the main theorem in the toral case with a properly modified formulation. This theorem yields the rigidity of measure–preserving conjugacies, centralizers and factors. These results are presented in Section 4; [**KKS**] contains applications of these results which use some subtle number–theoretic information. Apart from presenting complete proofs of the above theorem our arguments have an extra value because they elaborate an important method whose uses extend beyond this particular proof. For example, essentially the same method is used in the proof of rigidity of joinings in [**KaK**].

Before doing that in the next section we will explain some of the main features of the method by considering in a less formal way a special case of a $\mathbb{Z}^2$ action on the three–dimensional torus by hyperbolic automorphisms. In this example the underlying geometric structure is quite transparent and a number of complications which appear in more general situations do not show up. In more general cases in order to conclude the desired dichotomy (Haar on a rational subtorus or a homogeneous invariant submanifold, or zero entropy) one needs extra assumptions on the action (e.g. irreducibility in the toral case) or on the measure (such as K-property, or, in the symmetric space case, weak mixing). The conclusion in general is also somewhat weaker.

2. The simplest higher rank models

2.1. Cartan action on the three–dimensional torus. The following special case provides an excellent insight in the core features of the method used to study rigidity of invariant measures.

Let $A \in SL(3,\mathbb{Z})$ be any hyperbolic matrix with distinct real eigenvalues. By passing to a power, if necessary, we may assume that its eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$ are all positive. By the Dirichlet unit theorem the centralizer of A in $SL(3,\mathbb{Z})$ is a finite extension of $\mathbb{Z}^2$. See [**KKS**] for a more detailed discussion and [**BS**] for the number–theoretic background. In particular, this centralizer contains a subgroup isomorphic to $\mathbb{Z}^2$ which consist of hyperbolic matrices. Such a subgroup σ determines an action of $\mathbb{Z}^2$ on the three–dimensional torus $\mathbb{T}^3$ by hyperbolic automorphisms which we will denote by ρ. (See Section 1.2) All elements of σ are simultaneously diagonalizable. The eigenvectors define three one–dimensional ρ–invariant linear foliations on the torus, $W_1,\ W_2$ and W_3. Each of these foliations possesses the natural affine parameter which is preserved by ρ. This is just another way of saying that the Euclidean length along each foliation is multiplied by each element of the action ρ by a constant which is equal to the absolute value of the corresponding eigenvalue. While no element of the action has an eigenvalue one or minus one there are elements for which the eigenvalue is arbitrary close to one. A useful technical devise is the *suspension* construction which by passing from the torus to a certain five–dimensional solvmanifold (an extension of $\mathbb{T}^2$ by $\mathbb{T}^3$) produces an $\mathbb{R}^2$ action α quite closely reflecting the features of the action ρ. (See Section 1.2.2) In particular, the "fiber" $\mathbb{T}^3$ direction splits into three invariant directions generating invariant foliations which we will still denote by W_1, W_2 and W_3. For each of these foliations there is an *irrational* direction in $\mathbb{R}^2$ such that the action along this direction preserves the length along the foliation. This direction is that of the kernel of the corresponding Lyapunov characteristic exponent, i.e. the Lyapunov hyperplanes defined in 1.2.3 which are of course just lines in our case. Call this direction the *critical* direction for the corresponding foliation. See Section 5.2 below for pertinent definitions in much greater generality.

2.2. Rigidity of invariant measures for Cartan actions. Now consider an ergodic invariant measure for the action ρ. We will show that

($\mathcal{D}$) *Either every element of the action has zero entropy with respect to this measure or the measure is Lebesgue measure on* $\mathbb{T}^3$.

First, extend the measure in a canonical way to an (also ergodic) invariant measure μ of the suspension action α. Let W be one of the three invariant foliations described above. The action preserves the system of conditional measures on the leaves on W. (See Section 1.4). Our argument works if the action in the critical direction is ergodic with respect to μ, or, more generally,

($\mathcal{E}$) *The ergodic components of the action in the critical direction consist of the whole leaves of* W.

This assumption will be later justified for any measure ergodic with respect to ρ. Under assumption ($\mathcal{E}$) we will show that

($\mathcal{T}$) *For almost every leaf* L *of the foliation* W *the conditional measure* μ_L *is invariant under the set of translations of full* μ_L *measure.*

Naturally, translations are defined with respect to the Euclidean length parameter. Here we mean invariance in an exact sense, not up to a scalar multiple. In other words, the measure μ_L is defined up to a scalar multiple but for any choice of normalization the measure is invariant with respect to almost every translation.

2.2.1. Thus L can be identified with the real line. Let us show that property $(\mathcal{T})$ implies that

The measure μ_L is either concentrated in a single point, or is a counting measure concentrated on a certain arithmetic progression, or is invariant under all translations, i.e. is simply Lebesgue measure on the line.

For, the measure is either atomic or continuous (translations cannot mix two parts). In the former case the atoms all have the same measure and hence must be isolated since otherwise the measure would not be locally finite. If there is more than one atom then the support is invariant by *every* translation taking one atom into another, which hence have to form a lattice.

If the measure is nonatomic than the set of translations under which it is invariant contains two rationally independent translations. Any locally finite measure on the line invariant under two rationally independent translations generates a finite measure on the circle (the factor by the first translation) invariant under an irrational translation and is hence Lebesgue.

Notice that ergodicity of the action implies that only one of the possibilities is realized for almost every leaf L.

The lattice case is, in fact, impossible due to the presence of the elements in the actions which expand the length parameter along W. For, any such element would map the set of leaves with a given value of the progression into the set with a strictly bigger value contradicting the Poincaré recurrence theorem.

Now assume that for almost every leaf L the measure μ_L is concentrated in a single point. The foliation W is the strong stable foliation of a certain elements of our action. But this implies that the entropy of those elements is equal to zero. These elements form a convex open set in $\mathbb{R}^2$ (a *Weyl chamber*, see Section 5.2. The inverses of these elements also have zero entropy. Since entropy is a sub–additive function on $\mathbb{R}^2$ ([**H**], Theorem B), it is equal to zero identically on $\mathbb{R}^2$.

If on the other hand, the measures μ_L are Lebesgue then the same is true for the conditional measures on the linear foliation on the torus. A fixed translation along the leaves of such a foliation is the linear flow on the torus which is in our case uniquely ergodic due to the fact that our action is irreducible over rationals (no invariant rational subtori). Since the conditional measures are invariant under translations so is the global measure which produces those conditionals. Hence by unique ergodicity the measure is itself Lebesgue.

An alternative argument does not use unique ergodicity of irrational translations. Instead one notice that any of the three invariant foliations W_1, W_2 and W_3 can play the role of W in the above argument. Since for any of these there is an element of the action for which it is the whole stable foliation, if at least one of the three systems of conditional measures are atomic the entropy of this element and hence of any element of the action vanishes. On the other hand, if all three systems of conditional measures are Lebesgue, then by Fubini Theorem the global measure is also Lebesgue.

2.2.2. Now we will explain why property $(\mathcal{T})$ holds. This is done in two steps.

Step 1. The measures are invariant in the affine sense: the translations are proportional up to a scalar multiple. To that end, let us fix the normalization at μ–almost every point x in such a way that the conditional measure of the interval $I(x)$ of length one on the leaf $L(x)$ centered at x is equal to one. Identifying $I(x)$ with the standard interval $[0, 1]$ via the length parameter we obtain a Borel map from our space to the set of probability measures on the unit interval provided with the weak* topology. On a compact subset C of measure arbitrary close to one this map is continuous (Luzin Theorem). A typical compact piece of a leaf intersects such a subset by a set of almost full conditional measure. Now use our ergodicity assumption $(\mathcal{E})$. Starting from a typical point x on a typical leaf L and moving in the critical direction the interval $I(x)$ comes arbitrary close to μ_L almost every point $y \in L$. In particular, if one assumes that both x and y are typical points of one of the continuity sets C described above, one may assert that the returns also appear on the set C. But this implies that in the limit the images of the conditional measures on $I(x)$ weakly converge to the conditional measure on $I(y)$. On the other hand, since we move along the critical direction the interval $I(x)$ is simply translated. Hence, the normalized conditional measure on $I(y)$ coincides with a translation of the normalized conditional measure on $I(x)$.

Step 2. The normalization constant which we will denote $c(x, y)$, is in fact equal to one. This again follows from the Poincaré recurrence. For, obviously, this constant is equivariant with respect to the action: for $a \in \mathbb{R}^2$, $c(\alpha(a)x, \alpha(a)y) = c(x, y)$. Secondly this is cocycle: $c(x, y)c(y, z) = c(x, z)$. The latter condition implies that along a typical leaf the density is exponential with respect to the natural length parameter. Taking an element a for which $\alpha(a)$ contracts the foliation W we see that the exponent must grow contradicting the Poincaré recurrence Theorem again, unless it is zero, *i.e.* the conditional measure is indeed Lebesgue.

2.2.3. Finally, we need to check assumption $(\mathcal{E})$ for any α–invariant ergodic measure, namely to show that the ergodic components for the action in the critical direction consist of the whole leaves of W. To that end we will use the structure of stable and unstable foliations for different elements or the action. In fact, for any of the three foliations there exists an element $a \in \mathbb{R}^2$ such that this foliation is the stable foliation of $\alpha(a)$ and the sum of the other two is the unstable one. On the other hand, the classical Hopf argument shows that ergodic decomposition for any element consists of complete leaves of its stable and unstable foliations: the positive (negative) time average of any continuous function is constant along the stable (unstable) leaves. The remaining ingredient is an important observation that for a normally hyperbolic (generic) element of the action the measurable hulls of partitions into leaves of the stable and unstable foliation coincide, because each of them generates the Pinsker σ–algebra (maximal σ–algebra with zero entropy).

Let us denote by ξ_a the partition into ergodic component of the element $\alpha(a)$ and by $\xi(W)$ the measurable hull of the partition into the leaves of the foliation W. Now let a be a non–zero element in the critical direction, W' be the one–dimensional stable foliation of a, W'' be the remaining foliation and $b \in \mathbb{R}^2$ be a regular (non–critical) element such that W' is the stable foliation of the element $\alpha(b)$. Thus we have the following inequalities:

$$\xi_a \leq \xi(W') = \pi(\alpha(b)) = \xi(W \oplus W'') \leq \xi(W). \tag{2.2.1}$$

2.3. $\times 2$, $\times 3$ **and automorphisms of a solenoid.** A very similar situation appears for commuting expanding maps of the circle. The basic example is the action of $\mathbb{Z}_+^2$ generated by multiplications by 2 and by 3 (mod 1). The question about invariant measures of this actions was posed by Furstenberg in 1967 [**F**]; it was solved for measures with positive entropy by D. Rudolph, [**R**]. It was an attempt to understand Rudolph's result in a geometric fashion that led the second author to the consideration of the model on the three–torus described above. Now in order to prove that the only ergodic positive entropy invariant measure for the multiplications by 2 and by 3 is Lebesgue we pass to the natural extension for this action. The phase space for this natural extension is a solenoid, the dual group to the discrete group $\mathbb{Z}(1/2, 1/3)$. It is locally modeled on the product of $\mathbb{R}$ with the groups of 2–adic and 3–adic numbers. Thus while topologically the solenoid is one–dimensional, there are three Lyapunov exponents, one for the real direction, and two for the non–Archimedean ones. Since the multiplication by 2 is an isometry in the 3–adic norm and vice versa the critical lines in this case are the two axis and the line $x \log 2 + y \log 3 = 0$ which does not intersect the first quadrant. All the above arguments work in this case verbatim with the real foliation playing the role of W. The unique ergodicity of the flow of translations along the real direction follows form the construction of the solenoid.

This argument of course extends to multiplications by p and q unless for some natural numbers k and l, $p^k = q^l$.

2.4. Other types of rigidity. . The models discussed in this section are also very convenient for demonstrating a other types of rigidity phenomena which appear in actions of higher rank abelian groups.

The first type is rigidity of vector valued Hoelder and differentiable cocycles. Rigidity in these cases means that every cocycle from a given class is cohomologous to a constant coefficient cocycle, i.e. a homomorphism from the acting group to the vector space with the cohomology given by a transfer function from the same class or with moderate loss of regularity. See [**KS1**] for 1-cocycles over differentiable actions, [**KSch**] for 1-cocycles over actions by automorphisms of compact groups and [**KK**] for higher-order cocycles. A nice survey which also discusses other types of cocycles, such as those with values with compact abelian groups as well as nonabelian groups is [**NT**].

Another type of rigidity is local differentiable rigidity: any smooth action close to a given action in C^1 topology is differentiably conjugate (also maybe with a small loss of regularity) to the original action; in the continuous case up to an automorphism of $\mathbb{R}^k$ close to identity (see [**KS2**]). It is interesting to point that the proof of local differentiable rigidity involves a construction of a certain family of invariant geometric structures on certain invariant foliations of the perturbed action. In the simplest cases this structure is an affine connection. Then the structural stability implies existence of continuous conjugacy between the original and perturber action (up to a time change in the continuous case). The principal idea in the proof that the conjugacy must be smooth involves showing that it intertwines a standard structure on the leaves of an invariant foliation for the algebraic action with the corresponding structure of the perturbed one. These latter structure plays a role very similar to that of conditional measures in the setup of the present paper.

There are also global rigidity results which assume that an action has certain topological (e.g. homotopy type) and dynamical (i.e. Anosov) properties and assert conjugacy with a standard algebraic action [**KL, MQ**].