VECTOR DISTRIBUTIONS AND SUB-RIEMANNIAN SYSTEMS

Banach Center Mini-Workshop, 30 March – 2 April, Warszawa

ABSTRACTS

Bernard Bonnard (Dijon, France)

Riemannian metrics on two-spheres and extensions with applications to optimal control

We will discuss the relation between the Gauss curvature, the period mapping and the conjugate and cut-loci for a family of metrics arising as a deformation of the round sphere. Applications in space mechanics and optimal Monge transport on surfaces will be presented. Extensions related to quantum control will be discussed.

Andreas Cap (Vienna, Austria)

Parabolic geometries determined by distributions

Given a bracket generating distribution D on a smooth manifold M which satisfies a regularity condition, one may associate to each point $x \in M$ the so–called symbol algebra at x. This is a nilpotent graded Lie algebra, which can be thought of as the first order approximation of the distributions D. If the symbol algebras in all points of M are isomorphic to some fixed algebra $\mathfrak{n}$, then one obtains a natural frame bundle over M with structure group $Aut(\mathfrak{n})$, the group automorphisms of the graded Lie algebra $\mathfrak{n}$.

For a specific class of algebras $\mathfrak{n}$, the machinery of parabolic geometries can be used to construct a natural extension of this bundle and a canonical Cartan connection on this extended bundle. While in general dimensions, this covers only few distributions, generic distributions of rank 2 in dimension 5, rank 3 in dimension 6, rank 4 in dimension 7, and rank n in dimension $n(n+1)/2$ can be treated by this method. In my talk, I will outline how the class of algebras $\mathfrak{n}$ for which this works comes up and how the Cartan connection is constructed. I will also discuss the role of Lie algebra cohomology and Kostant's version of the Bott-Borel-Weil theorem in this theory.

Boris Doubrov (Minsk, Belarus)

Geometry of curves in parabolic geometries

The talk is devoted to the study of integral curves of constant type in parabolic homogeneous spaces. Generalizing the classical works of Wilczynski on projective geometry of curves, we construct a canonical moving frame for such curves and give the criterion when it turns out to be a Cartan connection. Generalizations to parametrized curves and to curved parabolic geometries will be discussed.

Maciej Dunajski (Cambridge, UK and Warsaw, Poland)

A geometry of cuspidal cubics

A cuspidal cubic, or Neil's parabola $y^2 = x^3$ belongs to a seven-dimensional orbit of the projective group SL(3). Thus all planar cuspidal cubics are integral curves of a certain 7th order ODE. I shall explain that the solution space of this ODE carries a natural geometric structure related to the exceptional group G_2.

Stefan Ivanov (Sofia, Bulgaria)

Quaternionic contact Yamabe problem and related geometric structures

The talk will present a special problem in sub-Riemannian geometry, where the horizontal distribution is endowed with a quaternionic structure, in addition.

Bronisław Jakubczyk (Warsaw)

Affine distributions and dynamic pairs
— curvatures and conjugate points
(common work with Wojciech Kryński)

We will discuss invariants of pairs (X, D) (called *dynamic pairs*), where X is a vector field and D is a vector distribution on a manifold M, both satisfying natural regularity conditions. Examples of such pairs are given by geodesic sprays on the tangent bundle, Hamiltonian vector fields on the cotangent bundle, control-affine systems, systems of ODEs of order $k > 1$. In the last case X is the total "time" derivative, while X and D span the Cartan distribution. For a control-affine system X is a distinguished drift.

A regular pair (X, D) defines a class of normal frames in D, defining parallelism along trajectories of X. Using these frames we define curvature operators of the pair. The curvature operators are responsible for the geometry of the transport of D by the flow of X. In the particular case of X being the geodesic spray the curvature operator plays the role of Riemann curvature. It determines the Jacobi equation and is responsible for the distribution (or absence) of conjugate points. We show that an analogous result holds in the general case of dynamic pairs of order 2 (an extension of Cartan-Hadamard theorem). In the Hamiltonian case we obtain results analogous to those of Agrachev, Gamkrelidze, Chtcherbakova and Zelenko, with a simpler definition of curvature.

In the second part of the talk we will sketch a solution of the equivalence problem for the pairs (X, D). This is done by constructing a Cartan connection. Examples of invariants obtained in this way will be presented.

Frederic Jean (Paris, France)

Nonholonomic control systems: nilpotent approximations, desingularization and steering by sinusoidal controls

In this talk, we discuss the problem of steering a nonholonomic control system from one point to another one, that is, in the context of SR geometry, of finding an horizontal path joining two given points. Three main issues have to be settled. The first one is to give a general iterative steering algorithm based on the use of nilpotent approximations. The second consists in the explicit algebraic construction, starting from the original control system, of a lifted control system which is regular. The third problem is to build an exact motion planning method for nilpotent systems, which makes use of sinusoidal control laws.

Wojciech Kryński (IHES, Bures-sur-Yvette, France and Warsaw, Poland)

On parabolic (3,5,6)-distributions

We consider a distribution D of rank 3 on a manifold of dimension 6 and assume that D has small growth vector: $(3, 5, 6)$. The study of this class of distributions was initiated by B. Doubrov (and his results were presented in September 2010 during the workshop in Banach Center). He showed that the class splits into three subclasses distinguished by the signature of a certain bilinear form associated to a distribution, i.e. D can be either parabolic, elliptic or hyperbolic. Doubrov concentrated on the elliptic and hyperbolic cases, which can be described in terms of $SL(4)$-geometry.

In our talk we concentrate on the parabolic case. This class splits into two subclasses: degenerated and non-degenerated. We prove that all degenerated parabolic $(3, 5, 6)$-distributions are locally equivalent. We also provide a classification of non-degenerated distributions and exhibit their connections with ODEs of order 3 and 4.

Piotr Mormul (Warsaw, Poland)

Nonholonomy degrees in [special] Monster tower for jets of plane curves. Question about entire small growth vectors in that tower

Monster manifold for the jets $J^k(1,2)$ is a compactification of the manifold $J^k(1,2)$ *together with* the canonical contact system $\mathcal{C}^k$ on $J^k(1,2)$. The distribution Δ^k on the compactification mimicks $\mathcal{C}^k$ on an open dense set, but is elsewhere rich in singularities. Varying k, one gets the tower of monster manifolds for all jets of maps $\mathbb{R}^1 \to \mathbb{R}^2$ (plane curves). Distributions Δ^k are everywhere completely nonholonomic, with the nonholonomy degree (= the length of the small growth vector) $k+1$ at jet-like points and with higher (and sometimes very high) nonholonomy degrees at singular points.

We recursively compute those nonholonomy degrees at the tangential points of the monsters, and word a question concerning the entire small growth vectors of Δ^k, $k \geq 1$.

Paweł Nurowski (Warsaw)

Differential equations and para-CR structures

Witold Respondek (INSA de Rouen, France)

Flatness of canonical Cartan distributions

An under-determined ordinary differential equation, i.e., a system of $n+m$ equations for m variables, is said to be flat if we can find m functions that parametrize (together with their time-derivatives) all solutions of the system.

In the first part, we study the problem of flatness of rank 2 distributions. Although it is known that flat distributions of rank 2 are those equivalent to the canonical Cartan distribution for $\mathbb{R}$-valued curves, the problems of describing all flat outputs and of calculating them is open. We show that all x-flat outputs are parametrized by an arbitrary function of three canonically defined variables. We also construct a system of 1st order PDE's whose solutions give all x-flat outputs of rank 2 distributions. We illustrate our results by describing all flat outputs of models of a rolling disk and a nonholonomic car.

In the second part, we propose a kinematic model of a system moving in $\mathbb{R}^{m+1}$ and consisting of n rigid bars attached successively to each other and subject to the nonholonomic constraints that the instantaneous velocity of the source point of each bar is parallel to that bar. A similar model has been studied by Jakubczyk and, recently, by Slayman and Pelletier. We prove that the associated control system is controllable and that the associated distribution is equivalent to the Cartan distribution for $\mathbb{R}^m$-valued curves around any regular configuration. Hence we deduce that the n-bar system is flat and the cartesian position of the source point of the last bar (together with its time-derivatives) parametrizes all solutions. The n-bar system is a natural generalization of the n-trailer system and we provide a comparison of flatness properties of both systems.

Presented results are based on a joint work with Shun-Jie Li.

Ben Warhurst (Warsaw, Poland)

Tangent cones are not quasiconformal invariants

Abstract: Pansu showed that if two Carnot groups are quasiconformally equivalent then they are isomorphic as groups. Margulis and Mostow generalised this by showing that the differential of a quasiconformal mapping between equiregular sub-Riemannian manifolds induces a Lie algebra isomorphism between tangent cones at corresponding points. One can consider the converse when the manifolds are strongly regular, that is the tangent cones at each point of the manifold are all isomorphic to a fixed Lie algebra. In this talk I will demonstrate with a counter example, that even in the strongly regular setting the converse is false.

Igor Zelenko (Texas A & M Univ., USA)

1. Jacobi symbols of vector distributions

The talk can be considered as a continuation of the talk of Boris Doubrov on geometry of curves in parabolic homogeneous spaces (or curves of generalized flags). We demonstrate the application of this theory to local geometry of vector distributions. First making a kind of symplectification of this equivalence problem we introduce the notion of Jacobi symbol of a distribution. Then we describe the construction of the canonical frames for all distributions with given Jacobi symbol, the most symmetric models among them and their symmetry groups. Previous results and techniques, including classical 1910 work of E. Cartan and works of N. Tanaka are restricted to the case of distributions with constant Tanaka symbol and therefore they give a complete picture only in few cases of low dimensions, where the Tanaka symbols (i.e. graded nilpotent algebras) can be classified and do not depend on continuous parameters. On the other hand, Jacobi symbols, being much simpler algebraic objects, can be explicitly classified. Besides, the assumption of constancy of Jacobi symbols of a distribution holds automatically (in a neighborhood of a generic points) due to the fact that the set of Jacobi symbols is discrete. Finally, the Jacobi symbol of a distribution is much coarser characteristic than its Tanaka symbols: distributions with different Tanaka symbols and even with different small growth vectors may have the same Jacobi symbol.

2. Rauch type comparison theorems in sub-Riemannian geometry

The classical Rauch comparison theorem in Riemannian geometry provides the estimation of the number of conjugate points along Riemannian geodesics in terms of bounds for the sectional curvature. We give a generalization of this theorem to extremals of a wide class of optimal control problems including sub-Riemannian extremals. The problem can be reformulated as the problem to estimate the number of conjugate points along a curve in a Lagrangian Grassmannian in terms of the invariants of this curve with respect to the natural action of the Linear Symplectic Group. Our treatment of this problem is based on the construction of the canonical bundle of moving frames and the complete system of symplectic invariants for curves in Lagrangian Grassmannians previously done in the joint works with Chengbo Li. We will explain how appropriately arranged bounds for these symplectic invariants effect the bounds for the number of conjugate points. The application for extremals of natural sub-Riemannian metrics on principal connections of principal bundles with one-dimensional fibers over Riemannian manifolds (i.e. magnetic fields on Riemannian manifolds) will be given.

3. On canonical frames for affine control systems spanning rank 2 distributions of maximal class

The talk is devoted to the state-feedback equivalence of affine control system with one input (equivalence of rank 1 affine subbundles of tangent bundles). Under certain mild genericity assumption that affine control systems with one input span a rank 2 distributions of so-called maximal class and using a kind of symplectification of the problem, we assign to each such system the canonical frame and we describe the most symmetric models among them. The consequences for micro-local equivalence of generic non-affine control systems with one input will be given as well.