POISSON GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 51
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

CLASSIFICATIONS OF STAR PRODUCTS AND DEFORMATIONS OF POISSON BRACKETS

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Abstract. On the algebra of functions on a symplectic manifold we consider the pointwise product and the Poisson bracket; after a brief review of the classifications of the deformations of these structures, we give explicit formulas relating a star product to its classifying formal Poisson bivector.

1. Introduction. In 1974 M. Flato, D. Sternheimer and A. Lichnerowicz [FLS] defined and studied the 1-differentiable deformations of the Poisson bracket on $\mathcal{C}^\infty(M)$ where M is a symplectic manifold. They distinguished a subset of deformations depending only on the geometry of M, which they called "inessential" (for physical reasons). Around twenty years later it was shown that this subset, quotiented by the natural equivalence relation, is exactly the classifying set of all the star products on a symplectic manifold [F1, NT] and, more generally, on a Poisson manifold [K].

After a short review of the results in the problem of classification that appeared since [FLS] we give explicit relations between 1-differentiable deformations of the Lie algebra $(\mathcal{C}^\infty(M), \{.,.\})$ and the general deformations of the associative algebra $(\mathcal{C}^\infty(M), .)$ (the "star products") for a symplectic manifold M. In particular, we try to see explicitly the Kontsevich isomorphism [K] between classes of star products and classes of formal Poisson bivectors.

2. Notation. (M, ω) is a symplectic manifold; $\sharp : \Gamma(M, \bigotimes^\bullet T^*M) \to \Gamma(M, \bigotimes^\bullet TM)$ is the canonical isomorphism given by ω; Λ is the Poisson bivector corresponding to ω by $\sharp$; d denotes the exterior derivative and $X_f = \sharp(df)$ is the hamiltonian vector field corresponding to $f \in \mathcal{C}^\infty(M)$.

2000 *Mathematics Subject Classification*: Primary 58H15; Secondary 81S10.
The paper is in final form and no version of it will be published elsewhere.

Definition 1. A *star product* on a smooth manifold M is an associative product on $\mathcal{C}^\infty(M)[[\hbar]]$, denoted by $*$, such that $f * g = f.g + \sum_{n\geq 1} \hbar^n C_n(f,g)$ where the C_n's are bidifferential operators.

If $\tilde{*}$ is another star product on M we say that $*$ is *equivalent* to $\tilde{*}$ ($* \approx \tilde{*}$) if there exists $T_\hbar = Id + \sum_{n\geq 1} T_n$, where the T_n's are differential operators, such that $T_\hbar(f * g) = T_\hbar(f) \tilde{*} T_\hbar(g)$.

Notice that if there exists a star product on M then $C_1(f,g) - C_1(g,f)$ is a Poisson bracket. It means that M is at least a Poisson manifold.

A star product on (M,ω) (or on a Poisson manifold (M,Λ)) is a star product beginning with $C_1(f,g) = -\frac{i\hbar}{2}\omega(X_f, X_g) = -\frac{i\hbar}{2}\Lambda(df, dg)$.

$Z^n_{DR}(M)$ is the space of de Rham n-cocycles (closed n-forms) on M and $H^n_{DR}(M)$ the n-th space of de Rham cohomology.

$H^n_{CE}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M))$ denotes the n-th space of Chevalley-Eilenberg cohomology.

3. Classifications

3.1. *Deformed Poisson brackets.* The idea of deformation quantization theory is to obtain a Lie bracket on the functions on the phase space corresponding to the Lie bracket of operators in quantum mechanics by deformation of the Poisson bracket. The first result in this direction is due to Moyal [M]. The first works on the theory (in the early 70's) did not consider star products but tried to deform directly the Poisson bracket on a symplectic manifold in the following way:

$$[f,g]_\hbar = \{f,g\} + \sum_{n\geq 1} \hbar^n P_n(f,g) \quad \text{where the } P_n\text{'s are bidifferential operators.} \tag{1}$$

$[.,.]_\hbar$ is said to be equivalent to $\langle .,. \rangle_\hbar$ if there exists $T_\hbar = Id + \sum_{n\geq 1} T_n$, where the T_n's are differential operators such that $T_\hbar[f,g]_\hbar = \langle T_\hbar(f), T_\hbar(g)\rangle_\hbar$. If we consider $\hbar$ as a formal parameter, we can use the Gerstenhaber theory [Ge] of formal algebraic deformations and then, as usual, the infinitesimal deformations (the P_1's) are classified by $H^2_{CE}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M))$.

One can also consider a subproblem: to find the deformations of type (1) taking for the P_n's bidifferential operators of order 1 in each variable (therefore vanishing on constants, v.c.). They were called (inessential) 1-differentiable deformations and studied in one of the first papers on deformation quantization [FLS]. It is easy to see that this kind of deformations coincides with the notion of formal Poisson bivectors on M with constant term equal to Λ, i.e. the elements of

$$\mathcal{P} = \{\Lambda_\hbar \in \Gamma(M, \textstyle\bigwedge^2 TM)[[\hbar]]; [\Lambda_\hbar, \Lambda_\hbar]_S = 0 \text{ and } \Lambda_\hbar = \Lambda + \hbar \ldots\}, \tag{2}$$

where $[.,.]_S$ denotes the Schouten bracket.

In this context, one can restrict the Chevalley-Eilenberg cohomology and obtain that

$$H^2_{1-diff.,v.c.}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) \simeq H^2_{DR}(M), \tag{3}$$

which classifies the infinitesimal deformations.

In 1987 Lecomte [Le] showed that, for M symplectic,

$$\mathcal{P}_{/\sim} \simeq H^2_{DR}(M)[[\hbar]], \tag{4}$$

where $\sim$ denotes the equivalence. The map is given by $\Lambda_\hbar = \mu_\hbar(\Omega^\hbar)$ for $\Omega^\hbar = \omega + \alpha^\hbar$, with $\alpha^\hbar \in \hbar H^2_{DR}(M)[[\hbar]]$, $\mu_\hbar$ being the canonical $\sharp$-isomorphism from $\Gamma(M, \bigotimes^\bullet T^*M)[[\hbar]]$ to $\Gamma(M, \bigotimes^\bullet TM)[[\hbar]]$ given by $\Omega^\hbar$. In [B] we define a composition law $\diamond$ on $\Gamma(M, \bigotimes^2 T^*M)[[\hbar]]$ such that we can explicitly write $\Lambda_\hbar = \Lambda - \sum_{p\geq 1} \hbar^p \sharp(\alpha^\hbar)^{\diamond p}$.

3.2. *Star products.* Considering the star products, it was early noticed [Gu, Li] that the possible "degrees of freedom" to continue a deformation were classified by $H^2_{DR}(M)$ in each power of $\hbar$. To obtain a complete result about the classification a general existence theorem was needed. That was done, for the symplectic manifolds, in [DWL](83) and in [F1] (in 94, with a completely different and more explicit method) and in [K] for the Poisson manifolds (i.e. the general case).

In 1995, Nest and Tsygan [NT] showed, using Fedosov construction [F1, F2], that

$$\mathcal{S}_{/\approx} \simeq H^2_{DR}(M)[[\hbar]], \tag{5}$$

where $\mathcal{S}$ is the set of all the star products on the symplectic manifold (M, ω) and $\approx$ denotes the usual equivalence. Actually, in [F1, F2] Fedosov constructs a unique star product for each element of $\omega + \hbar Z^2_{DR}(M)[[\hbar]]$.

Finally, we know by the recent and major work of Kontsevich [K] that formal Poisson bivectors are the good objects to classify the star products. He showed that there exists the following isomorphism:

$$\{\text{star products on M}\}_{/\approx} \simeq \{\Lambda_\hbar \in \Gamma(M, \textstyle\bigwedge^2 TM)[[\hbar]];\ [\Lambda_\hbar, \Lambda_\hbar]_S = 0\}_{/\sim}. \tag{6}$$

So (4) and (5) are now related in a more general setting: for a Poisson manifold (M, Λ), we have $\mathcal{S}_{/\approx} \simeq \mathcal{P}_{/\sim}$.

4. Explicit formulae for star products and 1-differentiable deformations. The idea of this work came from the comparison between (3) and (4) on one side and (5) on the other: we want to see how a given star product is constructed from an element of $\mathcal{P}$, i.e. from a 1-differentiable deformation. This brings us to consider the following question: can we read from the explicit formula of a star product its "defining" formal Poisson bivector? In terms of Kontsevich's work on the quantization of Poisson manifolds [K], it is an attempt to see more explicitly the isomorphism (6). Understanding this isomorphism in the symplectic case is a first step to obtain the same kind of results in the Poisson case.

The characteristic class of a star product is of the form

$$\Lambda_\hbar = \Lambda + \sum_{n\geq 1} \hbar^n \Lambda_n. \tag{7}$$

In order to locate the first non-zero Λ_n in the explicit formula of the star product we have the following theorem, from which, actually, (5) can be easily derived:

THEOREM 1 ([BCG]). *Let $*$ and $\tilde{*}$ be two star products on a symplectic manifold M. Suppose that $*$ and $\tilde{*}$ coincide up to the order k. Then the skewsymmetric part of their difference at the order $k+1$ is a closed 2-form α. Moreover we have: $* \approx \tilde{*}$ mod $\hbar^{k+2} \Leftrightarrow \alpha$ is exact.*

The next theorem shows that in fact $f * g - f \tilde{*} g$ contains a complete 1-differentiable deformation of Λ with $\sharp\alpha$ as the first non-zero term. As a corollary we obtain that (7) explicitly appears in the formula of the associated star product. Moreover, among all the 1-differentiable terms of a Fedosov star product, the formal Poisson bivector (7) can be characterized as follows.

Consider (M, ω) a symplectic manifold, ∇ a symplectic connection on M and R its curvature. Let $*$ be a Fedosov star product on M: for $f, g \in \mathcal{C}^\infty(M)$ we have

$$f * g = f.g - \frac{i\hbar}{2}\Lambda(f, g) + \sum_{n\geq 2} \hbar^n C_n(f, g). \tag{8}$$

Let $\Omega^\hbar \in \omega + \hbar Z^2_{DR}(M)[[\hbar]]$ be the *Weyl curvature* of $*$ (actually Fedosov [F1, F2] takes $-\Omega^\hbar$). The class of $*$ (in the sense of (5)) is the class of $\Omega^\hbar$ in $[\omega] + \hbar H^2_{DR}(M)[[\hbar]]$.

Let $\tilde{*}$ be the Fedosov star product of Weyl curvature $\tilde{\Omega}^\hbar = \Omega^\hbar + \hbar^k\alpha$, with $\alpha \in Z^2_{DR}(M)$, then

$$f\tilde{*}g = f.g - \frac{i\hbar}{2}\Lambda(f, g) + \sum_{n\geq 2} \hbar^n \tilde{C}_n(f, g). \tag{9}$$

Theorem 2. *The change $\Omega^\hbar \to \Omega^\hbar + \hbar^k\alpha$ adds the series $\frac{i}{2}\sum_{p\geq 1} \hbar^{pk+1}\sharp\alpha^{\diamond p}$ to the explicit expression of the star product $*$. The series $\Lambda - \sum_{p\geq 1} \hbar^p\sharp\alpha^{\diamond p}$ is a formal Poisson bracket and contains all the 1-differentiable terms of $\tilde{*}$ not depending explicitly on R, $\Omega^\hbar - \omega$ and derivatives of α.*

Therefore we would like to see the difference of two star products as an "addition" (in a sense to define) of a certain number of 1-differentiable deformations of the form $\Lambda_\hbar = \Lambda - \sum_{p\geq 1} \hbar^{pk+1}\sharp\alpha^{\diamond p}$ (which can be summed into a big one, given by the addition in $\Gamma(M, \bigwedge^2 T^*M)[[\hbar]]$). In order to make this more precise, we look (in [B]) at the 1-differentiable part of $f * g$ and find different series of terms of the form $\Lambda_\hbar(Y_f, Y_g)$ where Y_f and Y_g are 1-forms on M depending on f and g.

Proof (sketch). In Fedosov method we have to solve three successive equations to obtain the formula for the star product. We compute explicitly all the terms of the type described in the theorem along these three steps and we obtain a term $\sharp\alpha^{\diamond p}$ in each order $\hbar^{pk+1}$, $p \geq 1$. Finally, the coefficients of these terms are shown to be the coefficients of the Taylor series of $\frac{i}{2(1-x)}$. ∎

Theorem 2 can be reformulated in order to make the relation between this formal Poisson bracket and the characteristic class of the star product more explicit:

Corollary 3. *Let $*$ be the Fedosov star product of trivial Weyl curvature ω and $\tilde{*}$ the one with curvature $\Omega^\hbar = \omega + \alpha^\hbar$. $\Omega^\hbar$ appears explicitly in the formal Poisson form $\Lambda_\hbar$ as a part of the formula for $\tilde{*}$. $\Lambda_\hbar$ can be seen as all the 1-differentiable terms of $\tilde{*}$ not containing R or derivatives of $\alpha^\hbar$. We have:*

$$\begin{aligned} f\tilde{*}g &= f * g + \frac{i\hbar}{2}\sum_{p\geq 1} \sharp(\alpha^\hbar)^{\diamond p}(f, g) + \rho(f, g) \\ &= f.g - \frac{i\hbar}{2}\Big(\bar{\omega}(f, g) - \sum_{p\geq 1} \sharp(\alpha^\hbar)^{\diamond p}(f, g)\Big) + \sum_{n\geq 2} \hbar^n C_n(f, g) + \rho(f, g) \end{aligned}$$

$$= f.g - \frac{i\hbar}{2}\Lambda_\hbar(f,g) + \sum_{n\geq 2}\hbar^n C_n(f,g) + \rho(f,g), \tag{10}$$

where the terms occurring in the remainder ρ either depend explicitly on the curvature R or on derivatives of $\alpha^\hbar$, or are not 1-differentiable.

REMARK 1. Expression (10) shows that on (M,ω), for a star product $\tilde{*}$ of characteristic class $\Lambda_\hbar$, the corresponding bracket $\{f,g\}_{\tilde{*}} = \frac{i}{\hbar}(f\tilde{*}g - g\tilde{*}f)$ can be seen not only as a deformation of the Lie algebra $(\mathcal{C}(M),\Lambda)$ but also as a deformation of the "formal" Lie algebra $(\mathcal{C}(M)[[\hbar]],\Lambda_\hbar)$. Thus $\tilde{*}$ can be viewed as the star product of trivial characteristic class on the "formal" symplectic manifold $(M,\Omega^\hbar)$, i.e. M endowed with the formal symplectic structure given by $\Omega^\hbar$. We can also consider $\tilde{*}$ as a deformation of $*$ with $\frac{i}{2}\sum_{p\geq 1}\sharp(\alpha^\hbar)^{\diamond p}$ as infinitesimal deformation.

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