Invariance Principle for the Random Conductance Model with dynamic bounded Conductances

Sebastian Andres

University of Bonn

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The Static Random Conductance Model

Intuitive description

- Put i.i.d. random conductances (or weights) $\omega_e \in [0, \infty)$ on the edges of the Euclidean lattice $(\mathbb{Z}^d, E_d)$.
- Look at a continuous time Markov chain $X_t$ with jump probabilities proportional to the edge conductances.

$$P_{xy} = \frac{\omega_{xy}}{\sum_{z \sim x} \omega_{xz}}.$$ 

Bond conductivities: blue $\ll 1$, black $\approx 1$, red $\gg 1$. 
Definitions

- **Environment.** Let $\Omega = [0, \infty)^{E_d}$ be the space of environments, and let $\mathbb{P}$ be the probability law on $\Omega$ which makes the coordinates $\omega_e, e \in E_d$ i.i.d. random variables.
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- **Random walk.** Let $\Omega' = D([0, \infty), \mathbb{Z}^d)$. For each $\omega \in \Omega$ let $P^\omega_x$ be the probability law on $\Omega'$ which makes the coordinate process $X_t = X_t(\omega')$ a Markov chain with generator

$$
\mathcal{L}f(x) = \sum_{y \sim x} \omega_{xy}(f(y) - f(x)).
$$

Problems.

What we would like to have:

- Gaussian bounds (GB) on the heat kernel for $X$.
- Quenched functional CLT with diffusivity $\sigma^2$: Let $X^N_t = N^{-1}X^2_t$ and $W$ be a BM$(\mathbb{R}^d)$. Then for $P$-a.a. $\omega$, under $P^\omega_0$, $X^N_t \Rightarrow \sigma W$.

(In particular is $\sigma > 0$?)
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- **Random walk.** Let $\Omega' = D([0, \infty), \mathbb{Z}^d)$. For each $\omega \in \Omega$ let $P_x^\omega$ be the probability law on $\Omega'$ which makes the coordinate process $X_t = X_t(\omega')$ a Markov chain with generator

\[ \mathcal{L}f(x) = \sum_{y \sim x} \omega_{xy}(f(y) - f(x)). \]

- **Problems.** What we would like to have:
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Results on the static RCM

- “Elliptic”: $0 < C_1 \leq \omega_e \leq C_2 < \infty$. GB follow from results of Delmotte (1999). QFCLT proved by Sidoravicius and Sznitman (2004).
- “Supercritical Percolation”: $\omega_e \in \{0, 1\}$ (and $p_+ > p_c$.) GB proved by Barlow (2004). QFCLT proved by Sidoravicius and Sznitman (2004), Berger and Biskup (2007), Mathieu and Piatnitski (2007).
- “Bounded above”: $\omega_e \in [0, 1]$. Berger, Biskup, Hoffmann, Kozma (2008) showed GB may fail! QFCLT holds with $\sigma^2 > 0$: Biskup and Prescott (2007), Mathieu (2007).
- “Bounded below”: $\omega_e \in [1, \infty)$. GB and QFCLT proved by Barlow and Deuschel (2010).
- General i.i.d. $\omega_e \geq 0$. QFCLT by A., Barlow, Deuschel, Hambly (2011).
Dynamic Random Conductance Model

- **Environment.** Let $\Omega$ be the space of measurable mappings from $[0, \infty)$ into $[0, \infty)^{E_d}$, and let $\mathbb{P}$ be a probability law on $\Omega$. Write $\omega_e(t), e \in E_d, t \geq 0$, for the coordinates.
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- Random walk. Let $\Omega' = D([0, \infty), \mathbb{Z}^d)$. For each $\omega \in \Omega$ let $P_{s, x}^{\omega}$ be the probability law on $\Omega'$ which makes the coordinate process $X_t = X_t(\omega')$ a time-inhomogeneous Markov chain starting at $x$ at $t = s$ with time-dependent generator

$$\mathcal{L}_t^{\omega} f(x) = \sum_{y \sim x} \omega_{xy}(t) (f(y) - f(x)).$$

Denote by $p^{\omega}(s, x; t, y)$ the transition kernel.
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- **Random walk.** Let $\Omega' = D([0, \infty), \mathbb{Z}^d)$. For each $\omega \in \Omega$ let $P^{\omega}_{s,x}$ be the probability law on $\Omega'$ which makes the coordinate process $X_t = X_t(\omega')$ a time-inhomogeneous Markov chain starting at $x$ at $t = s$ with time-dependent generator

$$L_t^{\omega} f(x) = \sum_{y \sim x} \omega_{xy}(t) (f(y) - f(x)).$$

Denote by $p^{\omega}(s, x; t, y)$ the transition kernel.

- **Shift.** Let $\tau_{t,x}\omega$ be the environment obtained by shifting $\omega$ by $t$ in time and by $x$ in space.
QFCLT’s for RWRE with dynamical Environment

- Space-time i.i.d. environment: Boldrighini, Minlos, Pellegrinotti (2004); Rassoul-Agha, Seppäläinen (2005)
- Markovian in time, i.i.d. in space: Bandyopadhyay, Zeitouni (2006)
- Ergodic Markovian environment under some coupling conditions: Redig, Völlering (2011)
Assumptions

- **A1: Ergodicity.** The measure $\mathbb{P}$ is invariant and ergodic w.r.t. $(\tau_{t,x})$.
- **A2: Stochastic Continuity.** For any $\delta > 0$ and $f \in L^2(\mathbb{P})$ we have
  \[
  \lim_{h \to 0} \mathbb{P}[|f(\tau_h,0\omega) - f(\omega)| \geq \delta] = 0.
  \]
- **A3: Ellipticity.** There exist positive constants $C_l$ and $C_u$ such that
  \[
  \mathbb{P}[C_l \leq \omega_e(t) \leq C_u, \forall e \in E_d, t \geq 0] = 1.
  \]
Gaussian Bounds and Annealed Functional CLT

Theorem (Delmotte, Deuschel; 2005)

Under A1-A3, for \( \mathbb{P} \)-a.e. \( \omega \)

\[
p^{\omega}(s, x; t, y) \leq \frac{c_2}{(t - s)^{d/2}} \exp \left( -c_3 \frac{|x - y|^2}{t - s} \right), \quad \text{if} \ |x - y| \leq c_1(t - s)
\]

and similar lower bounds

Define annealed law \( \mathbb{P}^{s, x}_* = \int_{\Omega} P^{\omega}_{s, x} d\mathbb{P}(\omega) \).

Theorem (A., 2012)

Let \( d \geq 1 \). Under A1-A3 the law of \( X^{(N)} \) converges under \( \mathbb{P}^*_0,0 \) to the law of a Brownian motion on \( \mathbb{R}^d \) with a deterministic non-degenerate covariance matrix \( \Sigma \).
Quenched Functional CLT

- **A4: Time-Mixing.** There exists $p_1 > 1$ such that for all bounded $\varphi, \psi$ of the form $\varphi(\omega) = \tilde{\varphi}(\omega(t_1))$ and $\psi(\omega) = \tilde{\psi}(\omega(t_2))$, $|t_1 - t_2| \geq 1$, for some $\tilde{\varphi}, \tilde{\psi}$ depending on finitely many variables we have

  $$|E[\varphi \psi] - E[\varphi]E[\psi]| \leq c |t_1 - t_2|^{-p_1} \|\varphi\|_{L^\infty(P)} \|\psi\|_{L^\infty(P)}.$$ 

- **A5: Space-Mixing.** Let $d \geq 3$. There exists $p_2 > 2d/(d-2)$ such that for all $\varphi, \psi$ of the form $\varphi(\omega) = \tilde{\varphi}(\omega(t_0))$ and $\psi(\omega) = \tilde{\psi}(\omega(t_0))$ for some $\tilde{\varphi}, \tilde{\psi}$ depending on finitely many variables we have

  $$|E[\varphi(\omega)\psi(\tau_0, x_0)] - E[\varphi]E[\psi]| \leq c |x|^{-p_2} \|\varphi\|_{L^\infty(P)} \|\psi\|_{L^\infty(P)}.$$ 

**Theorem (A.,2012)**

Let $d \geq 3$. Under A1-A5, $P$-a.s. $X^{(N)}$ converges (under $P_{0,0}$) in law to a Brownian motion on $\mathbb{R}^d$ with a deterministic non-degenerate covariance matrix $\Sigma$. 

Sebastian Andres

Invariance Principle for the RCM with dynamic Conductances

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Overview of proofs

- Basic idea: Homogenization. Use the process of ‘the environment seen from the particle’

\[ \eta_t := \tau_t, x_t, \omega, \quad t \geq 0, \]

... to construct the time-dependent corrector function \( \chi \) such that

\[ X_t = M_t + \chi(t, X_t, \omega). \]

- Problem: \( \eta \) is not reversible!

- To control the corrector one needs to show

\[ \lim_{n \to \infty} n^{-1/2} \max_{k \leq n} |\chi(k, X_k, \omega)| = 0 \quad \text{in } P_0^\omega-\text{probability.} \]

- This follows from some fractional ergodic theorems by Derriennic and Lin if for some \( \delta > 0 \)

\[ \mathbb{E} E_{0,0}^\omega \left[ |\chi(n, X_n, \omega)|^2 \right] = O(n^{1-\delta}). \]
Local Limit Theorem

For the Gaussian heat kernel with diffusion matrix $\Sigma$ write

$$k_t(x) = \frac{1}{\sqrt{(2\pi t)^d \det \Sigma}} \exp(-x \cdot \Sigma^{-1} x/2t), \quad k_t(x, y) = k_t(0, y - x).$$

Combine the GB and the FCLT for $X$, following arguments from Barlow, Hambly (2009), to obtain

**Theorem**

Let $T > 0$. Under A1-A3 we have

$$\lim_{N \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T} \left| N^d \mathbb{E}[p^\omega(0, 0; N^2 t, \lfloor Nx \rfloor)] - k_t(x) \right| = 0.$$  

and under A1-A5,

$$\lim_{N \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T} \left| N^d p^\omega(0, 0; N^2 t, \lfloor Nx \rfloor) - k_t(x) \right| = 0, \quad \mathbb{P} \text{-a.s.}$$
Application: $\nabla \phi$-Interface Models

- Ginzburg-Landau interface models describe the separation of two thermodynamical phases.
- The interface is specified by a field of height variables $\phi_t(x), x \in \mathbb{Z}^d, t \geq 0$, given by

$$d\phi_t(x) = - \sum_{y:|x-y|=1} V'(\phi_t(x) - \phi_t(y))\, dt + \sqrt{2}dw_t(x),$$

with

- $\{w(x), x \in \mathbb{Z}^d\}$ collection of independent Brownian motions,
- potential $V \in C^2(\mathbb{R}, \mathbb{R}_+)$ even and strictly convex.

- In $d \geq 3$ there exists an ergodic Gibbs measure $\mu$ which is reversible for the dynamics.
Space-Time Covariances

- Helffer-Sjöstrand representation:

\[
\text{cov}_\mu(\phi_0(0), \phi_t(y)) = \int_0^\infty \mathbb{E}_\mu p^\phi(0, 0; t + s, y) \, ds,
\]

where \( p^\phi(s, x; t, y) \) is the transition kernel of a RW with generator

\[
\mathcal{L}^\phi_t f(x) = \sum_{y: |x-y|=1} V''(\phi_t(x) - \phi_t(y)) (f(y) - f(x)).
\]
Space-Time Covariances

- Helffer-Sjöstrand representation:

\[ \text{cov}_\mu(\phi_0(0), \phi_t(y)) = \int_0^\infty \mathbb{E}_\mu p_\phi(0, 0; t + s, y) \, ds, \]

where \( p_\phi(s, x; t, y) \) is the transition kernel of a RW with generator

\[ \mathcal{L}_t^\phi f(x) = \sum_{y: |x - y| = 1} V''(\phi_t(x) - \phi_t(y))(f(y) - f(x)). \]

- By the local limit theorem

\[ N^{d+2} \text{cov}_\mu(\phi_0(0), \phi_{N^2 t}(\lfloor Ny \rfloor)) = N^d \int_0^\infty \mathbb{E}_\mu p_\phi(0, 0; N^2(t + s), \lfloor Ny \rfloor) \, ds \]

\[ \xrightarrow{N \to \infty} \int_0^\infty k_{t+s}(y) \, ds. \]
Conclusion, Outlook and open questions

- For the static RCM we have a QFCLT in the case of general i.i.d. conductances.
- For the dynamic RCM we have a QFCLT under ellipticity and mixing assumptions.
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For the dynamic RCM we have a QFCLT under ellipticity and mixing assumptions.

Is it possible to obtain a QFCLT

- in the static case with stationary, ergodic conductances under some moment conditions? **Yes:** A., Deuschel, Slowik (in preparation)
- in the dynamic case without assuming ellipticity?
- for the static RCM on a half-lattice?
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Thank you!