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GEOMETRIC QUANTIZATION AND NO-GO THEOREMS

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Dedicated to the memory of Stanisław Zakrzewski

Abstract. A geometric quantization of a Kähler manifold, viewed as a symplectic manifold, depends on the complex structure compatible with the symplectic form. The quantizations form a vector bundle over the space of such complex structures. Having a canonical quantization would amount to finding a natural (projectively) flat connection on this vector bundle. We prove that for a broad class of manifolds, including symplectic homogeneous spaces (e.g., the sphere), such connection does not exist. This is a consequence of a "no-go" theorem claiming that the entire Lie algebra of smooth functions on a compact symplectic manifold cannot be quantized, i.e., it has no essentially nontrivial finite-dimensional representations.

1. Introduction. The quantization of a classical mechanical system is, in its most ambitious form, a representation R of some subalgebra A of the Lie algebra of smooth functions by self-adjoint operators on a Hilbert space Q. The Lie algebra structure on the space of functions is given by the Poisson bracket and the representation is usually assumed to satisfy some extra conditions which we will discuss later. It is generally accepted, however, that such a quantization does not exist when the algebra A is too large. (See, e.g., [Atk, Av1, Av2], and also [GGT, GGG] for a detailed discussion. We will return to this subject later.) In other words, the quantization problem in the strict form stated above has no solution. Results claiming that there are no such quantizations are often referred to as *no-go theorems*.

Thus, one often tries either to just construct the Hilbert space Q, without quantizing the functions, or to only find the algebra of "operators" representing A without a Hilbert space on which they would act. The latter program, which can successfully be carried

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out on symplectic manifolds, is called deformation quantization (see [We] for a review) and we are not concerned with it here. The former question, addressed by geometric quantization (see, e.g., [Wo]), is the subject of the present paper.

One of the main problems with geometric quantization, arising already for nice symplectic manifolds such as S^2, is that the construction of the geometric quantization space inevitably involves an extra structure (polarization). This leads to the question of whether the quantization spaces constructed for different polarizations can be naturally identified. (Under rather weak additional hypotheses the spaces are isomorphic.) In this paper we show that the answer to this question is *negative* for a broad class of manifolds including S^2. The problem of geometric quantization has no solution either!

Before we recall what geometric quantization is and outline our proof, let us return to the no-go theorems. The first such theorem is a classical result due to Groenewold and Van Hove stating that the algebra of polynomials on $\mathbb{R}^{2n}$ has no representation that would restrict to the Schrödinger representation of the Heisenberg algebra, i.e., the algebra of linear functions. (The Schrödinger representation is the unique unitary representation of the Heisenberg group; see, e.g., [LV] for more details and further references.) This result lies at the foundation of the general principle that a sufficiently large algebra of functions A cannot be quantized. (See [Atk, Av1, Av2, Gr, GGH, GGT, GGG], and also Section 3 for more details.)

The self-adjoint representations of A are required to satisfy certain extra conditions to warrant the title "quantizations". Although there is no consensus on what the conditions are, their main goal is to ensure that the representation is "small". For instance, in the majority of examples, the conditions include that the representation of the constant unit function is $const I$, where $const \neq 0$. (This is the case with the Groenewold–Van Hove theorem.) Such conditions exclude representations like the one arising from the natural action of the group of symplectomorphisms on the space of L^2-functions. When the symplectic manifold M in question is compact (and connected), its quantization is usually assumed to be finite–dimensional with the dimension equal to the Riemann–Roch number $\mathrm{RR}(M)$. A sufficiently large Lie algebra A of functions on M has no "essentially nontrivial" finite–dimensional representations, i.e., each such representation factors through a representation of $\mathbb{R} = A/\{A, A\}$. This rather well-known fact alone is sufficient to conclude that under some natural hypotheses about the manifold, M cannot be quantized in a canonical way. In other words, the geometric quantization spaces obtained for different polarizations cannot be naturally identified. (See Section 3).

We now return to the question of naturally identifying various quantization spaces. Our approach is inspired by recent results on quantization of moduli spaces of flat connections. (See, e.g., [ADPW, Ati, Hi] and references therein.) Given an integral compact symplectic manifold (M, ω), we consider the space $\mathcal{J}$ of all complex structures compatible with ω (i.e., complex polarizations). Then, for every $J \in \mathcal{J}$, the quantization $Q_J(M, k)$ is defined to be the space of J-holomorphic sections of the pre-quantum line bundle L^k. We take k sufficiently large to ensure that a vanishing theorem applies, so that $\dim Q_J(M, k) = \mathrm{RR}(M, k\omega)$. (By definition, L is a line bundle with a connection ∇ whose curvature is ω. The pair, ∇ and J, gives rise to the structure of a holomorphic line bundle on L, and so on L^k.)

Fix k, and consider the collection $\{Q_J(M,k)\}_{j\in\mathcal{J}}$ as a vector bundle E over $\mathcal{J}$. Here we ignore the fact that the lower bound on k necessary for the vanishing theorem may depend on J. (This leaves open the interesting question: Is there a universal J-independent bound?) An *identification* of quantizations (or their projectivizations) is the same as a (projectively) flat connection on E. The identification is *natural* if it is equivariant with respect to the group of symplectomorphisms Ham. Strictly speaking this group does not act on E, but it has a central extension $Cont_0$ which acts. The Lie algebra of $Cont_0$ is the algebra $A = C^\infty(M)$ with respect to the Poisson bracket $\{\ ,\ \}$. (The group $Cont_0$ is a subgroup of the group of contactomorphisms of the unit circle bundle associated with L.)

If it existed, a (projectively) flat $Cont_0$-invariant connection would give rise to a projective representation R of A on the fiber of E. Since this fiber is finite–dimensional, the representation R must factor through $A/\{A,A\} = \mathbb{R}$ as we pointed out above. On the other hand, such a representation R cannot exist if for some $J_0 \in \mathcal{J}$, the Kähler manifold (M,ω,J_0) has a continuous group G of Hamiltonian symmetries. For R would restrict to a non-trivial representation of the Lie algebra of G on $Q_{J_0}(M,k)$. This contradicts the fact that R factors through $A/\{A,A\}$. Hence, a $Cont_0$-invariant (projectively) flat connection does not exist for a broad class of manifolds M including homogeneous spaces and, in particular, S^2. The details are given in Section 2.

Of course, it may well happen that $\mathcal{J}$ is empty. In this case, instead of working with holomorphic sections of L^k, one considers the index of the $Spin^{\mathbb{C}}$-Dirac operator D or of the rolled-up $\bar\partial$ operator, [Du]. The index is a virtual space, which still has the right dimension $\mathrm{RR}(M,k\omega)$. For $\bar\partial$ and D there are again vanishing theorems (see [GU] and [BU]), ensuring that the index is a genuine vector space $Q_J(M,k)$. This space is equal to $H^0(M,\mathcal{O}(L^k))$ when the manifold is Kähler and k is large enough. Both of the operators depend on a certain extra structure on M, e.g., an almost complex structure for $\bar\partial$. These extra structures form a space serving, similarly to $\mathcal{J}$, as the base of the index vector bundle E, and the above argument applies word-for-word. (This can be viewed as an answer to the question asked in [Fr].)

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2. Natural flat connections on the vector bundle of quantizations. Let M be a compact Kähler manifold with symplectic form ω, which is assumed throughout this section to represent an integral cohomology class. As usual in geometric quantization, fix a Hermitian line bundle L over M with $c_1(L) = [\omega]$ (the prequantization line bundle) and a Hermitian connection on L whose curvature is ω. Consider the space $\mathcal{J}$ of all complex structures J on M which are compatible with ω in the sense that $\omega(\cdot, J\cdot)$ is a Riemannian metric on M. For every $J \in \mathcal{J}$, the connection on L gives rise to the structure of a holomorphic line bundle on L. Then, given a sufficiently large k, the vanishing theorem applies to the line bundle L^k for a fixed $J \in \mathcal{J}$. In other words, $H^q(M,\mathcal{O}(L^k)) = 0$ when $q > 0$ and $k \geq k_0$, where k_0 depends on J. Thus, we can take the space of J-holomorphic

sections $H^0(M, \mathcal{O}(L^k))$, $k \geq k_0$, of L^k as the quantization of M. Denote it by $Q_J(M,k)$ or just $Q_J(M)$ when k is fixed or irrelevant.

Let $\mathcal{J}_0$ be a C^1-small neighborhood of a fixed complex structure $J_0 \in \mathcal{J}$. It is not difficult to see that one can take the same k_0 for all $J \in \mathcal{J}_0$. Note that sometimes the same is true for the entire space $\mathcal{J}$. For example, this is the case when $\dim_{\mathbb{R}} M = 2$. Fixing $k \geq k_0$, we obtain a vector bundle E over $\mathcal{J}_0$ whose fiber over J is $Q_J(M,k)$.

Let *Ham* be the group of Hamiltonian symplectomorphisms of M. The elements of *Ham* are symplectomorphisms which can be given as time-one flows of time-dependent Hamiltonians. It is clear that *Ham* acts (locally) on $\mathcal{J}_0$.

To lift this action to E, consider the group *Cont* of diffeomorphisms of the unit circle bundle U of L which preserve the connection form θ. Clearly, θ is a contact form on U. Thus, *Cont* consists of those contact transformations which preserve the contact form θ itself (not just the contact field), and which, as a consequence, are also bundle automorphisms. Let $Cont_0$ be the identity connected component in *Cont*, i.e., the elements of $Cont_0$ are isotopic to *id* in *Cont*. Every element of $Cont_0$ naturally covers a symplectomorphism of M, which belongs to *Ham*. The projection $Cont_0 \to Ham$ is surjective, and it makes $Cont_0$ into a one-dimensional central extension of *Ham* by U(1). The Lie algebra of $Cont_0$ is just $C^\infty(M)$. Since $Cont_0$ acts on L, and so on L^k, it also acts (locally) on E and the latter action is a lift of the *Ham*-action on $\mathcal{J}_0$. A connection on E is said to be *natural* if it is invariant under the $Cont_0$-action.

Now we are in a position to state our main observation, which will be proved in the next section:

THEOREM 1. *Assume that the stabilizer G of J_0 in Ham has positive dimension and that the infinitesimal representation of G on $Q_{J_0}(M)$ is non-trivial. Then there is no natural (projectively) flat connection on E.*

When M is two-dimensional, the theorem applies to $M = S^2$ only, showing that the geometric quantizations of S^2 for different complex structures cannot be identified. Note that there are many (projectively) flat connections on E, for $\mathcal{J}$ and $\mathcal{J}_0$ are contractible, and many natural connections on E, but there is no connection which is simultaneously flat and natural.

REMARK 1. 1. As mentioned above, Theorem 1 extends word-for-word to compact symplectic, not necessarily Kähler, manifolds. In this case, $\mathcal{J}$ is the space of almost-complex structures compatible with the symplectic structure and $\mathcal{J}_0$ is a neighborhood of a given structure J_0 in $\mathcal{J}$. The quantization bundle E over $\mathcal{J}_0$ is defined using the vanishing theorems for either the $Spin^{\mathbb{C}}$-Dirac operator D or the rolled-up $\bar{\partial}$ operator (see [GU, BU]). Note also that in this case $\mathcal{J}$ is a contractible Fréchet manifold.

2. What makes this theorem somewhat surprising is a recent collection of constructions of projectively flat connections related to topological quantum field theory. Axelrod–Della Pietra–Witten [ADPW], and following them Atiyah [Ati] and Hitchin [Hi], constructed quantizations Q_J of the moduli space $\mathcal{M}_\Sigma$ of flat vector bundles over a Riemann surface Σ. Here the additional polarization data is a complex structure on Σ. Their connections are natural with respect to transformations of $\mathcal{M}_\Sigma$ induced by those of Σ, and not with

respect to all of $Cont_0(\mathcal{M}_\Sigma)$. Note also that our Theorem 1 seems to contradict what is said in [Ati], page 34–35.

3. Hodge theory for a compact manifold X associates the vector space H^p_g of g-harmonic p-forms on X to each Riemannian metric g on X. This space is canonically isomorphic to the p-th real cohomology of X. Consequently, Hodge theory defines a *flat* connection on the vector bundle $H^p \to \mathcal{M}$ over the space $\mathcal{M}$ of Riemannian metrics on X. This connection is $Diff(X)$ invariant. As a result, we have an induced representation of $Diff(X)$ on each H^p_g. Of course, this representation is trivial on the identity component $Diff_0(X)$ of X. Consequently, this induces the usual representation of the mapping class group $Diff(X)/Diff_0(X)$ on cohomology.

4. When the local action of Ham on $\mathcal{J}_0$ is free, it induces a projectively flat connection along the orbit of Ham. This connection is natural but does not seem to be of any interest for quantization.

3. No-go Theorems. Theorem 1 is an easy consequence of the general no-go theorems discussed in this section. Let (M, ω) be a connected symplectic manifold. Now ω is not assumed to be integral and M need not be compact. Let $A = C^\infty_c(M)$ be the Lie algebra of smooth compactly supported functions on M with respect to the Poisson bracket. Denote by A_0 the commutant $A_0 = \{A, A\}$ of A. In fact, A_0 is just the algebra of functions with zero mean and, therefore, A_0 is a maximal ideal of codimension one.

THEOREM 2. *The commutant A_0 is the only ideal of finite codimension in the Lie algebra A.*

This theorem has a long history. For a compact manifold, it is due to Avez, [Av2], who proposed a very interesting proof relying on the properties of the symplectic Laplacian. An algebraic version of Theorem 2, which applies to a broad class of Poisson algebras, has been obtained by Atkin [Atk]. This class includes the algebra of compactly supported functions and the algebra of (real) analytic functions when (M, ω) is (real) analytic. Furthermore, it appears that the reasoning and the key results of [Atk] (see Theorem 6.9 and Section 9) apply to the Poisson algebra of polynomial functions on a coadjoint orbit for a compact semisimple Lie algebra, which would give a generalization of the no-go theorem of [GGH]. A simple direct proof of Theorem 2 can be obtained by adapting the methods of [Om] (Chapter X), which, in turn, go back to Shanks and Pursell [SP].

REMARK 2. Theorem 2 is just a reflection of the general fact that the algebra A, like many infinite-dimensional algebras of vector fields, is in a certain sense "simple". This assertion should not be taken literally – A has many ideals of infinite codimension (functions supported within a given set) – but the Lie group of A is already simple in the algebraic sense [Ba]. (For more details see [Av1, Av2, ADL, Om, Atk], and references therein.)

In many of the papers quoted above, in varying generality, the following description of maximal ideals in A is given. For any $x \in M$, let I_x be the ideal of A formed by functions vanishing at x together with all their partial derivatives. It is well known and easy to see that I_x is a maximal ideal. In other words, the Lie algebra of formal power series with

Poisson bracket is simple. *These and A_0 are the only maximal ideals in A, i.e., every maximal ideal is either A_0 or I_x for some x.*

COROLLARY 3. *Any nontrivial finite-dimensional representation of A factors through a representation of $A/A_0 = \mathbb{R}$.*

Thus, if a quantization of A is to be understood as just a finite-dimensional representation, we conclude that there are no "non-trivial" quantizations. It is also worth noticing that the corollary still holds for representations R in a Hilbert space by bounded operators, provided that when M is compact $R(1)$ is a scalar operator [Av2].

Now we are in a position to prove Theorem 1 by reducing it to the no-go theorem (Theorem 2).

PROOF. Arguing by contradiction, assume that there is a natural projectively flat connection on E. This connection will be thought of as a flat connection on the projectivization bundle PE of E. Our goal is to construct, using this connection, a representation of $A = C^\infty(M)$, the Lie algebra of $Cont_0$, on the fiber $PQ = PQ_{J_0}(M)$ whose existence would contradict Theorem 2.

For $f \in A$, denote by $\tilde{\phi}_f^t$ the (local) flow on E generated by f in time t and by ϕ_f^t the (local) flow on $\mathcal{J}_0$ induced by the Hamiltonian flow of f on M in time t. (In fact, $\tilde{\phi}_f^t$ is induced by the contact flow of f on the unit circle bundle.) Let $\Pi(J_1, J_2)$ be the parallel transport from the fiber of PE over J_1 to the fiber over J_2. Since the connection on PE is flat, this operator is well defined. Finally, define a linear homomorphism $R(f)\colon PQ \to PQ$ as

$$R(f)(v) = \frac{d}{dt}\Pi\big(\phi_f^t(J_0), J_0\big)\tilde{\phi}_f^t(v)\bigg|_{t=0},$$

where $v \in PQ$. In other words, v is moved to the fiber over $\phi_f^t(J_0)$ using the group action and then transported back to PQ by means of the connection. We claim that R is a (projective) representation of A in Q, i.e.,

$$R(\{f, g\}) = [R(f), R(g)]$$

in the Lie algebra of the group of projective transformations of Q.

To see this, recall that

$$\tilde{\phi}_{\{f,g\}}^{\tau^2} = \tilde{\phi}_f^\tau \tilde{\phi}_g^\tau \tilde{\phi}_f^{-\tau} \tilde{\phi}_g^{-\tau} + O(\tau^3).$$

Furthermore, $\Pi(\phi_{\{f,g\}}^{\tau^2}(J_0), J_0)$ is equal, up to $O(\tau^3)$, to the parallel transport from the fiber over $\phi_f^\tau \phi_g^\tau \phi_f^{-\tau} \phi_g^{-\tau}(J_0)$ to PQ. Thus,

$$R(\{f, g\}) = \lim_{\tau \to 0} \frac{1}{\tau^2}\Pi\big(\phi_f^\tau \phi_g^\tau \phi_f^{-\tau} \phi_g^{-\tau}(J_0), J_0\big)\tilde{\phi}_f^\tau \tilde{\phi}_g^\tau \tilde{\phi}_f^{-\tau} \tilde{\phi}_g^{-\tau}.$$

Let us now focus on $[R(f), R(g)]$. By definition,

$$[R(f), R(g)] = \lim_{\tau \to 0} \frac{1}{\tau^2}(\text{commutator}),$$

where

$$\begin{aligned}\text{commutator} = \{&(\Pi(\phi_f^\tau(J_0), J_0)\tilde{\phi}_f^\tau)(\Pi(\phi_g^\tau(J_0), J_0)\tilde{\phi}_g^\tau)\\ &\times(\Pi(\phi_f^\tau(J_0), J_0)\tilde{\phi}_f^\tau)^{-1}(\Pi(\phi_g^\tau(J_0), J_0)\tilde{\phi}_g^\tau)^{-1}\}.\end{aligned}$$

To calculate the commutator, we use the assumption that the connection is natural, i.e., $Cont_0$-invariant. Explicitly, this assumption means that

$$\Pi(J_1, J_2)\tilde{\phi}_h^t = \tilde{\phi}_h^t \Pi(\phi_h^t J_1, \phi_h^t J_2)$$

for any $h \in A$ and $t \in \mathbb{R}$. Observing also that $\Pi(J_1, J_2)^{-1} = \Pi(J_2, J_1)$, we transform the commutator on the right hand side of the expression for $[R(f), R(g)]$ as follows:

$$\begin{aligned}
\text{commutator} &= \Pi\big(\phi_f^\tau(J_0), J_0\big)\tilde{\phi}_f^\tau \Pi\big(\phi_g^\tau(J_0), J_0\big)\tilde{\phi}_g^\tau \\
&\qquad \times \tilde{\phi}_f^{-\tau}\Pi\big(J_0, \phi_f^\tau(J_0)\big)\tilde{\phi}_g^{-\tau}\Pi\big(J_0, \phi_g^\tau(J_0)\big) \\
&= \Pi\big(\phi_f^\tau(J_0), J_0\big)\Pi\big(\phi_f^\tau\phi_g^\tau(J_0), \phi_f^\tau(J_0)\big)\tilde{\phi}_f^\tau\tilde{\phi}_g^\tau \\
&\qquad \times \tilde{\phi}_f^{-\tau}\Pi\big(J_0, \phi_f^\tau(J_0)\big)\tilde{\phi}_g^{-\tau}\Pi\big(J_0, \phi_g^\tau(J_0)\big) \\
&= \Pi\big(\phi_f^\tau(J_0), J_0\big)\Pi\big(\phi_f^\tau\phi_g^\tau(J_0), \phi_f^\tau(J_0)\big) \\
&\qquad \times \Pi\big(\phi_f^\tau\phi_g^\tau\phi_f^{-\tau}(J_0), \phi_f^\tau\phi_g^\tau(J_0)\big)\tilde{\phi}_f^\tau\tilde{\phi}_g^\tau\tilde{\phi}_f^{-\tau}\tilde{\phi}_g^{-\tau} \\
&\qquad \times \Pi\big(J_0, \phi_g^\tau(J_0)\big) \\
&= \Pi\big(\phi_f^\tau(J_0), J_0\big)\Pi\big(\phi_f^\tau\phi_g^\tau(J_0), \phi_f^\tau(J_0)\big) \\
&\qquad \times \Pi\big(\phi_f^\tau\phi_g^\tau\phi_f^{-\tau}(J_0), \phi_f^\tau\phi_g^\tau(J_0)\big) \\
&\qquad \times \Pi\big(\phi_f^\tau\phi_g^\tau\phi_f^{-\tau}\phi_g^{-\tau}(J_0), \phi_f^\tau\phi_g^\tau\phi_f^{-\tau}(J_0)\big) \\
&\qquad \times \tilde{\phi}_f^\tau\tilde{\phi}_g^\tau\tilde{\phi}_f^{-\tau}\tilde{\phi}_g^{-\tau} \\
&= \Pi\big(\phi_f^\tau\phi_g^\tau\phi_f^{-\tau}\phi_g^{-\tau}(J_0), J_0\big)\tilde{\phi}_f^\tau\tilde{\phi}_g^\tau\tilde{\phi}_f^{-\tau}\tilde{\phi}_g^{-\tau}.
\end{aligned}$$

Comparing this with the formula for $R(\{f, g\})$, we see that R is indeed a representation. ∎

4. Concluding remarks. One natural connection on E seems to be of a particular interest. For the sake of simplicity, we describe it for the case when M is a Kähler manifold and, thus, $\mathcal{J}_0$ is the space of complex structures compatible with a fixed symplectic form.

Let s be a section of E and $J(t)$ a path in $\mathcal{J}_0$. Observe that every fiber E_J is a linear subspace in the linear space $C^\infty(M; L)$ of smooth sections of the prequantization line bundle L over M. We set

$$\nabla_{\dot{J}(0)} s(0) = Ps'(0),$$

where $s'(0) \in C^\infty(M; L)$ is the derivative of $s(J(t))$ with respect to t at $t = 0$ and P is the orthogonal projection to $E_{J(0)}$, the space of holomorphic sections of L for $J(0)$. It is easy to check that ∇ is indeed a connection. (A similar connection can be defined for the vector bundle of quantizations in the almost complex case.) The following two questions on the properties of ∇ appear interesting already for $M = S^2$:

- Is there an explicit expression for the curvature of ∇?

The curvature of ∇ evaluated on the vectors $\partial/\partial t_1$ and $\partial/\partial t_2$ tangent to a two-parameter family $J(t_1, t_2)$ is equal, as is easy to see, to $-[\partial P/\partial t_1, \partial P/\partial t_2]$ where $P = P(t_1, t_2)$ is the orthogonal projection to $E_{J(t_1,t_2)}$. (This holds only when M is Kähler.) By an explicit expression we mean a formula which can be used, for example, to see directly that the curvature is nonzero. From a different perspective Theorem 1 shows that the vector bundle

$E \to \mathcal{J}_0$ is not $Cont_0$-equivariantly trivial. Then an explicit expression for the curvature may yield some information on the $Cont_0$-equivariant Chern classes of E.

To state the second question, inspired to some extend by the results of [Gu], consider the curvature for E with fiber $Q_J(M,k)$ over J as a function of k.

- Is it true that the curvature of ∇ goes to zero as $k \to \infty$?

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