Right-Angled Artin Groups

Bedlewo 2012

Lectures 3 & 4

Right-Angled Artin Groups (RAAGs)

Coxeter group

$$W = \langle V \mid v_i^{\,2} = 1,\ (v_i v_j)^{m_{i,j}} = 1 \rangle$$

Artin group

$$A_\Gamma = \langle V \mid v_i v_j v_i \ldots = v_j v_i v_j \ldots \rangle$$

A Coxeter/Artin group is right-angled if all $m_{i,j} \in \{2, \infty\}$.

In a right-angled Artin group, all relators are commutators:

$$v_i v_j = v_j v_i.$$

To describe such a group, we need only specify which generators commute.

Notation:

Γ = finite, simplicial graph "defining graph"

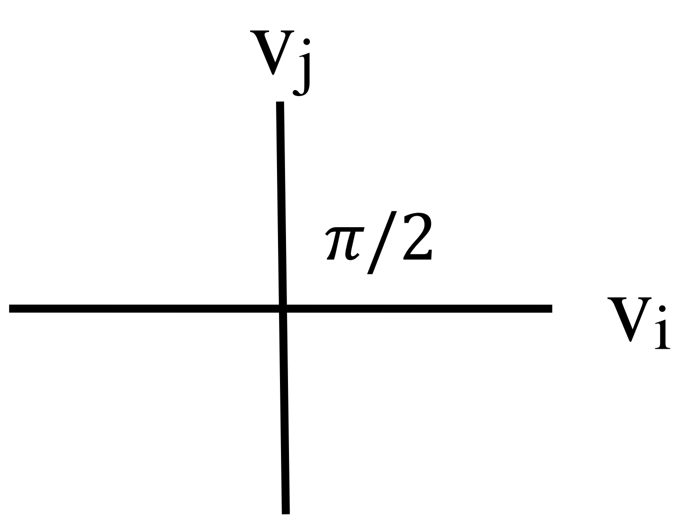
$V = \{v_1, \ldots, v_n\}$ = vertex set

$A_\Gamma = \langle V \mid v_i v_j = v_j v_i$, iff v_i, v_j are adjacent in $\Gamma \rangle$

Why the name "right-angled" Artin group?

Because in the corresponding Coxeter group, reflection planes meet at right angles.

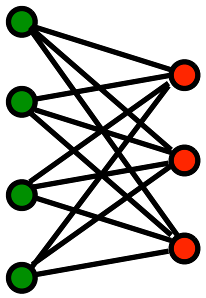
Also known as "graph groups" or "partially commutative groups".

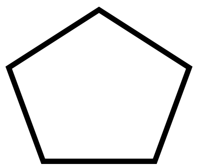


Examples:

Γ discrete $\Rightarrow$ A_Γ = free group

Γ complete $\Rightarrow$ A_Γ = free abelian group

Γ complete bipartite  $\Rightarrow$ $A_\Gamma = F_n \times F_m$

Γ =  $\Rightarrow$ A_Γ = ??

Properties of RAAGs.

(1) $K(A_\Gamma,1)$-spaces:

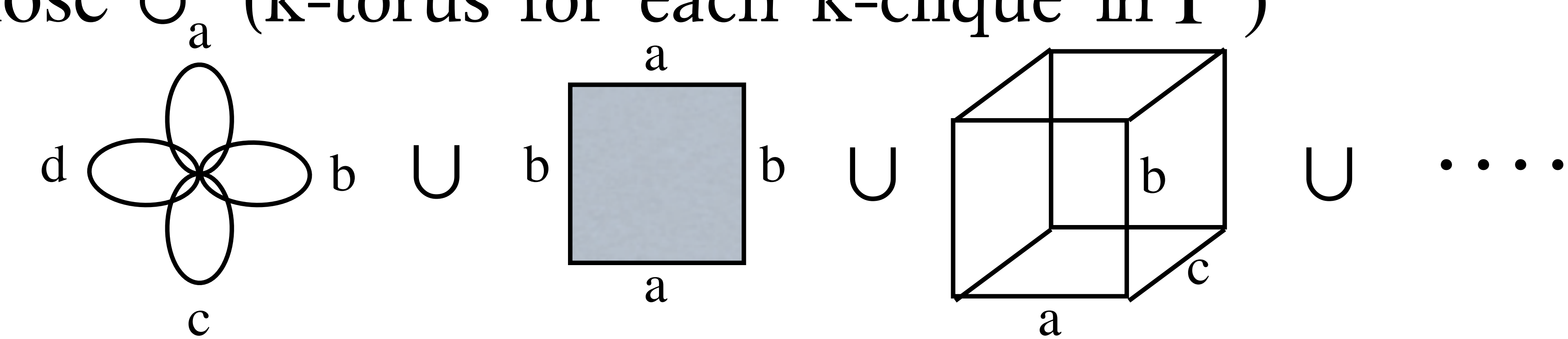
Last time: finite cell complex, Salvetti complex

$K(\pi,1)$-Conjecture: Salvetti complex is a $K(A,1)$-space.

Salvetti complex for A_Γ : $S_\Gamma = Sal(A_\Gamma)$

W_T is finite iff the generators in T all commute (so T spans a clique in Γ). In this case, the Coxeter cell C_T is a k-cube, $k=|T|$.

S_Γ = Rose $\cup$ (k-torus for each k-clique in Γ)



Examples:

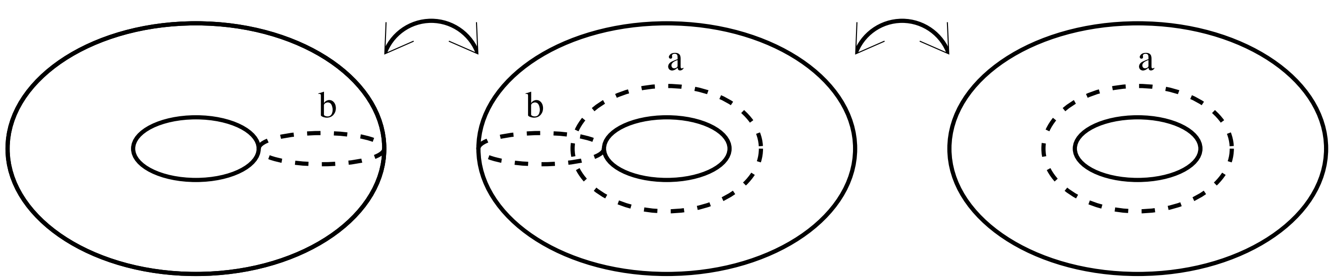
A_Γ = free group (Γ discrete)

$\Rightarrow$ S_Γ = Rose, $\widetilde{S}_\Gamma$ = tree

A_Γ = free abelian group (Γ complete)

$\Rightarrow$ S_Γ = n-torus, $\widetilde{S}_\Gamma = R^n$

Γ = 

S_Γ = 

Thm: S_Γ is a locally CAT(0) cube complex, with fundamental group A_Γ, hence it is a $K(A_\Gamma,1)$-space.

Immediate consequences:

1. A_Γ is a CAT(0) group
2. A_Γ satisfies the $K(A_\Gamma,1)$-conjecture.
3. A_Γ is torsion-free
4. A_Γ has cohomological dim = dim S_Γ
5. $H^*(A_\Gamma) = H^*(S_\Gamma) = \Lambda\,[V]\,/\,I$, $\quad I = \langle v_i v_j \mid \text{non-commuting} \rangle$
6. A_Γ is biautomatic (Hermiller-Meier, Niblo-Reeves)

$\widetilde{S}_\Gamma$ = universal cover of S_Γ is interesting in its own right.

Example: Boundaries of CAT(0) spaces.

X = Gromov hyperbolic metric space. Define

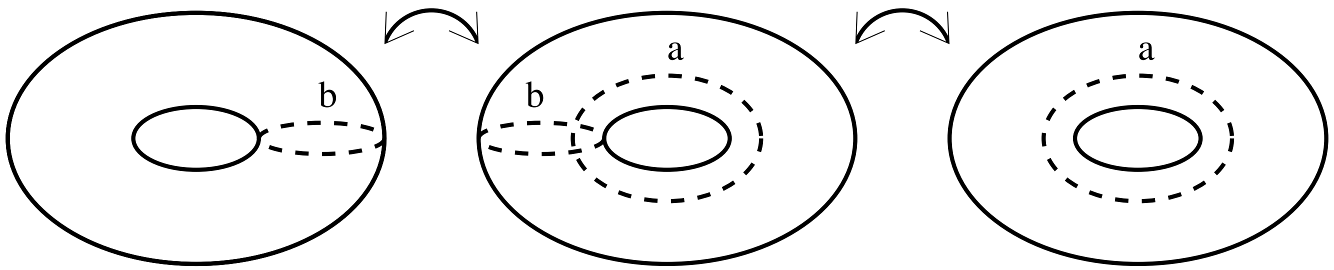
∂X = {equivalence classes of rays}

Any quasi-isometries $X \to Y$ between 2 such spaces induces a homeomorphisms $\partial X \simeq \partial Y$.

Q: Is the same true for CAT(0) spaces?

A: (Croke-Kliener) No.

Take Γ = 

S_Γ = 

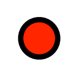
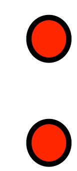
Take $X = \widetilde{S}_\Gamma$ and $Y = \widetilde{S}_\Gamma$ with skewed metric

(2) Subgroups and supergroups:

Every Artin group surjects onto a Coxeter group.

Right-angled Artin groups also inject into Coxeter groups.

Thm (Davis-Januszkiewicz): For any Γ, there exists Γ' such that A_Γ embeds as a finite index subgroup in $W_{\Gamma'}$.

Eg. $A_\Gamma = Z \rightarrowtail D_\infty = W_{\Gamma'}$ $\quad \Gamma =$  $\quad \Gamma' =$ 

Cor: For any Γ, A_Γ is linear.

RAAGs appear naturally as subgroups of many other groups.

- (Crisp-Paris) Let A be arbitrary Artin group with generating set S, then the subgroup generated by $(s_1)^{k_1}, \ldots\ldots, (s_n)^{k_n}$ is a RAAG, for all $k_i \geq 2$.

- (Koberda, Clay-Leininger-Mangahas) Given a finite set of elements in the mapping class group of a surface, under some fairly general conditions, high enough powers of these elements generate a RAAG.

RAAGs contain interesting (and sometimes exotic) subgroups.

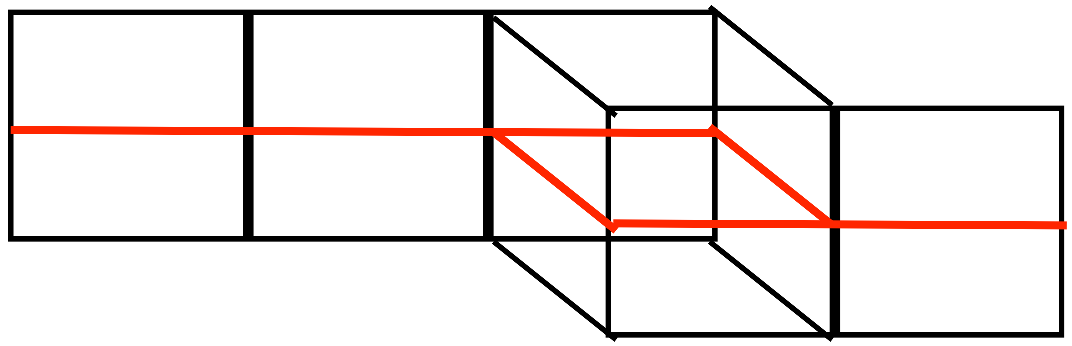
- (Droms-Servatius-Servatius, S-H Kim, Crisp-Wiest) Many RAAGs contain surface groups. Every surface group (with 3 exceptions) embeds in some RAAG.

- (Crisp-Wiest) Graph braid groups (fundamental groups of configurations spaces of n points on a graph) embed in RAAGs.

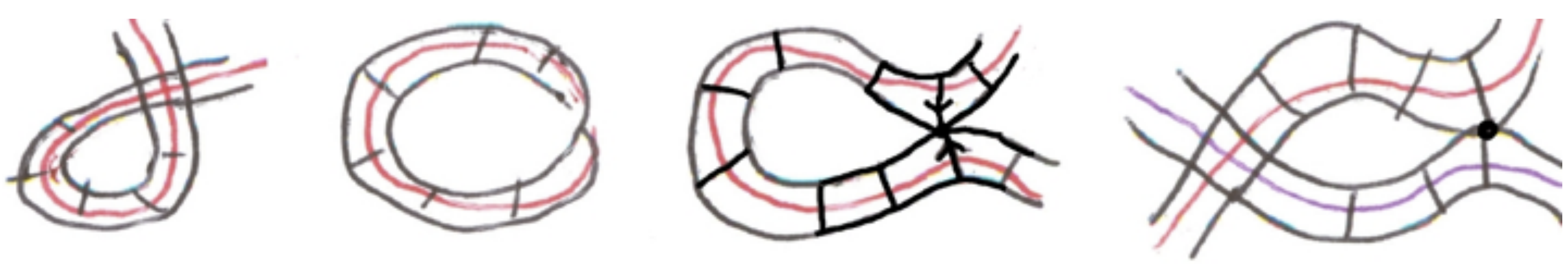
- (Bestvina-Brady) Constructed counterexamples to a conjecture about homological finiteness conditions using subgroups of RAAGs.

• (Haglund-Wise) Introduce the notion of a special cube complex. These are locally CAT(0) cube complexes in which walls are well-behaved.

wall:



not allowed:



Thm: (Haglund-Wise) If X is a special cube complex, then there exists a locally isometric embedding of X into a Salvetti complex S_Γ, for some Γ. Hence $\pi_1(X)$ injects into A_Γ.

(3) Residual properties:

A group G is residually (P) if $\forall g \neq 1 \in G$, there exists a group H satisfying (P) and a surjection $G \to H$ such that $g \mapsto h \neq 1$.

Residual properties of G are inherited by every subgroup of G.

Residual properties of A_Γ:

- A_Γ is residually finite (follows from linearity of A_Γ)
- (Duchamp-Krob) A_Γ is residually torsion-free nilpotent ($\Leftrightarrow$ A_Γ has a Magnus expansion)
- (Agol) A_Γ is residually finite rational solvable (RFRS)

Thm: (Agol) If M is a compact, irreducible, hyperbolic 3-manifold whose fundamental group $\pi_1(M)$ is RFRS, then M has a finite-sheeted covering which fibers over a circle.

Thurston's Virtual Fibering Conjecture: Every hyperbolic 3-manifold M has a finite-sheeted covering which fibers over a circle.

Thm: (Agol) If M is a compact, irreducible, hyperbolic 3-manifold whose fundamental group $\pi_1(M)$ is RFRS, then M has a finite-sheeted covering which fibers over a circle.

Thm: (Haglund-Wise) If X is a special cube complex, then $\pi_1(X)$ injects into A_Γ, so $\pi_1(X)$ is RFRS.

Thm: (Wise) If M is has a (geometrically finite) incompressible surface, then some finite index subgroup of $\pi_1(M)$ can be realized as the fundamental group of a special cube complex, hence Thurston's conjecture holds.

Automorphisms of
Right-Angled Artin Groups

Joint work with

Karen Vogtmann

Right-angled Artin groups

- have nice geometry
- contain interesting subgroups and supergroups
- interpolate between free groups and free abelian groups, hence their automorphism groups interpolate between

$$\mathrm{Out}(F_n) \xleftrightarrow{\mathrm{Out}(A_\Gamma)} GL_n(Z)$$

Many properties are known to hold for

$\mathrm{Out}(F_n)$ and $GL_n(Z)$

Which of these properties hold

for *all* $\mathrm{Out}(A_\Gamma)$?

Two techniques:

- induction on dim A_Γ
- construct an analogue of "outer space"

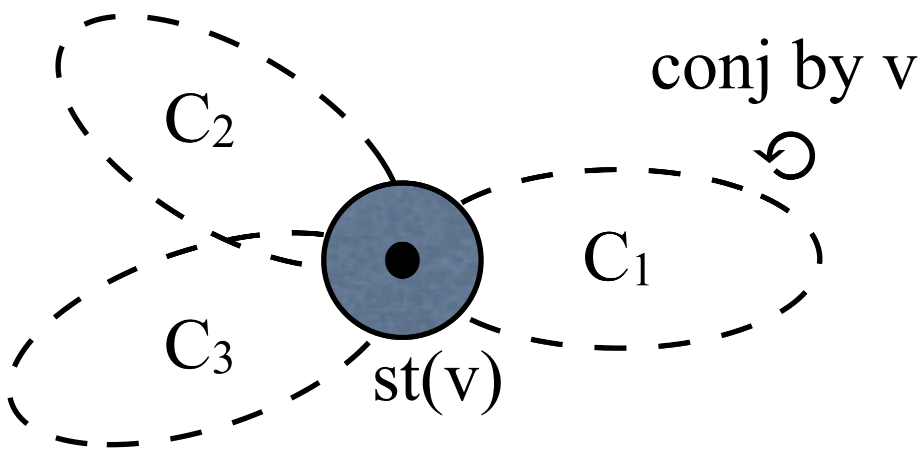
$\dim A_\Gamma = \dim S_\Gamma$ = size of maximal clique in Γ

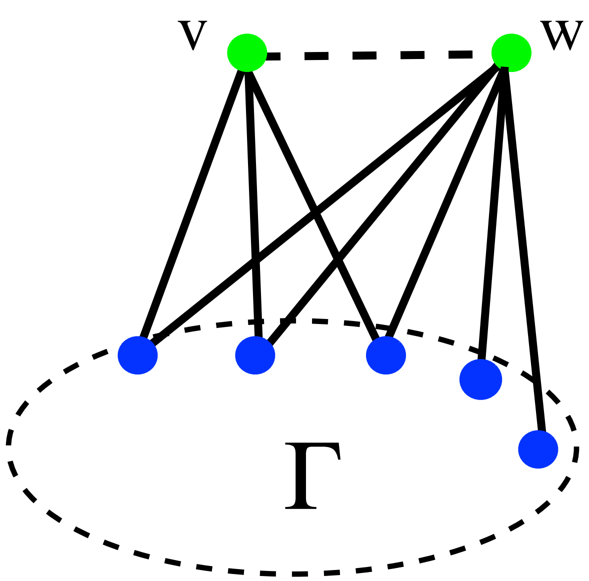
- $\dim = 1 \Rightarrow A_\Gamma$ = free group
- $\dim = n = |V| \Rightarrow A_\Gamma$ = free abelian group

A generating set for Out(A_Γ)

Servatius ('89), Laurence ('95): Out(A_Γ) has a finite generating set consisting of:

- Graph symmetries: $\Gamma \to \Gamma$
- Inversions: $v \to v^{-1}$
- Partial conjugations: conjugate a connected component of $\Gamma \backslash st(v)$ by v.
- Transvections: $v \to vw$, providing $lk(v) \subset st(w)$





Define $Out^0(A_\Gamma)$ = subgroup generated by inversions, partial conjugations, transvections

Inductive technique

Definition: Let $\Theta \subset \Gamma$ be a full subgraph. Say Θ is *characteristic* if every automorphism of A_Γ preserves A_Θ up to conjugacy (and graph symmetry).

Let $\Theta \subset \Gamma$ be characteristic. Then

$$A_\Theta \hookrightarrow A_\Gamma \twoheadrightarrow A_{\Gamma\setminus\Theta}$$

induces restriction and exclusion homomorphisms:

$$\mathrm{Out}\,(A_\Theta) \xleftarrow{R_\Theta} \mathrm{Out}\,(A_\Gamma) \xrightarrow{E_\Theta} \mathrm{Out}\,(A_{\Gamma\setminus\Theta})$$

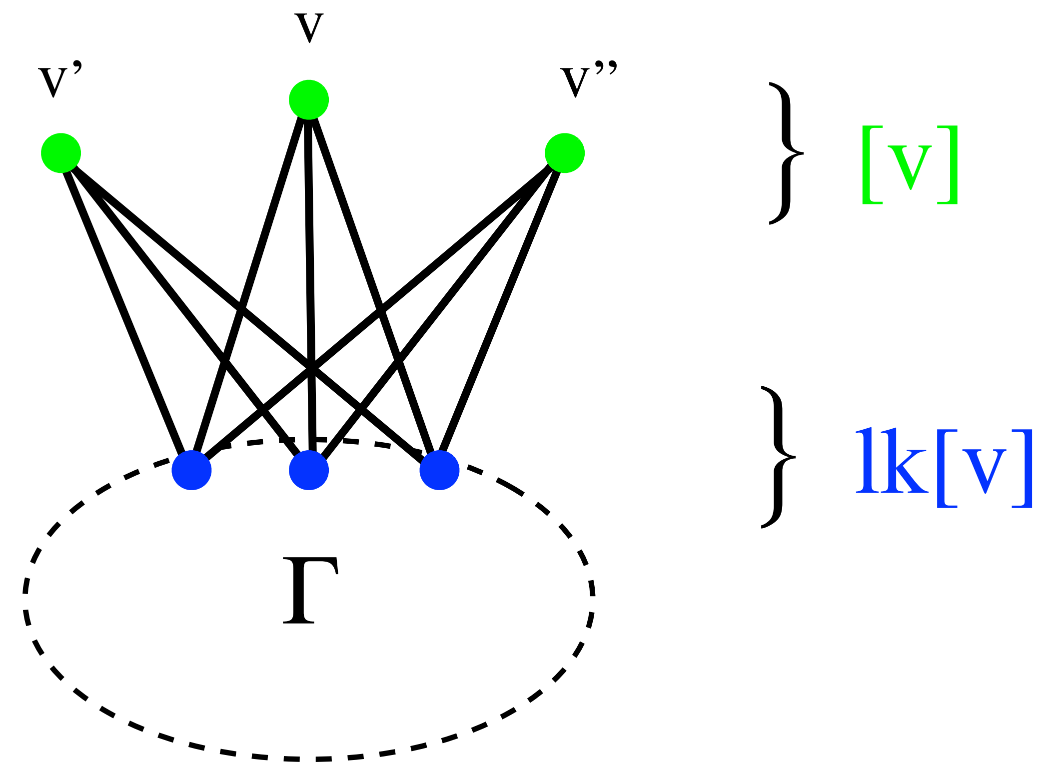
Main idea: use these to reduce questions about $\mathrm{Out}(A_\Gamma)$ to questions about some smaller $\mathrm{Out}(A_\Theta)$ and use induction.

Need to find characteristic subgraphs.

Define a partial ordering on vertices of Γ

$v \leq w$ if $lk(v) \subset st(w)$

$v \sim w$ if $v \leq w$ and $w \leq v$



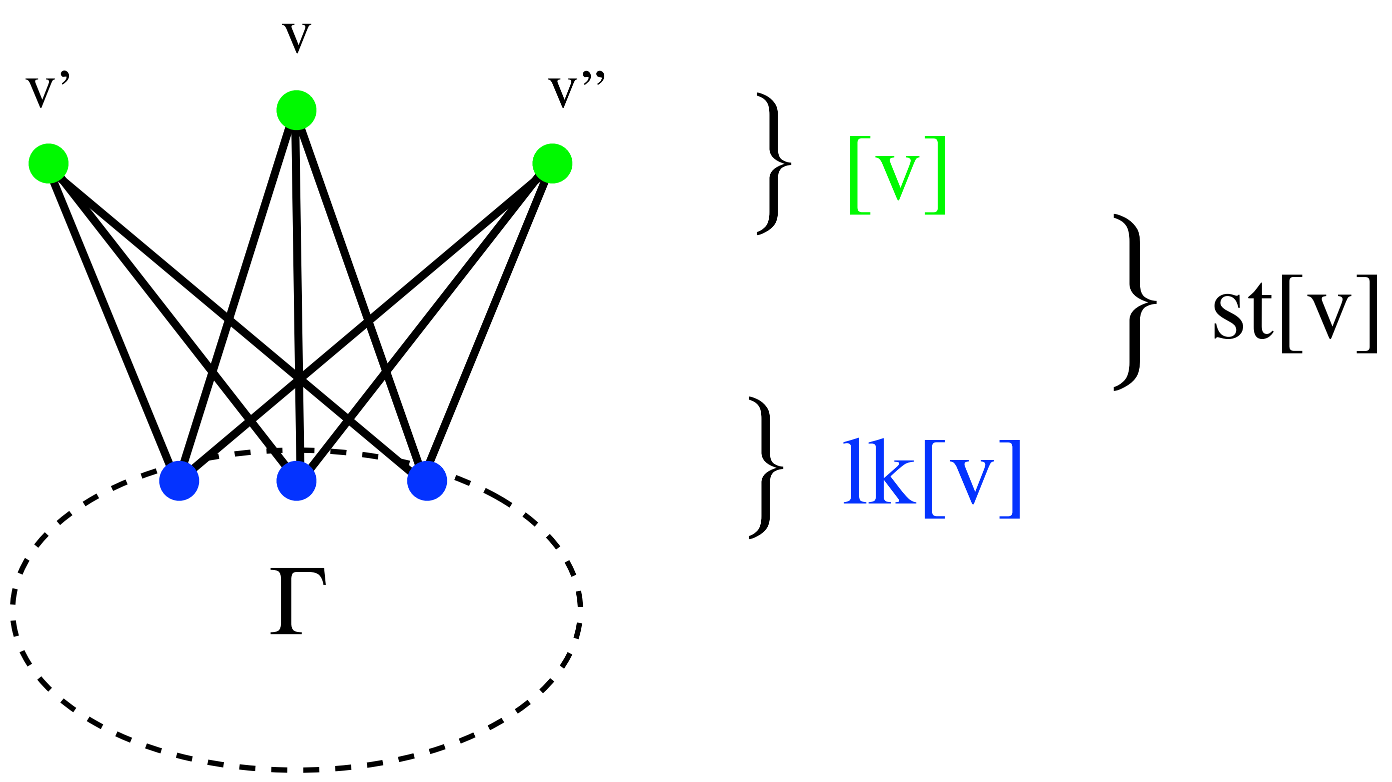
Let $[v]$ = equivalence class of v

$$st[v] = \bigcup_{w \sim v} st(w)$$

$$lk[v] = st[v] \setminus [v]$$

If [v] is maximal, then [v] and st[v] are characteristic!

Proof: check that each of the Servatius-Laurence generators preserves $A_{[v]}$ and $A_{st[v]}$ up to conjugacy.



So if [v] is maximal, we have a homomorphism

$$P_{[v]}\colon \mathrm{Out}^0(A_\Gamma) \xrightarrow{R} \mathrm{Out}^0(A_{st[v]}) \xrightarrow{E} \mathrm{Out}^0(A_{lk[v]})$$

Key Lemma: If Γ is connected, then the kernel K of

$$1 \to K \to \mathrm{Out}^0(A_\Gamma) \xrightarrow{p} \prod \mathrm{Out}^0(A_{lk[v]})$$

is a finitely generated free abelian group. (We give explicit generating set for K.)

Key Lemma: If Γ is connected, then the kernel K of

$$1 \to K \to Out^0(A_\Gamma) \to \prod Out^0(A_{lk[v]})$$

is a finitely generated free abelian group.

Theorem: (C-Crisp-Vogtmann, C-Vogtmann) For all right-angled Artin groups A_Γ, $Out(A_\Gamma)$ is virtually torsion-free and has finite virtual cohomological dimension (vcd).

Proof: Induction on dim A_Γ.
dim $A_\Gamma = 1$ means dim A_Γ = free group. True by Culler-Vogtmann.
Say dim $A_\Gamma > 1$. Note that dim $A_{lk[v]} <$ dim A_Γ for all [v].
So by induction, $Out(A_\Gamma)$ is virtually torsion-free and has finite vcd, *providing* Γ *is connected*.
If Γ is disconnected, A_Γ is a free product and can use results of Guirardel-Levitt on Out(free products).

Also get bounds on the vcd.

Theorem: (C-Bux-Vogtmann) If Γ is a tree, then

$$\mathrm{vcd}(\mathrm{Out}(A_\Gamma)) = e + 2l - 3$$

where e = # edges and l = # leaves.

Theorem: (C-Vogtmann) For all A_Γ, $\mathrm{Out}(A_\Gamma)$ is residually finite.

Proof: Use Key Lemma as before,

$$1 \to K \to \mathrm{Out}^0(A_\Gamma) \xrightarrow{P} \prod \mathrm{Out}^0(A_{lk[v]})$$

to show that its true for connected Γ. Use results of Minasyan-Osin for free products.

(Also proved using different methods by Minasyan.)

Tits Alternative

A group G satisfies the Tits Alternative if every subgroup of G is either virtually solvable or contains F_2.

A group G satisfies the Strong Tits Alternative if every subgroup of G is either virtually abelian or contains F_2.

A_Γ = free group, $Out(A_\Gamma)$ satisfies the Strong Tits Alternative

A_Γ = free abelian, $Out(A_\Gamma)=Gl(n, Z)$ satisfies the Tits Alternative and has non-abelian solvable subgroups.

What about the Tits Alternative for other $Out(A_\Gamma)$?

Try to prove Tits Alternative for $Out(A_\Gamma)$
by induction as above.
Problem: can't get from connected $\Rightarrow$ disconnected Γ

Question: If $G = G_1 * \ldots * G_k$ and $Out(G_i)$ satisfies the Tits Alternative for all i, does the same hold for $Out(G)$?

Definition: Γ is homogeneous of dim 1 if Γ is discrete.
Γ is homogeneous of dim n if Γ is connected and lk(v) is homogeneous of dim n-1 for all v.

Example: The 1-skeleton of any triangulation of a n-manifold is homogeneous of dimesnion n.

Theorem: (C-Vogtmann) Assume Γ is homogeneous of dim n. Then

1. $Out(A_\Gamma)$ satisfies the Tits Alternative.
2. The derived length of every solvable subgroup is $\leq$ n.
3. $\widetilde{Out}(A_\Gamma)$ satisfies the Strong Tits Alternative.

(where $\widetilde{Out}(A_\Gamma)$ is the subgroup generated by all of the Servatius-Laurence generators, *except* adjacent transvections.)

Corollary: If Γ is a connected graph with no triangles and no leaves, then $Out(A_\Gamma) = \widetilde{Out}(A_\Gamma)$ satisfies the Strong Tits Alternative.

Proof: (1) and (2) follow from key lemma and induction.
To prove (3), must show virtually solvable $\Rightarrow$ virtually abelian.
Conner, Gersten-Short: true if every ∞-order element has positive translation length, $\tau(g) = \lim_{k\to\infty} \frac{\| g^k \|}{k} > 0$.

Work in Progress

Find an "outer space" for $\mathrm{Out}(A_\Gamma)$

Outer space for F_n,

$\mathcal{CV}(F_n)$ = {minimal, free actions of F_n on a tree}

What is the analogue for $\mathrm{Out}(A_\Gamma)$?

$\mathcal{CV}(A_\Gamma)$ = {minimal, free actions of A_Γ on n-dim'l CAT(0) rectangle complexes} ??