{a}$$
$$\|d\alpha_a(\xi)\| \leq e^{-k}\|\xi\|, \text{ for } \xi \in \mathfrak{n}^+,$$
$$\|d\alpha_a(\xi)\| \leq e^{k}\|\xi\| \text{ for } \xi \in \mathfrak{n}^-,$$

where $k = \min\{\lambda(\log a) | \lambda \in \Lambda^+\}$ is a positive constant depending only on $\mathfrak{a}$. □

Notice that the Weyl chambers in our sense are the same as those in the classical theory of simple Lie groups.

Remark 2.3.2 *One needs Γ to be torsion-free only to assure that N is a manifold. Treating the orbifold case can be done in a similar way.*

If the group G is $\mathbb{R}$-*split*, i.e., its real rank equals its complex rank, then $M = \{\mathrm{Id}\}$. In this case the Weyl chamber flow acts on G/Γ.

In the non-split case the action of A on the whole group G is a compact group extension of the Weyl chamber flow and hence is partially hyperbolic with the zero Lyapunov exponent of extra multiplicity $\dim M$.

2.3.4 Examples of Weyl chamber flows

2.3.4.1 Weyl chamber flow on $SL(n, \mathbb{R})$

The Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ of $SL(n, \mathbb{R})$ can be identified with the set of $n \times n$ matrices of trace zero. The subgroup $D_n^+ \subset SL(n, \mathbb{R})$ of matrices with positive diagonals is the connected component of the identity in a maximal Cartan subgroup H of $SL(n, \mathbb{R})$. The diagonal entries of $d \in D_n^+$ can be written as exponentials e^{t_i}, $i = 1, \ldots, n$, where $t_1 + \cdots + t_n = 0$. Thus it is convenient to parameterize elements A in the Lie algebra $\mathfrak{h}$

of D_n^+ by coordinates $t_1, \ldots, t_n$ satisfying the relation $t_1 + \cdots + t_n = 0$. The dimension of $\mathfrak{h}$ is $n-1$.

Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a co-compact lattice. We describe the invariant foliations for the Weyl chamber flow. Note that the derivative of the right multiplication by elements from H on $\Gamma \setminus M$ coincides with the inverse on H compose to the adjoint representation. To find the Lyapunov exponents it is enough to find the eigenvalues of this map. The invariant foliations can be obtained as before by taking the exponential of the subspaces generated by eigenvectors.

The following basis in $\mathfrak{sl}(n, \mathbb{R})$ consists of eigenvectors for $ad(A), A = diag(t_1, \ldots, t_n)$

$$N_i = v_{i+1,i+1} - v_{i,i}, 1 \leq i \leq n-1,$$
$$C_{i,j} = v_{i,j}, 1 \leq i, j \leq n, i \neq j.$$

A direct computation shows that

$$ad(A)(N_i) = AN_i - N_i A = 0,$$
$$ad(A)(C_{i,j}) = AC_{i,j} - C_{i,j} A$$
$$= diag(t_1, \ldots, t_n) v_{i,j} - v_{i,j} diag(t_1, \ldots, t_n) = (t_i - t_j) C_{i,j}.$$

Thus we can give the following description for the Lyapunov exponents.

Proposition 2.3.3 *Non-zero Lyapunov exponents for an element $a = diag(e^{t_1}, \ldots, e^{t_n})$ of the Weyl chamber flow on $SL(n, \mathbb{R})/\Gamma$ are $t_i - t_j$ where $i \neq j$ and $1 \leq i, j \leq n$. Zero Lyapunov exponent comes only from the orbit foliation and hence has multiplicity $n-1$. Consequently any matrix $d \in D_n^+$ whose elements are pairwise different acts normally hyperbolically on $SL(n, \mathbb{R})/\Gamma$ and hence is regular.*

For every $i \neq j$ the equation $t_i = t_j$ defines a Lyapunov hyperplane $H_{i,j} \subset D_n^+$. The connected components of

$$D_n^+ \setminus \bigcup_{i \neq j} H_{i,j}$$

are the Weyl chambers of the flow α. We recall that any element belonging to a Weyl chamber is regular element. The picture of the Weyl chambers for $n = 3$ is shown in Figure 2.1. The signs that appear in each chamber are the signs of half of the Lyapunov exponents of a regular element from the chamber with respect to a certain fixed basis. For this action, the Lyapunov exponents appear in pairs of opposite signs.

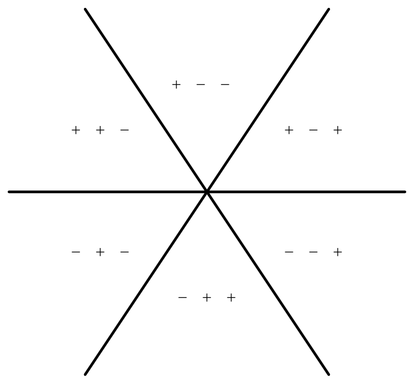

Fig. 2.1. Weyl chambers for $SL(3,\mathbb{R})$

One should note that the Weyl chamber flows on certain factors of $SL(n,\mathbb{R})$, especially on $SL(n,\mathbb{Z})/SL(n,\mathbb{R})$, which is identified with the space of all unimodular lattices in $\mathbb{R}^n$, appear in several problems in number theory. The former space, however, it is not compact.

Notice the highly resonant character of Lyapunov exponents, in contrast to the case of Cartan actions on the torus. Out of $n(n-1)$ exponents there are only $n-1$ independent ones, say $t_i - t_{i+1},\ i = 1,\dots,n-1$. The most important resonances geometrically are those which bring pairs of exponents differing by sign. Thus every Lyapunov hyperplane is the kernel of two exponents. This is in fact a general feature of all Weyl chamber flows.

2.3.4.2 Weyl chamber flow on Sp$(n,\mathbb{R})$

The symplectic group $\mathrm{Sp}(n,\mathbb{R})$ is the group of matrices that leaves invariant the exterior form:

$$x_1 \wedge x_{n+1} + x_2 \wedge x_{n+2} + \cdots + x_n \wedge x_{2n}.$$

Equivalently, it is the set of $2n \times 2n$ matrices g with real entries that satisfy $g^t J_n g = J_n$, where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $I_n \in \mathrm{GL}(n,\mathbb{R})$ is the identity. $\mathrm{Sp}(n,\mathbb{R})$ has a natural embedding in $SL(n,\mathbb{R})$ and its Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ can be identified with a Lie subalgebra of $\mathfrak{sl}(n,\mathbb{R})$:

$$\mathfrak{sp}(n,\mathbb{R}) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^t \end{pmatrix} \right\},$$

where X_1, X_2, X_3 are $n \times n$ matrices with real entries and in addition X_2, X_3 are symmetric matrices, that is $X_2^t = X_2, X_3^t = X_3$, where A^t is the transpose of the matrix A. The dimension of $\mathrm{Sp}(n,\mathbb{R})$ is $\frac{3n^2+n}{2}$.

Let H be the maximal $\mathbb{R}$-split torus in $\mathrm{Sp}(n,\mathbb{R})$ that is the exponential of the n dimensional Cartan subalgebra $\mathfrak{h}$ which has a basis consisting of the elements

$$v_{1,1} - v_{n+1,n+1}, v_{2,2} - v_{n+2,n+2}, \ldots, v_{n,n} - v_{2n,2n}.$$

The elements $A \in \mathfrak{h}$ and $\exp(A) \in H$ are diagonal matrices

$$\begin{aligned} A &= diag(t_1, t_2, \ldots, t_n, -t_1, \ldots, -t_n) \\ \exp(A) &= diag(e^{t_1}, \ldots, e^{t_n}, e^{-t_1}, \ldots, e^{-t_n}) \end{aligned} \tag{2.3.2}$$

with $t_1, \ldots, t_n$ real numbers.

Let $\Gamma \subset \mathrm{Sp}(n,\mathbb{R})$ be a co-compact lattice. We describe the invariant foliations for the Weyl chamber flow. We use the same procedure as before. The following basis in $\mathfrak{sp}(n,\mathbb{R})$ consists of eigenvectors for $ad(A), a \in \mathfrak{h}$ given by formula (2.3.2):

$$\begin{aligned} N_i &= v_{i,i} - v_{n+i,n+i}, 1 \le i \le n \\ C^1_{i,j} &= v_{i,j} - v_{n+j,n+i}, 1 \le i,j \le n, i \ne j \\ C^2_i &= v_{i,n+i}, 1 \le i \le n \\ C^3_{i,j} &= v_{i,n+j} + v_{j,n+i}, 1 \le i,j \le n, i > j \\ C^4_i &= v_{i+n,i}, 1 \le i \le n \\ C^5_{i,j} &= v_{i,j+n} + v_{j,i+n}, 1 \le i,j \le n, i > j. \end{aligned}$$

An easy computation shows that

$$\begin{aligned} ad(A)(N_i) &= 0 \\ ad(A)(C^1_{i,j}) &= (t_i - t_j)C^1_{i,j} \\ ad(A)(C^2_i) &= (2t_i)C^2_i \\ ad(A)(C^3_{i,j}) &= (t_i + t_j)C^3_{i,j} \\ ad(A)(C^4_i) &= (-2t_i)C^4_i \\ ad(A)(C^5_{i,j}) &= (-t_i - t_j)C^5_{i,j}. \end{aligned}$$

Thus we can give the following description for the Lyapunov exponents.

Proposition 2.3.4 *Non-zero Lyapunov exponents for a transformation* $a = diag(e^{t_1}, \ldots, e^{t_n}, e^{-t_1}, \ldots, e^{-t_n}) \in H$ *of the Weyl chamber flow on* $\Gamma \setminus Sp(n,\mathbb{R})$ *are* $t_i - t_j, t_j - t_i, 2t_i, -2t_i, t_i + t_j, -t_i - t_j$ *where* $i > j$

and $1 \le i, j \le n$. Zero Lyapunov exponent comes only from the orbit foliation and hence has multiplicity n. Consequently any matrix $a \in H$ whose elements are pairwise different in absolute value and different from 1 acts normally hyperbolically on $\Gamma \setminus Sp(n, \mathbb{R})$ and hence is regular.

2.3.4.3 Weyl chamber flow on $SO(n, n, \mathbb{R})^\circ$

The orthogonal group $\mathrm{SO}(n, n, \mathbb{R})$ has a natural embedding in $\mathrm{SL}(2n, \mathbb{R})$ as the group of matrices that keep invariant the quadratic form

$$-x_1^2 - \cdots - x_n^2 + x_{n+1}^2 + \cdots + x_{2n}^2.$$

Equivalently, it is the set of $2n \times 2n$ matrices g with real entries that satisfy $g^t I_{n,n} g = I_{n,n}$, where $I_{n,n} = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$ and $I_n \in \mathrm{GL}(n, \mathbb{R})$ is the identity. Its Lie algebra is denoted $\mathfrak{so}(n, n, \mathbb{R})$ and can be viewed as a subalgebra of $\mathfrak{sl}(2n, \mathbb{R})$:

$$\mathfrak{so}(n, n, \mathbb{R}) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{pmatrix} \right\},$$

where X_1, X_2, X_3 are $n \times n$ matrices with real entries and in addition X_1, X_3 are skew symmetric, that is $X_1^t = -X_1, X_3^t = -X_3$, where A^t is the transpose of the matrix A. The dimension of $\mathfrak{so}(n, n, \mathbb{R})$ is $\frac{n^2 - n}{2}$.

Let H be the maximal $\mathbb{R}$-split torus in $\mathrm{SO}(n, n, \mathbb{R})^\circ$ that is the exponential of the n dimensional Cartan subalgebra $\mathfrak{h}$ which has a basis consisting of the elements

$$v_{1,n+1} + v_{n+1,1}, v_{2,n+2} + v_{n+2,2}, \ldots, v_{n,2n} + v_{2n,n}.$$

Let $\Gamma \subset \mathrm{SO}(n, n, \mathbb{R})^\circ$ be a co-compact lattice. We describe the invariant foliations for the Weyl chamber flow. We use the same procedure once more.

Consider now the following vectors in $\mathfrak{so}(n, n, \mathbb{R})$

$$\begin{aligned} C_{i,j}^1 &= v_{i,j} - v_{j,i}, 1 \le i, j \le n, i < j \\ C_{i,j}^2 &= v_{i+n,j} - v_{j+n,i}, 1 \le i, j \le n, i < j \\ C_{i,j}^3 &= v_{j+n,i} + v_{i,j+n}, 1 \le i, j \le n, i < j \\ C_{i,j}^4 &= v_{i+n,j} + v_{j,i+n}, 1 \le i, j \le n, i < j. \end{aligned}$$

The following basis in $\mathfrak{so}(n, n, \mathbb{R})$ consists of eigenvectors for $ad(A)$,

$A \in \mathfrak{h}$, $A = \sum_{i=1}^{n} t_i N_i$:

$$\begin{aligned} N_i &= v_{i,n+i} + v_{n+i,i}, 1 \leq i \leq n, \\ D^1_{i,j} &= C^1_{i,j} + C^2_{i,j} + C^3_{i,j} + C^4_{i,j} \\ D^2_{i,j} &= C^1_{i,j} - C^2_{i,j} - C^3_{i,j} + C^4_{i,j} \\ D^3_{i,j} &= C^1_{i,j} + C^2_{i,j} - C^3_{i,j} - C^4_{i,j} \\ D^4_{i,j} &= C^1_{i,j} - C^2_{i,j} + C^3_{i,j} - C^4_{i,j}. \end{aligned}$$

An easy computation shows that

$$\begin{aligned} ad(A)N_i &= 0 \\ ad(A)D^1_{i,j} &= (t_i - t_j)D^1_{i,j} \\ ad(A)D^2_{i,j} &= (t_i + t_j)D^2_{i,j} \\ ad(A)D^3_{i,j} &= (-t_i + t_j)D^3_{i,j} \\ ad(A)D^4_{i,j} &= (-t_i - t_j)D^4_{i,j}. \end{aligned}$$

We can give now the following description for the Lyapunov exponents.

Proposition 2.3.5 *Non-zero Lyapunov exponents for a transformation* $a = \exp(A), A \in H, A = \sum_{i=1}^{n} t_i N_i$ *of the Weyl chamber flow on* $\Gamma \setminus SO(n,n,\mathbb{R})^\circ$ *are* $t_i - t_j, t_j - t_i, t_i + t_j, -t_i - t_j$ *where* $i \neq j$ *and* $1 \leq i, j \leq n$. *Zero Lyapunov exponent comes only from the orbit foliation and hence has multiplicity* n. *Consequently any matrix* $a \in H$ *whose eigenvalues are pairwise different in absolute value and different from* 1 *acts normally hyperbolically on* $\Gamma \setminus SO(n,n,\mathbb{R})^\circ$ *and hence is regular.*

2.3.4.4 Weyl chamber flow on $SU(m,n)$

Let $m \geq n$ positive integers. The unitary group $\mathrm{SU}(m,n)$ has a natural embedding in $\mathrm{SL}(m+n,\mathbb{C})$ as the group of matrices of determinant 1 that keep invariant the Hermitian form H:

$$-x_1\bar{x}_1 - \cdots - x_m\bar{x}_m + x_{m+1}\bar{x}_{m+1} + \cdots + x_{m+n}\bar{x}_{m+n}.$$

Equivalently, it is the set of $(m+n) \times (m+n)$ matrices g with complex entries that satisfy $g^t I_{m,n} \bar{g} = I_{m,n}$, where $I_{m,n} = \begin{pmatrix} -I_m & 0 \\ 0 & I_n \end{pmatrix}$ and $I_n \in \mathrm{GL}(n,\mathbb{R})$ is the identity.

In what follows we denote by $\bar{A}^t$ the complex conjugate transpose of the matrix A. The Lie algebra $\mathfrak{su}(m,n)$ of $\mathrm{SU}(m,n)$ can be expressed as the set of $2(m+n) \times 2(m+n)$ matrices:

$$\begin{pmatrix} X_1 & X_2 \\ \bar{X}_2^t & X_3 \end{pmatrix},$$

where X_1, X_2, X_3, X_4 are complex matrices, X_2 is arbitrary of order $m \times n$, X_1 of order $m \times m$ and $\bar{X}_1^t = X_1$, X_3 of order $n \times m$ and $\bar{X}_3^t = X_3$, and in addition $\mathrm{Tr}(X_1) + \mathrm{Tr}(X_3) = 0$.

Alternatively, and this is the approach we use in the sequel, one can choose a base $(e_i)_{i=1}^{m+n}$ in $\mathbb{C}^{m+n}$ (under a linear transformation that has real coefficients) for which

$$\begin{aligned} &H(e_i, e_{i+n}) = 1, \quad 1 \le i \le n, \\ &H(e_j, e_j) = 1, \quad 2n+1 \le j \le m+n, \\ &H(e_i, e_j) = 0, \quad \text{otherwise.} \end{aligned}$$

Using this base, the Lie algebra $\mathfrak{su}(m,n)$ can be expressed as the set of $(m+n) \times (m+n)$ matrices

$$\begin{pmatrix} X_1 & X_2 & Y_1 \\ X_3 & X_4 & Y_2 \\ Z_1 & Z_2 & W \end{pmatrix},$$

where all entries are complex matrices, X_1, X_2, X_3, X_4 are $n \times n$ matrices, Y_1, Y_2 are $n \times (m-n)$ matrices, Z_1, Z_2 are $(m-n) \times n$ matrices and W is a $(m-n) \times (m-n)$ matrix. In addition

$$\begin{aligned} &X_1 = -\bar{X}_4^t, \quad \bar{X}_2^t = -X_2, \quad \bar{X}_3^t = -X_3, \\ &\bar{W}^t = -W, \quad Y_1 = -\bar{Z}_2^t, \quad Y_2 = -\bar{Z}_1^t, \\ &\mathrm{Tr}(X_1) + \mathrm{Tr}(X_4) + \mathrm{Tr}(W) = 0. \end{aligned}$$

The $\mathbb{R}$-dimension of $\mathfrak{su}(m,n)$ is $(m+n)^2 - 1$.

We denote by $v_{i,j}$ the $(m+n) \times (m+n)$ matrix that has the entry (i,j) equal to 1 and all other entries equal to zero. Let H be the maximal $\mathbb{R}$-split torus in $\mathrm{SU}(m,n)$ that is the exponential of the n dimensional Cartan subalgebra $\mathfrak{h}$ which has a basis consisting of the elements

$$N_i = v_{i,i} - v_{n+i,n+i}, 1 \le i \le n. \tag{2.3.3}$$

Let $\Gamma \subset \mathrm{SU}(m,n)$ co-compact lattice. We describe now the foliations for the Weyl chamber flow induced by H on $\Gamma \setminus \mathrm{SU}(m,n)$. We use the same approach as for the other Weyl chamber flows.

The following elements in $su(m,n)$ are eigenvectors for $ad(A), A \in$

$\mathfrak{h}, A = \sum_{i=1}^{n} t_i N_i$:

$$\begin{aligned}
&A^1_{i,j} = v_{i,j+n} - v_{j,i+n}, 1 \leq i,j \leq n, i < j,\\
&A^2_{i,j} = i(v_{i,j+n} + v_{j,i+n}), 1 \leq i,j \leq n, i < j,\\
&B^1_{i,j} = v_{i,j} - v_{j+n,i+n}, 1 \leq i,j \leq n, i \neq j,\\
&B^2_{i,j} = i(v_{i,j} + v_{j+n,i+n}), 1 \leq i,j \leq n, i \neq j,\\
&C^1_{i,j} = v_{j+n,i} - v_{i+n,j}, 1 \leq i,j \leq n, i < j,\\
&C^2_{i,j} = i(v_{j+n,i} + v_{i+n,j}), 1 \leq i,j \leq n, i < j,\\
&D^1_{i,l} = v_{i,2n+l} - v_{2n+l,i+n}, 1 \leq i \leq n, l \leq m-n,\\
&D^2_{i,l} = i(v_{i,2n+l} + v_{2n+l,i+n}), 1 \leq i \leq n, l \leq m-n,\\
&E^1_{i,l} = v_{i+n,2n+l} - v_{2n+l,i+n}, 1 \leq i \leq n, l \leq m-n,\\
&E^2_{i,l} = i(v_{i+n,2n+l} + v_{2n+l,i+n}), 1 \leq i \leq n, l \leq m-n,\\
&F_i = iv_{i+n,i}, 1 \leq i \leq n,\\
&G_i = iv_{i,i+n}, 1 \leq i \leq n.
\end{aligned} \tag{2.3.4}$$

One can complete the set of vectors given by (2.3.3) and (2.3.4) to a base for $\mathfrak{su}(m,n)$ by adding vectors in a base for the compact part of the centralizer H.

An easy computation shows that

$$\begin{aligned}
ad(A)N_i &= 0\\
ad(A)A^1_{i,j} &= (t_i + t_j)A^1_{i,j}\\
ad(A)A^2_{i,j} &= (t_i + t_j)A^2_{i,j}\\
ad(A)B^1_{i,j} &= (t_i - t_j)B^1_{i,j}\\
ad(A)B^2_{i,j} &= (t_i - t_j)B^2_{i,j}\\
ad(A)C^1_{i,j} &= (-t_i - t_j)C^1_{i,j}\\
ad(A)C^2_{i,j} &= (-t_i - t_j)C^2_{i,j}\\
ad(A)D^1_{i,l} &= t_i D^1_{i,l}\\
ad(A)D^2_{i,l} &= t_i D^2_{i,l}\\
ad(A)E^1_{i,l} &= -t_i E^1_{i,l}\\
ad(A)E^2_{i,l} &= -t_i E^2_{i,l}\\
ad(A)F_i &= -2t_i F_i\\
ad(A)G_i &= 2t_i G_i.
\end{aligned}$$

We give the description for the Lyapunov exponents of the Weyl chamber flow.

Proposition 2.3.6 *Non-zero Lyapunov exponents for a transformation* $a = \exp(A), A \in H, A = \sum_{i=1}^{n} t_i N_i$ *of the Weyl chamber flow on* $\Gamma \setminus SU(m,n)$ *are* $t_i + t_j, t_i - t_j, t_j - t_i, -t_i - t_j, t_i, -t_i$*, all appearing with multiplicity* 2*, and* $2t_i, -2t_i$*, appearing with multiplicity* 1*, where* $i \neq j$ *and* $1 \leq i, j \leq n$*. If* $m = n$ *the exponents* $t_i, -t_i$ *do not appear. Zero Lyapunov exponent comes from the orbit foliation and from the compact part of the centralizer of the maximal* $\mathbb{R}$*-split Cartan subgroup, and has multiplicity* $2n - 1 + (m-n)^2$*. Consequently, any matrix* $a \in H$ *whose eigenvalues are pairwise different in absolute value and different from* 1 *acts only partially hyperbolically on* $\Gamma \setminus SU(m,n)$*.*

2.3.5 Twisted Weyl chamber flows and further extensions

Now we describe another class of algebraic Anosov action of $\mathbb{R}^n$ which is obtained from the Weyl chamber flows by a very special extension procedure. This example appears in [78], Example 2.7.

Let G, Γ, A and M be as in Section 2.3.3.2. Let $\rho : \Gamma \to SL(n, \mathbb{Z})$ be a representation of Γ which is irreducible over $\mathbb{Q}$. Then Γ acts on the n-torus $\mathbb{T}^n$ via ρ and hence on the product space $(M \setminus G) \times \mathbb{T}^n$ via

$$\gamma(x,t) = (x\gamma^{-1}, \rho(\gamma)(t)).$$

Let $N = (M \setminus G \times \mathbb{T}^n)/\Gamma$ be the orbit space of this action. As the action of A on $M \setminus G \times \mathbb{T}^n$ given by $a(x,t) = (ax, t)$ commutes with the Γ-action, it induces an action of A on N.

Assume that $\rho(\gamma)$ for some element $\gamma \in \Gamma$ is an Anosov diffeomorphism on $\mathbb{T}^n$. The image under ρ of the center of Γ is finite by Schur lemma. Hence Γ may be assumed to be a lattice in a semisimple Lie group with finite center. By Margulis' superrigidity theory [110], semisimplicity of the algebraic hull H of $\rho(\Gamma)$ and existence of a hyperbolic element $\rho(\gamma)$, which guarantee that the image is not compact, the representation ρ of Γ extends to a homomorphism $G \to H_{ad}$, where H_{ad} is the adjoint group of H. Assume now that $\rho(\Gamma)$ has a trivial center (otherwise one can consider the case of an orbifold). Since Γ is co-compact, γ is a semisimple element of G. Let $\gamma = k_\gamma s_\gamma$ be the decomposition of γ into compact and split semisimple parts. Then s_γ is conjugate to an element $a \in A$. As ρ extends to G, it follows that $\rho(s_\gamma)$ and $\rho(a)$ have no eigenvalues of absolute value 1. Moreover, one can pick a such that $\log a$ belongs to an open Weyl chamber of the Lie algebra $\mathfrak{g}$. Then it follows from Theorem 2.3.1 that a acts normally hyperbolic on $M \setminus G/\Gamma$.

One shows now that the action of a on N is hyperbolic as well. Let $(x,t) \in N$. Since Γ is co-compact, there is a uniformly bounded sequence of elements $u_n \in G$ such that $x^{-1}a^n x = u_n(x)\gamma_n(x)$, for some $\gamma_n(x) \in \Gamma$. Since $u_n(x)$ is uniformly bounded in x and n, the stable tangent vectors for $x^{-1}ax$ are exponentially contracted by $\gamma(x_n)$ with estimates uniform in x. Same conclusion applies to unstable vectors. Thus

$$a^n(x,t) = (x(x^{-1}a^n x), t) = (xu_n, \rho(\gamma_n)t)$$

and since a acts normally hyperbolic on $M \setminus G/\Gamma$, it follows that a is normally hyperbolic with respect to the orbit foliation of A.

The above construction can be generalized by considering extensions by automorphisms of a torus of other higher rank actions for which one of the monodromy elements is Anosov. For example, using a twisted Weyl chamber flow as above as the base we obtain nilmanifold extensions of the Weyl chamber flow. A. Starkov pointed out that one can also start with the product of a Weyl chamber flow with a transitive action of some $\mathbb{R}^l$ on a torus and produce a toral extension which is Anosov and no finite cover splits as a product. These two extension constructions can be combined and iterated.

2.3.6 Reduction of scalars

For completeness we give a brief presentation of the available methods of constructing lattices in a semisimple Lie group G. More details can be found in [110], [178] and [179]. In particular, this discussion will allow us to show an explicit example of a twisted Weyl chamber flow.

If G is a semisimple Lie group, and, in addition, a matrix group, the simplest way to construct a lattice is to take the integer points in G. For example take $SL(n,\mathbb{Z})$ in $SL(n,\mathbb{R})$. This construction always gives non-compact lattices because of the existence of unipotent elements in the lattice. To obtain co-compact lattices, one can use a standard construction called in the literature "restriction of scalars" [176].

Recall that an algebraic variety over a closed field K is the set of zeroes in K^l of a finite family of polynomials in $K[x_1,\dots,x_l]$. If V is an algebraic variety in K^l, and A is a subring of K, then we denote by V_A the set of A-points in V, that is, $V \cap A^l$. If $k \subset K$ is a subfield, an algebraic variety V over K is called k-variety if the polynomials defining the variety have all coefficients in k. Suppose now that k is an algebraic number field, that is, $\mathbb{Q} \subset k \subset \mathbb{C}$ and $d = [k : \mathbb{Q}] < \infty$, and G is an algebraic k-group. Then there are d distinct field embeddings $\sigma_1 =$identity,

$\sigma_2, \ldots, \sigma_d$ of k into $\mathbb{C}$ which are linearly independent in the $\mathbb{C}$-vector space of all functions $k \to \mathbb{C}$, and such that, if $\alpha \in k$, then $\sigma_i(\alpha) = \alpha$ for all i if and only if $\alpha \in \mathbb{Q}$. Any embedding $\sigma : k \to \mathbb{C}$ can be extended to an automorphism of $\mathbb{C}$, and moreover, defines a $\mathbb{C}$-isomorphism from $\mathbb{C}^l$ into itself. We denote by G^σ the image of G under σ, which is an algebraic $\sigma(k)$-group.

For $k, \sigma_1, \ldots, \sigma_d$ as above, let $R_{k/\mathbb{Q}}(G) = \prod_{i=1}^d G^{\sigma_i}$, and for $g \in G_k$ let $g' = (\sigma_1(g), \ldots, \sigma_d(g))$. Let $(G_k)' = \{g' | g \in G_k\}$. Then $R_{k/\mathbb{Q}}(G)$ is isomorphic to an algebraic $\mathbb{Q}$-group such that $(R_{k/\mathbb{Q}}(G))_\mathbb{Q} = (G_k)'$ and $(R_{k/\mathbb{Q}}(G))_\mathbb{Z} = (G_\mathcal{O})'$, where $\mathcal{O} \subset k$ is the subring of algebraic integers in k. The projection map $R_{k/\mathbb{Q}}(G) \to G$ onto the first factor is defined over k and defines bijections $(R_{k/\mathbb{Q}}(G))_\mathbb{Q} \to G_k$ and $(R_{k/\mathbb{Q}}(G))_\mathbb{Z} \to G_\mathcal{O}$. In particular $G_\mathcal{O}$ is isomorphic to a lattice in $(R_{k/\mathbb{Q}}(G))_\mathbb{R}$.

We illustrate the restrictions of scalars by the following example.

Example 2.3.7 *Let $G = SO(4,2)$, which is viewed as imbedded naturally in $SL(6,\mathbb{R})$ as the group of transformations that preserves the bilinear form*

$$< x, y >= -\sqrt{2}(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4) + x_5y_5 + x_6y_6.$$

This form and therefore G are defined over the field $\mathbb{Q}(\sqrt{2})$ which has a conjugation σ defined by $\sigma(\sqrt{2}) = -\sqrt{2}$. Define $\Gamma = G(\mathbb{Z}[\sqrt{2}])$. Then there exists an embedding of Γ into $H = SO(4,2) \times SO(6)$ given by $\gamma \in \Gamma$ to $(\gamma, \sigma(\gamma))$. It is easy to check that this embedding is discrete. Moreover, Γ is embedded as integral points for the rational structure on H with rational points $(m, \sigma(m))$, where $m \in G(\mathbb{Q}(\sqrt{2}))$. This implies that Γ is a lattice in H. Moreover, Γ projects to a lattice in G, since G is co-compact in H, and it follows from the general theory in [110] that this lattice is actually co-compact in H.

Observe now that $SO(4,2)$ has real rank 2 and two connected components. Choose the maximal abelian Lie subalgebra $\mathfrak{h} \subset so(4,2)$ generated by

$$v_{1,6} + v_{6,1}, v_{2,5} + v_{5,2},$$

with the corresponding abelian subgroup $H = \exp(\mathfrak{h})$, and let $M = SO(2)$

be embedded in $SO(4,2)$ as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the action of H on $(M \setminus SO(4,2)^o)/\Gamma$ induces a Weyl chamber flow. To show an example of twisted Weyl chamber flow, observe that $\Gamma \subset SL(6, \mathbb{Z}[\sqrt{2}]$ has an embedding in $SL(12, \mathbb{Z})$ if we identify $\mathbb{Z}^{12}$ with the image in $\mathbb{R}^{12}$ of $\mathbb{Z}[\sqrt{2}]^6$ via the embedding $x \to (x, \sigma(x))$. Then the action of H on the manifold $N = (M \setminus SO(4,2)^o \times \mathbb{T}^{12})/\Gamma$ induces a twisted Weyl chamber flow.

2.4 Affine actions beyond tori and nilmanifolds

2.4.1 Noninvertible examples on tori and nilmanifolds and relations to solenoids

Let α be a $\mathbb{Z}_+^k$ action by endomorphisms of a torus $\mathbb{T}^m$ defined by matrices $A_1, \ldots, A_k$. The natural extension of can be identified with a $\mathbb{Z}^k$ action $\alpha^* : \mathbb{Z}^k \to \mathrm{Aut}(S)$ by automorphisms of a *solenoid* S. The solenoid is a compact abelian group modeled locally on the product of an Euclidean space (the so called Archimedean directions) and several additive groups of p-adic integers (the so called non-Archimedean directions). The solenoid can be realized as a subset of $(\mathbb{T}^m)^{\mathbb{Z}^k}$ as follows. Let σ_i be the i'th shift on $\mathbb{Z}^k$, that is, $\sigma_i(a_1, \ldots, a_i, \ldots, a_k) = (a_1, \ldots, a_i + 1, \ldots, a_k)$, and define

$$S = \{\omega \in (\mathbb{T}^m)^{\mathbb{Z}^k} | \omega_{\sigma_i j} = A_i \omega_j\}.$$

The solenoid is a compact subgroup of $(\mathbb{T}^m)^{\mathbb{Z}^k}$ considered with the product topology. Its dual is a subgroup of $\mathbb{Q}^m$ and is included in $(\mathbb{Z}(p_1, \ldots, p_l))^m$, where $p_1, \ldots, p_l$ are the prime integers that appear in the prime decomposition of the determinants of at least one of the $A_1, \ldots, A_k$ and $\mathbb{Z}(p_1, \ldots, p_l)$ is the set of rational numbers having denominators with prime divisors only $p_1, \ldots, p_l$. The group $\mathbb{Z}^k$ acts on S by coordinates shifts. The solenoid is a fibration over $\mathbb{T}^m$ with the projection given by $S \ni \omega \to \omega(0, \ldots, 0) \in \mathbb{T}^m$. The fibers are Cantor sets. The projection intertwines the restriction of α^* to $\mathbb{Z}_+^k$ and the action α.

On the solenoid there exists a Hölder structure that can be introduced using any metric on the product space of the form:

$$d_\lambda(\omega, \omega') = \sum_{j\in\mathbb{Z}^k} \frac{d(\omega_j, \omega'_j)}{\lambda^{\|j\|}},$$

where $\lambda > 1$ and d is a metric on the torus. The Hölder structure is independent of the constant λ. The structure can be used to define exponential convergence along the fibers and hence Lyapunov exponents. Note that the Lyapunov exponents split into Archimedean and non-Archimedean. The Weyl chambers analysis extends to this case although the space of the action is no longer a manifold.

One of the simplest and most famous example is Furstenberg's $\times 2, \times 3$ action on a circle [40].

Example 2.4.1 *The action $E_{2,3}$ of $\mathbb{Z}^2_+$ on the circle is generated by the endomorphisms:*

$$E_2 : S^1 \to S^1, x \mapsto 2x, \quad (\ mod\ 1)$$

and

$$E_2 : S^1 \to S^1, x \mapsto 3x, \quad (\ mod\ 1).$$

Closed orbits of this actions are those of rational numbers whose denominators are relatively prime with 2 and 3. Thus, in particular the orbit of $\frac{1}{2^n-1}$ is closed if $2^n \equiv 2,\ (\mathrm{mod}\ 3)$. Its E_2 orbit has period $n-1$ and consists of the points $\frac{2^k}{2^n-1}$ $k = 1, \ldots, n-1$, and hence for large n is concentrated mostly around 0 while E_3 orbit typically is almost uniformly distributed.

The natural extension $S_{2,3}$ of $E_{2,3}$ acts on the dual group of the discrete group $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$. Topologically it is a connected but not locally connected one-dimensional compact space locally modeled on the direct product of $\mathbb{R}$ and the Cantor set. As a group it is an extension of S^1 with the product of dyadic integers and 3-adic integers $\mathbb{Z}_2 \times \mathbb{Z}_3$ in the fiber.

One can identify the discrete time $\mathbb{Z}^2$ with the integer lattice in the plane $\mathbb{R}^2$ with coordinates s, t. There are three Lyapunov exponents for $S_{2,3}$: one Archimedean:

$$t \log 2 + s \log 3$$

and two non-Archimedean:

$$-t\log 2 \text{ and } -s\log 3.$$

This can be seen from the observation that multiplication by two acts an an isometry on $\mathbb{Z}_3$ and as a contraction with constant coefficient of contraction $1/2$ on $\mathbb{Z}_2$, and correspondingly, the multiplication by three acts as an isometry on $\mathbb{Z}_2$ and as a contraction with coefficient $1/3$ on $\mathbb{Z}_3$. Thus in this example there are three Lyapunov lines in general position:

$$\begin{aligned} t\log 2 + s\log 3 &= 0 \\ t &= 0 \\ s &= 0 \end{aligned} \tag{2.4.1}$$

and six Weyl chambers. Combinatorially, the picture looks exactly as for any Cartan action of $\mathbb{Z}^2$ on $\mathbb{T}^3$. The positive quadrant constitutes a Weyl chamber, namely the one where the Archimedean exponent is positive and the other two non-Archimedean exponents are negative.

Similar examples can by constructed on nilmanifolds. The simplest one is three dimensional.

Example 2.4.2 *Let H be the Heisenberg group of 3×3 upper diagonal matrices, that is*

$$N = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R}.$$

$\Lambda \subset H$ be the subgroup of integer matrices, $\rho_2 : H \to H$ be the automorphism

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2x & 4y \\ 0 & 1 & 2z \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\rho_2(\Lambda) \subset \Lambda$ and ρ_2 projects to a non-invertible map on the compact nilmanifold $X = H/\Lambda$ [67, Section 17.3]. One can similarly define the automorphism $\rho_3 : H \to H$ by replacing the multiplications by 2 and 4 with multiplications by 3 and 9. The projections of ρ_2 and ρ_3 to X define an expanding action of $\mathbb{Z}_+^2$.

In the previous example there are two Archimedean Lyapunov exponents. Using the coordinates from the previous example they can be written as

$$\chi_- = t\log 2 + s\log 3$$

and

$$\chi_+ = t \log 4 + s \log 9.$$

The exponent χ_- has multiplicity 2 and χ_+ is simple. The Lyapunov distribution of χ_- is nonintegrable. The relation between the exponents $\chi_+ = 2\chi_-$ is a simple example of resonance. Note that the projection of the abelian action to the center gives the action from example 2.4.1.

2.4.2 Automorphisms of other compact abelian groups

In this section we discuss $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups. These actions are natural generalizations of abelian actions by automorphisms of tori and solenoids discusses above. The key to their study is the interplay with commutative algebra and algebraic geometry. This is a well developed area of dynamics. Our goal is to briefly describe some basic results and techniques, and to present several examples of such actions that do not fall in the category of automorphisms of a torus. This will allow the reader for a comparison with the theory and examples presented earlier in this chapter. While the phase spaces for those actions are not any more finite-dimensional manifolds or even finite-dimensional objects in the sense the solenoids are, some of them exhibit similar rigidity properties. For a more complete discussion of the topics presented in the section we refer the reader to the monograph [157].

Definition 2.4.3 *Let X be a compact abelian group and $d \geq 1$. An* algebraic $\mathbb{Z}^d$-action *on X, $\mathbf{n} \to \alpha^{\mathbf{n}}$, is a $\mathbb{Z}^d$ action by continuous automorphisms of X.*

Any algebraic action α on the compact abelian group X has a natural invariant measure, namely the normalized Haar measure λ_X. Thus one can introduce in the standard way the notions of ergodicity, mixing, or Bernoulli with respect to this invariant measure.

We start to describe the relationship between abelian compact group actions and commutative algebra. For proofs of basic results in commutative algebra we refer to [90].

Let $d \geq 1$ integer, and let $R_d = \mathbb{Z}[u_1^{\pm}, \ldots, u_d^{\pm}]$ be the ring of Laurent polynomials with integral coefficients in the commuting variables

$u_1, \ldots, u_d$. An element $f \in R_d$ can be described as

$$f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \tag{2.4.2}$$

where $u^{\mathbf{n}} = u_1^{n_1} \ldots u_d^{n_d}$, $c_f(\mathbf{n}) \in \mathbb{Z}^d$ for every $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, and $c_f(\mathbf{n}) = 0$ for all but finitely many $\mathbf{n}$.

The dual group $M = \hat{X}$, consisting of the characters of X, and viewed as an additive group, is a module over the ring R_d with the scalar multiplication given by

$$f \cdot a = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) \widehat{\alpha^{\mathbf{n}}}(a)$$

for $f \in R_d, a \in M$, where $\widehat{\alpha^{\mathbf{n}}}$ is the automorphism of $\hat{X}$ dual to $\alpha^{\mathbf{n}}$. In particular $u^{\mathbf{n}} \cdot a = \widehat{\alpha^{\mathbf{n}}}(a)$. The module M is called the *dual module* of the action α.

Conversely, any R_d-module M determines an algebraic $\mathbb{Z}^d$ action α_M on the compact abelian group $X_M = \widehat{M}$ with $\widehat{\alpha}_M^{\mathbf{n}}$ being the dual to multiplication by $u^{\mathbf{n}}$ on M for every $\mathbf{n} \in M$.

The simplest examples of R_d-modules that can be used to construct abelian actions are the cyclic ones, that is those of the form $M = R_d/I$, where $I \subset R_d$ is an ideal. We recall that a module is called *Noetherian* if every strictly increasing sequence of submodules $M \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots$ is finite. Since the ring R_d, as a module over itself, is Noetherian, a module over R_d is Noetherian if and only if it is finitely generated. In particular, M is Noetherian if and only if there exist elements $a_1, \ldots, a_n$ such that $M = R_d a_1 + \cdots + R_d a_n$, and every cyclic R_d-module is Noetherian.

We describe the abelian action induced by a Noetherian R_d-module.

Example 2.4.4 *Let $M = R_d$. The module R_d is isomorphic as a group to the direct sum $\oplus_{\mathbb{Z}^d}\mathbb{Z}$, and its dual group $\widehat{R}_d$ is isomorphic to the cartesian product $\mathbb{T}^{\mathbb{Z}^d}$ of copies of the torus $\mathbb{T}$. If $x = (x_n)_{n\in\mathbb{Z}^d}$ is a generic element in $\oplus_{\mathbb{Z}^d}\mathbb{Z}$, one can identify $\widehat{R}_d$ and $\mathbb{T}^{\mathbb{Z}^d}$ via the duality bracket*

$$< x, f >= e^{2\pi i \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}}},$$

where f is given by (2.4.2). Under this identification the $\mathbb{Z}^d$-action α_{R_d} on $\mathbb{T}^{\mathbb{Z}^d}$ becomes the shift action given by:

$$[(\alpha_{R_d})^{\mathbf{n}}(x)]_m = x_{n+m} \text{ for all } m, n \in \mathbb{Z}.$$

Example 2.4.5 *More general, let $I \subset R_d$ be an ideal and $M = R_d/I$. Note that any ideal is a submodule, hence an $\widehat{\alpha}_{R_d}$-invariant subgroup. The dual group $\widehat{M}$ is the α_{R_d}-invariant subgroup of $\widehat{R}_d = \mathbb{T}^{\mathbb{Z}^d}$*

$$\begin{aligned}\widehat{M} &= \{x \in \mathbb{T}^{\mathbb{Z}^d} :< x, f >= 1 \text{ for all } f \in I\} \\ &= \{x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}} \text{ (mod 1) for all } f \in I, \mathbf{n} \in \mathbb{Z}^d\}\end{aligned} \tag{2.4.3}$$

and the action $\alpha_{R_d/I}$ is the restriction of the action α_{R_d} to the shift invariant subgroup $\widehat{M}$.

Conversely, one can start with $X \subset \mathbb{T}^{\mathbb{Z}^d}$ closed subgroup, and let

$$X^{ann} = \{f \in R_d :< x, f >= 1 \text{ for every } x \in X\}$$

be the annihilator of X in $\widehat{R}_d$. Then X is shift invariant if and only if X^{ann} is an ideal in R_d.

Example 2.4.6 *Let $d = 1$ and let I be the ideal generated by p, some prime number p. Then the dual action induced on R_1/I is the two-sided full shift on p symbols.*

Example 2.4.7 *Even more general, let M be a R_d-Noetherian module, and let $\{a_1, ..., a_n\}$ be a set of generators for M. The surjective homomorphism $(f_1, \ldots, f_k) \to f_1 a_1 + \cdots + f_k a_k$ from R_d^k to M induces a dual injective homomorphism $\phi : \widehat{M} \to \widehat{R_d^k} \cong (\mathbb{T}^k)^{\mathbb{Z}^d}$ such that $\phi \alpha_m^{\mathbf{n}} = \alpha_{R_d^k}^{\mathbf{n}} \phi$ for all $\mathbf{n} \in \mathbb{Z}^d$. In particular ϕ embeds $\widehat{M}$ as a closed shift invariant subgroup of $(\mathbb{T}^k)^{\mathbb{Z}^d}$.*

We observed that an algebraic $\mathbb{Z}^d$-action α is completely determined by the dual module M. Consequently, one can try to express the dynamical properties of α in terms of algebraical properties of the module M.

Definition 2.4.8 *A prime ideal $\mathfrak{p} \subset R_d$ is said to be associated with the R_d-module M if*

$$\mathfrak{p} = \{f \in R_d : f \cdot a = 0_M\}$$

for some $a \in M$. The set of prime ideals associated with a Noetherian R_d-module M is finite.

Definition 2.4.9 *If $I \subset R_d$ is an ideal we denote by $V_{\mathbb{C}}(I)$ the variety of I, that is*

$$V_{\mathbb{C}}(I) = \{c = (c_1, \ldots, c_d) \in (\mathbb{C}^*)^d : f(c) = 0 \text{ for all } f \in I\}$$

where $\mathbb{C}^* = \mathbb{C} - \{0\}$.

The following basic results are part of Theorem 6.5 in [157].

Theorem 2.4.10 *Let $\mathfrak{p} \subset R_d$ be a prime ideal and $\alpha = \alpha_{R_d/\mathfrak{p}}$ be the algebraic $\mathbb{Z}^d$ action on $X := \widehat{R_d/\mathfrak{p}}$.*

(i) *For every $\mathbf{n} \in \mathbb{Z}^{\mathbf{d}}$ the following are equivalent:*
   (a) *$\alpha^{\mathbf{n}}$ is ergodic;*
   (b) *$\mathfrak{p} \cap \{u^{l\mathbf{n}} - 1 : l \geq 1\} = \emptyset$.*

(ii) *The following conditions are equivalent:*
   (a) *α is ergodic;*
   (b) *$\alpha^{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^{\mathbf{d}}$;*
   (c) *$\{u^{\mathbf{n}} - 1 : \mathbf{n} \in \Gamma\} \nsubseteq \mathfrak{p}$ for every subgroup of finite index $\Gamma \subset \mathbb{Z}^d$.*

(iii) *The following conditions are equivalent:*
   (a) *α is mixing (either topologically or w.r.t. Haar measure λ_X);*
   (b) *for every nonzero $\mathbf{n} \in \mathbb{Z}^n$, $\alpha^{\mathbf{n}}$ is ergodic;*
   (c) *for every nonzero $\mathbf{n} \in \mathbb{Z}^n$, $\alpha^{\mathbf{n}}$ is mixing;*
   (d) *$\mathfrak{p} \cap \{u^{\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} = \{0\}$.*

(iv) *The following are equivalent:*
   (a) *α is expansive;*
   (b) *$V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S}^d = \emptyset$, where $\mathbb{S} \subset \mathbb{T}$ is the unit circle.*

We show now more examples of abelian actions and determine their dynamics using the last theorem. We first consider the case of a toral automorphism and $d = 1$, which is instructive in itself.

Example 2.4.11 *Let α be the automorphism of $\mathbb{T}^2$ determined by the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in GL(n, \mathbb{Z})$ with characteristic irreducible polynomial $f(u) = u^2 - u - 1$. Note that A is equal to the companion matrix of f. It follows from Example 2.4.5 that*

$$\widehat{R_1/(f)}$$
$$= \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : x_n + x_{n+1} - x_{n+2} = 0 \text{ (mod 1) for all } n \in \mathbb{Z}\},$$

where $(f) = fR_1 \subset R_1$ *is the principal prime ideal generated by* f *and the action* $\alpha_{R_1/(f)}$ *is the shift on* $\widehat{R_1/(f)} \subset \mathbb{T}^{\mathbb{Z}}$*. The projection on the coordinates* $0, 1$*,* $\pi_{0,1} : \widehat{R_1/(f)}(\subset \mathbb{T}^{\mathbb{Z}}) \to \mathbb{T}^2$*, gives an algebraic conjugacy between the actions* α *and* $\alpha_{R_1/(f)}$*. Theorem 2.4.10, (3),d), immediately implies that* α *is mixing and expansive.*

Example 2.4.12 *More general, let* α *be an automorphism of* $\mathbb{T}^n$ *determined by a matrix* $A \in GL(n, \mathbb{R})$*, and let* f *be the characteristic polynomial of* A*. The associated prime ideals of the module* $\widehat{\mathbb{T}}^n$ *are in one to one correspondence to the principal prime ideals arising from the irreducible divisors of* f*. Theorem 2.4.10 implies now that* A *is ergodic (or mixing) if and only if* A *does not have any eigenvalue that is root of unity. Moreover,* α *is expansive if* f *has no eigenvalue of absolute value 1.*

We recall that if $f = c_0 + \cdots + c_{n-1}u^{n-1} + u^n$ *is the characteristic polynomial of* A*, then its companion matrix is*

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -c_0 & -c_1 & -c_3 & \dots & -c_{n-2} & -c_{n-1} \end{pmatrix}.$$

If A *is equal to its companion matrix, then the projection*

$$\phi : \widehat{R_1/(f)}(\subset \mathbb{T}^{\mathbb{Z}}) \to \mathbb{T}^n$$

on the coordinates $0, 1, \dots, n-1$ *is an algebraic conjugacy between the actions* $\alpha_{R_1/(f)}$ *and* α*.*

In general the matrix A *is conjugate to the companion matrix of* f *over* $\mathbb{Q}$*, but not over* $\mathbb{Z}$*. The dual module* $M = \widehat{\mathbb{T}^n}$ *of* α *has a submodule of finite index* $N \subset M$ *that is isomorphic to* $R_1/(f)$*. From this it follows that there are continuous surjective finite-to-one group homomorphisms* $\phi_1 : \mathbb{T}^n \to \widehat{R_1/(f)}$ *and* $\phi_2 : \widehat{R_1/(f)} \to \mathbb{T}^n$ *such that* $\phi_1\alpha = \alpha_{R_1/(F)}\phi_1$ *and* $\alpha\phi_2 = \phi_2\alpha_{R_1/(f)}$*. See Section 9 in [157] for details.*

Example 2.4.13 *Let* I *be the ideal generated in* R_2 *by* $u_1 - 2$ *and* $u_2 - 3$*. Then*

$$\widehat{R_2/I} = \{x \in \mathbb{T}^{\mathbb{Z}^2} | x_{m+1,n} = 2x_{m,n}, x_{m,n+1} = 3x_{m,n} \text{ for all } m, n \in \mathbb{Z}\}.$$

This system is the invertible extension of the $\times 2, \times 3$ *abelian action in Example 2.4.1.*

We show now an example of higher rank algebraic $\mathbb{Z}^d$-action that is not given by homomorphisms of a torus.

Example 2.4.14 *Let*

$$X = \{x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{m,n} + x_{m+1,n} + x_{m,n+1} = 0 \text{ for all } m, n \in \mathbb{Z}\}.$$

Equivalently, X *is the dual group of* R_2/I*, where* I *is the ideal generated by* 2 *and* $1 + u_1 + u_2$*.*

The set X *is a compact totally disconnected group which is invariant under both horizontal and vertical shifts on* $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$*. We define* α *to be the* $\mathbb{Z}^2$*-action induced on* X *by these shifts.*

It was shown by Ledrappier [93] that α *is mixing of order two but not mixing of order three. This is in contrast to the case of a* $\mathbb{Z}^d$*-action by automorphisms of a torus. Indeed, it was shown by Schmidt and Ward [159] that every mixing* $\mathbb{Z}^d$*-action by automorphisms of a compact, connect, abelian group is mixing of all orders.*