POISSON GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 51
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

DIFFERENTIAL BATALIN-VILKOVISKY ALGEBRAS ARISING FROM TWILLED LIE-RINEHART ALGEBRAS

JOHANNES HUEBSCHMANN

Université des Sciences et Technologies de Lille
UFR de Mathématiques
F-59 655 Villeneuve d'Ascq Cedex, France
E-mail: Johannes.Huebschmann@univ-lille1.fr

Abstract. Twilled L(ie-)R(inehart)-algebras generalize, in the Lie-Rinehart context, complex structures on smooth manifolds. An almost complex manifold determines an "almost twilled pre-LR algebra", which is a true twilled LR-algebra iff the almost complex structure is integrable. We characterize twilled LR structures in terms of certain associated differential (bi)graded Lie and G(erstenhaber)-algebras; in particular the G-algebra arising from an almost complex structure is a (strict) d(ifferential) G-algebra iff the almost complex structure is integrable. Such G-algebras, endowed with a generator turning them into a B(atalin-)V(ilkovisky)-algebra, occur on the B-side of the mirror conjecture. We generalize a result of Koszul to those dG-algebras which arise from twilled LR-algebras. A special case thereof explains the relationship between holomorphic volume forms and exact generators for the corresponding dG-algebra and thus yields in particular a conceptual proof of the Tian-Todorov lemma. We give a differential homological algebra interpretation for twilled LR-algebras and by means of it we elucidate the notion of a generator in terms of homological duality for differential graded LR-algebras.

Introduction. In a series of seminal papers [2], [3], [4], Batalin and Vilkovisky studied the quantization of constrained systems and for that purpose introduced certain differential graded algebras which have later been christened *Batalin-Vilkovisky algebras*. Batalin-Vilkovisky algebras have recently become important in string theory and elsewhere, cf. e.g. [1], [14], [19], [23], [25], [30], [36], [42], [46]. String theory leads to the mysterious mirror conjecture. A version thereof involves certain differential Batalin-Vilkovisky algebras arising from a Calabi-Yau manifold. These differential Batalin-Vilkovisky alge-

2000 *Mathematics Subject Classification*: Primary 17B55, 17B56, 17B65, 17B66, 17B70, 17B81; Secondary 32V05, 32G05, 53C05, 53C15, 81T70.

Key words and phrases: Lie-Rinehart algebra, twilled Lie-Rinehart algebra, Lie bialgebra, Gerstenhaber algebra, Batalin-Vilkovisky algebra, differential graded Lie algebra, mirror conjecture, Calabi-Yau manifold.

The paper is in final form and no version of it will be published elsewhere.

bras involve what is referred to in the literature as the *Tian-Todorov* lemma which, in turn, implies the unobstructedness of the deformations of the complex structure of a Calabi-Yau manifold. This fact was first stated and proved by *Bogomolov*, in [5] for the special case of a symplectic complex Kähler manifold and in [6] for the general case. (At some places in the literature, this unobstructedness of the complex structure is referred to as the Bogomolov-Tian-Todorov result.) Here we will give a leisurely introduction to a thorough study of such differential Batalin-Vilkovisky algebras and generalizations thereof in the framework of Lie-Rinehart algebras, trying to avoid technicalities; these and more details may be found in [21].

A *Gerstenhaber* algebra is a graded commutative algebra together with a bracket which (i) yields an ordinary graded Lie bracket once the underlying module (or vector space) has been regraded down by 1 and which (ii) satisfies a certain derivation property. Such a bracket occurs in Gerstenhaber's paper [12]. See Section 2 below for details. A differential Batalin-Vilkovisky algebra is a differential Gerstenhaber algebra together with an exact generator, and the underlying Gerstenhaber algebras of interest for us, in turn, arise as (bigraded) algebras of forms on *twilled Lie-Rinehart algebras* (which we introduce below). In the Lie-Rinehart context, a twilled Lie-Rinehart algebra generalizes, among others, the notion of a complex structure on a smooth manifold. One of our results, Theorem 2.3 below, says that an "almost twilled Lie-Rinehart algebra" is a true twilled Lie-Rinehart algebra if and only if the corresponding Gerstenhaber algebra is a differential Gerstenhaber algebra. This implies, for example, that the integrability condition for an almost complex structure on a smooth manifold may be phrased as a condition saying that a certain operator on the corresponding Gerstenhaber algebra turns the latter into a differential Gerstenhaber algebra. Now a theorem of Koszul [29] establishes, on an ordinary smooth manifold, a bijective correspondence between generators for the Gerstenhaber algebra of multi-vector fields and connections in the top exterior power of the tangent bundle in such a way that exact generators correspond to flat connections. In Theorem 2.7 below we generalize this bijective correspondence to the differential Gerstenhaber algebras arising from twilled Lie-Rinehart algebras; such Gerstenhaber algebras come into play, for example, in the mirror conjecture. What corresponds to a flat connection on the line bundle in Koszul's theorem is now a holomorphic volume form—its existence is implied by the Calabi-Yau condition—and our generalization of Koszul's theorem shows in particular how a holomorphic volume form determines a generator for the corresponding differential Gerstenhaber algebra turning it into a differential Batalin-Vilkovisky algebra. The resulting differential Batalin-Vilkovisky algebra then generalizes that which underlies what is called the B-model. In particular, as a consequence of our methods, we obtain a new proof of the Tian-Todorov lemma. We also give a differential homological algebra interpretation of twilled Lie-Rinehart algebras and, furthermore, of a generator for a differential Batalin-Vilkovisky algebra in terms of a suitable notion of homological duality. This relies on results in our earlier papers [18] and [19] as well as on various generalizations therof.

I am indebted to Y. Kosmann-Schwarzbach and K. Mackenzie for discussions, to J. Stasheff and A. Weinstein for some e-mail correspondence about various topics related with the paper, and to the referee for his careful reading of the manuscript and

for suggesting a number of improvements of the exposition. At the "Poissonfest", Y. Kosmann-Schwarzbach introduced me to the recent manuscript [40] which treats topics somewhat related to the present paper. There is little overlap, though. It is a pleasure to thank the organizers of the "Poissonfest" for the opportunity to present these results; they are in fact related to some of the work of the late S. Zakrzewski; see Remark 4.3 below. We respectfully dedicate this paper to his memory.

1. Twilled Lie-Rinehart algebras. Let R be a commutative ring. A *Lie-Rinehart algebra* (A, L) consists of a commutative R-algebra A and an R-Lie algebra L together with an A-module structure $A \otimes_R L \to L$ on L, written $a \otimes_R \alpha \mapsto a\alpha$, and an action $L \to \mathrm{Der}(A)$ of L on A (which is a morphism of R-Lie algebras and) whose adjoint $L \otimes_R A \to A$ is written $\alpha \otimes_R a \mapsto \alpha(a)$; here $a \in A$ and $\alpha \in L$. These mutual actions are required to satisfy certain compatibility properties modeled on $(A, L) = (C^\infty(M), \mathrm{Vect}(M))$ where $C^\infty(M)$ and $\mathrm{Vect}(M)$ refer to the algebra of smooth functions and to the Lie algebra of smooth vector fields, respectively, on a smooth manifold M. In general, the compatibility conditions read:

$$(a\alpha)(b) = a\alpha(b), \quad a, b \in A,\ \alpha \in L, \tag{1.1}$$

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta], \quad a \in A,\ \alpha, \beta \in L. \tag{1.2}$$

For a Lie-Rinehart algebra (A, L), following [39], we will refer to L as an (R, A)-*Lie algebra*. In differential geometry, (R, A)-Lie algebras arise as spaces of sections of Lie algebroids. Lie-Rinehart algebras have been studied before Rinehart by Herz [16] under the name "pseudo-algèbre de Lie" as well as by Palais [38] who used the terminology "d-Lie-ring". We have chosen to refer to these objects as *Lie-Rinehart algebras* since Rinehart subsumed their cohomology under standard homological algebra and established a Poincaré-Birkhoff-Witt theorem for them [39]. In particular, Rinehart has shown how to describe de Rham cohomology in the language of Ext-groups. In a sense, the homological algebra interpretations of differential Batalin-Vilkovisky algebras to be given below push these observations of Rinehart's further.

Given two Lie-Rinehart algebras (A, L') and (A, L''), together with mutual actions $\cdot\colon L' \otimes_R L'' \to L''$ and $\cdot\colon L'' \otimes_R L' \to L'$ which endow L'' and L' with an (A, L')- and (A, L'')-module structure, respectively, we will refer to (A, L', L'') as an *almost twilled Lie-Rinehart algebra*; we will call it a *twilled Lie-Rinehart algebra* provided the direct sum A-module structure on $L = L' \oplus L''$, the sum $(L' \oplus L'') \otimes_R A \to A$ of the adjoints of the L'- and L''-actions on A, and the bracket $[\cdot, \cdot]$ on $L = L' \oplus L''$ given by

$$[(\alpha', \alpha''), (\beta', \beta'')] = [\alpha', \beta'] + [\alpha'', \beta''] + \alpha' \cdot \beta'' - \beta'' \cdot \alpha' + \alpha'' \cdot \beta' - \beta' \cdot \alpha'', \tag{1.3}$$

where $\alpha', \beta' \in L'$, $\alpha'', \beta'' \in L''$, turn (A, L) into a Lie-Rinehart algebra. We then write $L = L' \bowtie L''$ and refer to (A, L) as the *twilled sum* of (A, L') and (A, L'').

For illustration, consider a smooth manifold M with an almost complex structure, let A be the algebra of smooth complex functions on M, L the $(\mathbb{C}, A)$-Lie algebra of complexified smooth vector fields on M, and consider the ordinary decomposition of the complexified tangent bundle $\tau_M^{\mathbb{C}}$ as a direct sum $\tau'_M \oplus \tau''_M$ of the *almost holomorphic* and *almost antiholomorphic* tangent bundles τ'_M and τ''_M, respectively; write L' and

L'' for their spaces of smooth sections. Then (A, L', L''), together with the mutual actions coming from L, is a twilled Lie-Rinehart algebra if and only if the almost complex structure is integrable, i.e. a true complex structure; τ'_M and τ''_M are then the ordinary *holomorphic* and *antiholomorphic* tangent bundles, respectively. The precise analogue of an almost complex structure is what we call an *almost twilled pre-Lie-Rinehart algebra* structure; this notion is weaker than that of almost twilled Lie-Rinehart algebra. The basic difference is that, for an almost twilled pre-Lie-Rinehart algebra, instead of having mutual actions $\cdot\colon L' \otimes_R L'' \to L''$ and $\cdot\colon L'' \otimes_R L' \to L'$, we only require that there be given R-linear pairings $\cdot\colon L' \otimes_R L'' \to L''$ and $\cdot\colon L'' \otimes_R L' \to L'$, which endow L'' and L' with an (A, L')- and (A, L'')-connection, respectively; see [21] for details. A situation similar to that of a complex structure on a smooth manifold and giving rise to a twilled Lie-Rinehart algebra arises from a smooth manifold with two transverse foliations as well as from a Cauchy-Riemann structure (cf. [7]); see [21] for some comments about Cauchy-Riemann structures. Lie bialgebras provide another class of examples of twilled Lie-Rinehart algebras; Kosmann-Schwarzbach and Magri refer to these objects, or rather to the corresponding twilled sum, as *twilled extensions of Lie algebras* [28]; Lu and Weinstein call them *double Lie algebras* [31]; and Majid uses the terminology *matched pairs* of Lie algebras [35]. Spaces of sections of suitable pairs of Lie algebroids with additional structure lead to yet another class of examples of twilled Lie-Rinehart algebras; these have been studied in the literature under the name *matched pairs of Lie algebroids* by Mackenzie [32] and Mokri [37].

An almost twilled Lie-Rinehart algebra (A, L'', L') is a true twilled Lie-Rinehart algebra if and only if (A, L'', L') satisfies three compatibility conditions; these are spelled out in [21] (Proposition 1.7). This proposition is merely an adaptation of earlier results in the literature to our more general situation. Another interpretation of the compatibility conditions involves certain annihilation properties of the two operators d' and d'' which, for an almost twilled pre-Lie-Rinehart algebra (A, L', L''), are given by exactly the same formulas as the ordinary Lie-Rinehart differentials with respect to L' and L'', respectively, on the bigraded algebra $\mathrm{Alt}_A(L'', \mathrm{Alt}_A(L', A))$ (with the obvious bigrading); the only difference is that, instead of true (A, L')- and (A, L'')-module structures, at first we only have (A, L')- and (A, L'')-connections on L'' and L', respectively, whence the resulting operators d' and d'' are not necessarily exact. To explain this interpretation, we will say that an A-module M has *property* P provided that, given $x \in M$, $\phi(x)$ is zero for every $\phi\colon M \to A$ only if x is zero. For example, a projective A-module has property P or a reflexive A-module has this property or, more generally, an A-module M such that the canonical map from M into its double A-dual is injective. On the other hand, for example, for a smooth manifold X, the $C^\infty(X)$-module D of formal (= Kähler) differentials does *not* have property P: On the real line, with coordinate x, consider the functions $f(x) = \sin x$ and $g(x) = \cos x$. The formal differential $df - gdx$ is non-zero in D; however, the $C^\infty(X)$-linear maps from D to $C^\infty(X)$ are the smooth vector fields, whence every such $C^\infty(X)$-linear map annihilates the formal differential $df - gdx$.

We now have the following, cf. Theorem 1.15 in [21].

THEOREM 1.4. *If (A, L', L'') is a twilled Lie-Rinehart algebra, the operators d' and d'' turn the bigraded algebra $\mathrm{Alt}_A(L'', \mathrm{Alt}_A(L', A))$ into a differential bigraded algebra*

which then necessarily computes the cohomology $\mathrm{H}^*(\mathrm{Alt}_A(L,A))$ *of the twilled sum* L *of* L' *and* L''*. Conversely, given an almost twilled pre-Lie-Rinehart algebra* (A,L',L'')*, if the operators* d' *and* d'' *turn the bigraded algebra* $\mathrm{Alt}_A(L'',\mathrm{Alt}_A(L',A))$ *into a differential bigraded algebra and if* L' *and* L'' *have property* P*,* (A,L',L'') *is a true twilled Lie-Rinehart algebra.*

For example, *for the twilled Lie-Rinehart algebra arising from the holomorphic and antiholomorphic tangent bundles of a complex manifold, the resulting differential bigraded algebra* $(\mathrm{Alt}_A(L'',\mathrm{Alt}_A(L',A)),d',d'')$ *comes down to the ordinary de Rham bicomplex* which is sometimes referred to as the $\partial\overline{\partial}$-complex (but we avoid this notation since it conflicts with our notation ∂ employed below).

2. Twilled Lie-Rinehart algebras, Gerstenhaber-, and dBV-algebras. Our present aim is to give other characterizations of twilled Lie-Rinehart algebras which explain the structure of certain differential Batalin-Vilkovisky algebras. Section 2 of [21] is devoted to more details about differential graded Lie-Rinehart algebras.

Given a commutative algebra A and an action of an ordinary Lie algebra $\mathfrak{g}$ on A by derivations, the Lie bracket on $\mathfrak{g}$ extends to a bracket on $A\otimes\mathfrak{g}$ which, together with the obvious pairing $(A\otimes\mathfrak{g})\otimes A\to A$, turns $(A,A\otimes\mathfrak{g})$ into a Lie-Rinehart algebra, called the *crossed product Lie-Rinehart* algebra; its structure is uniquely determined by (1.1) and (1.2). We now give an extension of this construction which is tailored to our purposes.

Let (A,L) be a Lie-Rinehart algebra, and let $\mathcal{A}$ be a graded commutative A-algebra which is endowed with a graded (A,L)-module structure in such a way that (i) L acts on $\mathcal{A}$ by derivations—this is equivalent to requiring the structure map from $\mathcal{A}\otimes_A\mathcal{A}$ to $\mathcal{A}$ to be a morphism of graded (A,L)-modules—and that (ii) the canonical map from A to $\mathcal{A}$ is a morphism of left (A,L)-modules. Let $\mathcal{L}=\mathcal{A}\otimes_A L$, and define a bigraded bracket

$$[\cdot,\cdot]\colon\mathcal{L}\otimes_R\mathcal{L}\to\mathcal{L} \tag{2.1.1}$$

of bidegree $(0,-1)$ by means of the formula

$$[\alpha\otimes_A x,\beta\otimes_A y]=(\alpha\beta)\otimes_A[x,y]+\alpha(x(\beta))\otimes_A y-(-1)^{|\alpha||\beta|}\beta(y(\alpha))\otimes_A x \tag{2.1.2}$$

where $\alpha,\beta\in\mathcal{A}$ and $x,y\in L$. A calculation shows that, for every $\beta\in\mathcal{A}$ and every $x,y,z\in L$,

$$[[x,y],\beta\otimes_A z]-([x,[y,\beta\otimes_A z]]-[y,[x,\beta\otimes_A z]])=\big([x,y](\beta)-x(y(\beta))-y(x(\beta))\big)\otimes_A z,$$

whence (2.1.1) is a graded Lie bracket if and only if the structure map $L\otimes_R\mathcal{A}\to\mathcal{A}$ is a Lie algebra action. Here we identify in notation x and y with $1\otimes x$ and $1\otimes y$, respectively. Moreover, let

$$\mathcal{A}\otimes_R\mathcal{L}\to\mathcal{L} \tag{2.1.3}$$

be the obvious graded left $\mathcal{A}$-module structure arising from extension of scalars, that is from extending L to a (graded) $\mathcal{A}$-module, and define a pairing

$$\mathcal{L}\otimes_R\mathcal{A}\to\mathcal{A} \tag{2.1.4}$$

by

$$(\alpha\otimes_A x)\otimes_R\beta\mapsto(\alpha\otimes_A x)(\beta)=\alpha(x(\beta)). \tag{2.1.5}$$

Then $(\mathcal{A}, \mathcal{L})$, together with (2.1.1), (2.1.3) and (2.1.4), constitutes a graded Lie-Rinehart algebra. We refer to $(\mathcal{A}, \mathcal{L})$ as the *(graded) crossed product* of $\mathcal{A}$ and (A, L) and to the corresponding $(R, \mathcal{A})$-Lie algebra $\mathcal{L}$ as the *crossed product* of $\mathcal{A}$ and L. More details about this notion of graded crossed product Lie-Rinehart algebra may be found in [21] (2.8). We will see shortly that (what is called) the Kodaira-Spencer algebra yields an example of a graded crossed product Lie-Rinehart algebra.

Remark 2.1.6. We must be a little circumspect here: The three terms on the right-hand side of (2.1.2) are *not* well defined individually; only their sum is well defined. For example, if we take ax instead of x, where $a \in A$, on the left-hand side, $\alpha \otimes_A (ax)$ equals $(\alpha a) \otimes_A x$ but $(\alpha\beta) \otimes_A [ax, y]$ differs from $(\alpha a\beta) \otimes_A [x, y]$.

Let (A, L'', L') be an almost twilled Lie-Rinehart algebra having L' finitely generated and projective as an A-module. Write $\mathcal{A}'' = \mathrm{Alt}_A(L'', A)$ and $\mathcal{L}' = \mathrm{Alt}_A(L'', L')$. Now $\mathcal{A}''$ is a graded commutative A-algebra and, endowed with the Lie-Rinehart differential d'' (which corresponds to the (R, A)-Lie algebra structure on L''), $\mathcal{A}''$ is a differential graded commutative R-algebra. Moreover, from the (A, L'')-module structure on L', $\mathcal{L}'$ inherits an obvious differential graded $\mathcal{A}''$-module structure. Furthermore, the (A, L')-structure on L'' induces an action of L' on $\mathcal{A}''$ by graded derivations. Since L' is supposed to be finitely generated and projective as an A-module, the canonical A-module morphism

$$\mathcal{A}'' \otimes_A L' \to \mathcal{L}' = \mathrm{Alt}_A(L'', L')$$

is an isomorphism of graded A-modules, in fact of graded $\mathcal{A}''$-modules. Applying the graded crossed product construction explained above to $L = L'$ and $\mathcal{A} = \mathcal{A}''$, together with the mutual structure of interaction just explained, we obtain the graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}')$. Now the (R, A)-Lie algebra structure on L'' and the (A, L'')-module structure on L' determine the corresponding Lie-Rinehart differential on $\mathcal{L}' = \mathrm{Alt}_A(L'', L')$; we denote it by d''. By symmetry, when L'' is finitely generated and projective as an A-module, we have the same structure, with L' and L'' interchanged.

Theorem 2.1. *As an A-module, L' being supposed to be finitely generated and projective, the statements* (i), (ii), *and* (iii) *below are equivalent:*
(i) (A, L'', L') *is a true twilled Lie-Rinehart algebra;*
(ii) $(\mathcal{L}', d'') = (\mathrm{Alt}_A(L'', L'), d'')$ *is a differential graded R-Lie algebra;*
(iii) $(\mathcal{A}'', \mathcal{L}'; d'')$ *is a differential graded Lie-Rinehart algebra.*
Thus, under these circumstances, there is a bijective correspondence between twilled Lie-Rinehart algebra and differential graded Lie-Rinehart algebra structures.

For a proof of this result and for more details, see (3.2) in [21]. We note that, in the situation of Theorem 2.1, the Lie bracket on $\mathcal{L}' = \mathrm{Alt}_A(L'', L')$ does not just come down to the shuffle product of forms on L'' and the Lie bracket on L'; in fact, such a bracket would not even be well defined since the Lie bracket of L' is not A-linear, i.e., in the usual differential geometry context, does not behave as a "tensor".

When (A, L', L'') is the twilled Lie-Rinehart algebra arising from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold M, the differential graded Lie algebra $(\mathcal{L}', d'') = (\mathrm{Alt}_A(L'', L'), d'')$ is that occurring in [24]; it controls the

infinitesimal deformations of the complex structure on M and is sometimes called the *Kodaira-Spencer* algebra in the literature. The bracket on $\mathcal{L}'$ is then referred to as the *Frölicher-Nijenhuis* bracket; it introduced in [10, 11]. The cohomology $\mathrm{H}^*(L'', L')$ then inherits a graded Lie algebra structure and the (infinitesimal) obstruction to deforming the complex structure is the map $\mathrm{H}^1(L'', L') \to \mathrm{H}^2(L'', L')$ which sends $\eta \in \mathrm{H}^1(L'', L')$ to $[\eta, \eta] \in \mathrm{H}^2(L'', L')$.

Recall that a *Gerstenhaber algebra* is a graded commutative R-algebra $\mathcal{A}$ together with a graded Lie bracket from $\mathcal{A} \otimes_R \mathcal{A}$ to $\mathcal{A}$ of degree -1 (in the sense that, if $\mathcal{A}$ is regraded down by one, $[\cdot,\cdot]$ is an ordinary graded Lie bracket) such that, for each homogeneous element a of $\mathcal{A}$, $[a,\cdot]$ is a derivation of $\mathcal{A}$ of degree $|a|-1$ where $|a|$ refers to the degree of a; see [13] where these objects are called G-algebras, or [19, 25, 30, 46]. Likewise a *differential* Gerstenhaber algebra $(\mathcal{A}, [\cdot,\cdot], d)$ consists of a Gerstenhaber algebra $(\mathcal{A}, [\cdot,\cdot])$ together with a differential d of degree 1 which endows $\mathcal{A}$ with a differential graded R-algebra structure [25, 46]; we will say that $(\mathcal{A}, [\cdot,\cdot], d)$ is *strict* provided d behaves as a derivation for the Gerstenhaber bracket $[\cdot,\cdot]$. In our paper [19], we worked out an intimate link between Gerstenhaber's paper [12] and Rinehart's paper [39] which involves the notion of Gerstenhaber bracket. In a sense, we now extend this link to the differential graded situation.

Given a bigraded commutative R-algebra $\mathcal{A}$, we will say that a bigraded bracket $[\cdot,\cdot]\colon \mathcal{A} \otimes_R \mathcal{A} \to \mathcal{A}$ of bidegree $(0,-1)$ is a *bigraded Gerstenhaber bracket* provided $[\cdot,\cdot]$ is an ordinary bigraded Lie bracket when the second degree of $\mathcal{A}$ is regraded down by one, the first one being kept, such that, for each homogeneous element a of $\mathcal{A}$ of bidegree (p,q), $[a,\cdot]$ is a derivation of $\mathcal{A}$ of bidegree $(p,q-1)$; a bigraded R-algebra with a bigraded Gerstenhaber bracket will be referred to as a *bigraded Gerstenhaber algebra*. Moreover, given a bigraded Gerstenhaber algebra $(\mathcal{A}, [\cdot,\cdot])$ together with a differential d of bidegree $(1,0)$ which endows $\mathcal{A}$ with a differential graded R-algebra structure we will say that $(\mathcal{A}, [\cdot,\cdot])$ and d constitute a *differential bigraded* Gerstenhaber algebra (or differential bigraded G-algebra), written $(\mathcal{A}, [\cdot,\cdot], d)$; in the same vein as above, we will say that $(\mathcal{A}, [\cdot,\cdot], d)$ is *strict* provided d behaves as a derivation for the bigraded Gerstenhaber bracket $[\cdot,\cdot]$, that is,

$$d[x,y] = [dx,y] - (-1)^{|x|}[x,dy], \quad x,y \in \mathcal{A},$$

where the total degree $|x|$ is the sum of the two bidegree components.

Recall that, given a Lie-Rinehart algebra (A, L), the Lie bracket on L determines a Gerstenhaber bracket on the exterior A-algebra $\Lambda_A L$ on L; for $\alpha_1, \ldots, \alpha_n \in L$, the bracket $[u,v]$ in $\Lambda_A L$ of $u = \alpha_1 \wedge \ldots \wedge \alpha_\ell$ and $v = \alpha_{\ell+1} \wedge \ldots \wedge \alpha_n$ is given by the expression

$$[u,v] = (-1)^\ell \sum_{j \le \ell < k} (-1)^{(j+k)} [\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_k} \ldots \wedge \alpha_n, \tag{2.2.1}$$

where $\ell = |u|$ is the degree of u, cf. [19] (1.1).

We now return to a general almost twilled Lie-Rinehart algebra (A, L', L'') having L' finitely generated and projective as an A-module and consider the graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}')$. The graded Lie-Rinehart bracket on $\mathcal{L}'(= \mathrm{Alt}_A(L'', L'))$ extends to a (bigraded) bracket on $\mathrm{Alt}_A(L'', \Lambda_A L')$ which turns the latter into a bigraded

Gerstenhaber algebra; as a bigraded algebra, $\mathrm{Alt}_A(L'', \Lambda_A L')$ could be thought as of the exterior $\mathcal{A}''$-algebra on $\mathcal{L}'$, and we write sometimes

$$\Lambda_{\mathcal{A}''}\mathcal{L}' = \mathrm{Alt}_A(L'', \Lambda_A L').$$

With reference to the graded Lie bracket $[\cdot,\cdot]$ on $\mathcal{L}'$ and the L'-action on $\mathcal{A}''$, the bigraded Gerstenhaber bracket

$$[\cdot,\cdot]\colon \Lambda_{\mathcal{A}''}\mathcal{L}' \otimes_R \Lambda_{\mathcal{A}''}\mathcal{L}' \to \Lambda_{\mathcal{A}''}\mathcal{L}' \tag{2.2.2}$$

on $\Lambda_{\mathcal{A}''}\mathcal{L}'$ may be described by the formulas

$$\begin{aligned} [\alpha\beta,\gamma] &= \alpha[\beta,\gamma] + (-1)^{|\alpha||\beta|}\beta[\alpha,\gamma], \quad \alpha,\beta,\gamma \in \Lambda_{\mathcal{A}''}\mathcal{L}', \\ [x,a] &= x(a), \quad x \in L',\ a \in \mathcal{A}'', \end{aligned} \tag{2.2.3}$$

and its graded skew symmetricity amounts to the identity

$$[\alpha,\beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,\alpha], \quad \alpha,\beta \in \Lambda_{\mathcal{A}''}\mathcal{L}', \tag{2.2.4}$$

where as above $|\cdot|$ refers to the total degree. The bracket (2.2.2) is in fact the *(bigraded) crossed product bracket extension* of the Gerstenhaber bracket on $\Lambda_A L'$, and $\Lambda_{\mathcal{A}''}\mathcal{L}'$ may be viewed as the *(bigraded) crossed product Gerstenhaber algebra* of $\mathcal{A}''$ with the ordinary Gerstenhaber algebra $\Lambda_A L'$. See Section 4 of [21] for details.

The Lie-Rinehart differential d'' which corresponds to the Lie-Rinehart structure on L'' and the graded (A, L'')-module structure on $\Lambda_A L'$ induced by the (A, L'')-module structure on L' turn $\mathrm{Alt}_A(L'', \Lambda_A L')$ into a differential (bi)-graded commutative R-algebra. By symmetry, when L'' is finitely generated and projective as an A-module, we have the same structure, with L' and L'' interchanged. For details about the following result and its proof see Theorem 4.4 in [21].

THEOREM 2.3. *The almost twilled Lie-Rinehart algebra (A, L'', L') is a true twilled Lie-Rinehart algebra if and only if $(\Lambda_{\mathcal{A}''}\mathcal{L}', d'')$ $(= (\mathrm{Alt}_A(L'', \Lambda_A L'), d''))$ is a strict differential (bi)-graded Gerstenhaber algebra.*

When (A, L', L'') arises from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold M, the resulting strict differential Gerstenhaber algebra $(\mathrm{Alt}_A(L'', \Lambda_A L'), d'')$ is that of forms of type $(0,*)$ with values in the holomorphic multivector fields, the operator d'' being the Cauchy-Riemann operator (which is more usually written $\overline{\partial}$). This strict differential Gerstenhaber algebra comes into play in the mirror conjecture; it was studied by Barannikov-Kontsevich [1], Manin [36], Witten [45], and others.

Let now (A, L'', L') be a twilled Lie-Rinehart algebra having L' finitely generated and projective as an A-module of constant rank n (say), and write $\Lambda_A^n L'$ for the top exterior power of L' over A. Consider the strict differential bigraded Gerstenhaber algebra $(\mathrm{Alt}_A(L'', \Lambda_A L'), d'')$. Our next aim is to study generators thereof. To this end, we observe that, when $\mathrm{Alt}_A(L', \Lambda_A^n L')$ is endowed with the obvious graded (A, L'')-module structure induced from the left (A, L'')-module structure on L' which is part of the twilled Lie-Rinehart algebra structure, the canonical isomorphism

$$\mathrm{Alt}_A(L'', \Lambda_A L') \to \mathrm{Alt}_A(L'', \mathrm{Alt}_A(L', \Lambda_A^n L')) \tag{2.4}$$

of graded A-modules is compatible with the differentials which correspond to the Lie-

Rinehart structure on L'' and the (A, L'')-module structures on the coefficients on both sides of (2.4); abusing notation, we denote each of these differentials by d''.

For a bigraded Gerstenhaber algebra $\mathcal{A}$ over R, with bracket operation written $[\cdot,\cdot]$, an R-linear operator Δ on $\mathcal{A}$ of bidegree $(0,-1)$ will be said to *generate* the Gerstenhaber bracket provided, for every homogeneous $a, b \in \mathcal{A}$,

$$[a,b] = (-1)^{|a|}(\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b)); \tag{2.5}$$

the operator Δ is then called a *generator*. A generator Δ is said to be *exact* provided $\Delta\Delta$ is zero, that is, Δ is a differential; an exact generator will henceforth be written ∂. A bigraded Gerstenhaber algebra $\mathcal{A}$ together with a generator Δ will be called a *weak bigraded Batalin-Vilkovisky* algebra (or weak bigraded BV-algebra); when the generator is exact, we will denote it by ∂, and we will refer to $(\mathcal{A}, \partial)$ (more simply) as a *bigraded Batalin-Vilkovisky* algebra (or bigraded BV-algebra).

It is clear that a generator determines the bigraded Gerstenhaber bracket. An observation due to Koszul [29] (p. 261) carries over to the bigraded case: for any bigraded Batalin-Vilkovisky algebra $(\mathcal{A}, [\cdot,\cdot], \partial)$, the operator ∂ (which is exact by assumption) behaves as a derivation for the bigraded Gerstenhaber bracket $[\cdot,\cdot]$, that is,

$$\partial[x,y] = [\partial x, y] - (-1)^{|x|}[x, \partial y], \quad x, y \in \mathcal{A}. \tag{2.6}$$

In view of (2.5), a generator, even if exact, behaves as a derivation for the multiplication of $\mathcal{A}$ only if the bracket $[\cdot,\cdot]$ is zero.

Let $(\mathcal{A}, \Delta)$ be a weak bigraded Batalin-Vilkovisky algebra, write $[\cdot,\cdot]$ for the bigraded Gerstenhaber bracket generated by Δ, and let d be a differential of bidegree $(+1, 0)$ which endows $(\mathcal{A}, [\cdot,\cdot])$ with a strict differential bigraded Gerstenhaber algebra structure. Consider the graded commutator $[d, \Delta] = d\Delta + \Delta d$ on $\mathcal{A}$; it is an operator of bidegree $(1,-1)$ and hence of total degree zero. We will say that $(\mathcal{A}, \Delta, d)$ is a *weak differential* bigraded Batalin-Vilkovisky algebra provided the commutator $[d, \Delta]$ is zero. In particular, a weak differential bigraded Batalin-Vilkovisky algebra $(\mathcal{A}, \Delta, d)$ which has Δ exact is called a *differential bigraded Batalin-Vilkovisky algebra*. Thus a differential bigraded Batalin-Vilkovisky algebra $(\mathcal{A}, \partial, d)$ is a strict differential bigraded Gerstenhaber algebra $(\mathcal{A}, [\cdot,\cdot], d)$ together with an exact generator ∂ for the Gerstenhaber bracket $[\cdot,\cdot]$ such that $[d, \partial]$ is zero.

We now reproduce the statement of Theorem 5.4.6 in [21].

THEOREM 2.7. *The isomorphism* (2.4) *furnishes a bijective correspondence between generators of the bigraded Gerstenhaber structure on the left-hand side of* (2.4) *and* (A, L')*-connections on* $\Lambda^n_A L'$ *in such a way that exact generators correspond to* (A, L')*-module structures (i.e. flat connections). Under this correspondence, generators of the strict differential bigraded Gerstenhaber structure on the left-hand side correspond to* (A, L')*-connections on* $\Lambda^n_A L'$ *which are compatible with the* (A, L'')*-module structure on* $\Lambda^n_A L'$.

Thus, in particular, exact generators of the strict differential bigraded Gerstenhaber structure on the left-hand side correspond to (A, L'')-compatible (A, L')-module structures on $\Lambda^n_A L'$.

When L'' is zero and L' the Lie algebra of smooth vector fields on a smooth manifold, the statement of this theorem comes down to the result of Koszul [29] mentioned earlier. Our result not only provides many examples of differential Batalin-Vilkovisky algebras but also explains how every differential Batalin-Vilkovisky algebra having as underlying bigraded A-algebra one of the kind $\mathrm{Alt}_A(L'', \Lambda_A L')$ arises.

When (A, L', L'') is the twilled Lie-Rinehart algebra which comes from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold M as explained earlier, the theorem gives a bijective correspondence between generators of the strict differential bigraded Gerstenhaber algebra $(\mathrm{Alt}_A(L'', \Lambda_A L'), d'')$ of forms of type $(0, *)$ with values in the holomorphic multi-vector fields, the differential d'' being the Cauchy-Riemann operator, and holomorphic connections on the highest exterior power of the holomorphic tangent bundle in such a way that exact generators correspond to flat holomorphic connections. In particular, suppose that M is a *Calabi-Yau* manifold so that, in particular, it admits a holomorphic volume form Ω (say). This holomorphic volume form identifies the highest exterior power of the holomorphic tangent bundle with the algebra of smooth complex functions on M as a module over the Lie algebra $L = L'' \oplus L'$ of smooth (complexified) vector fields on M, hence induces a flat holomorphic connection thereupon and thence an exact generator ∂_Ω for $(\mathrm{Alt}_A(L'', \Lambda_A L'), d'')$, turning the latter into a differential (bi)graded Batalin-Vilkovisky algebra. This is the differential (bi)graded Batalin-Vilkovisky algebra coming into play on the B-side of the mirror conjecture and studied in the cited sources. That the holomorphic volume form induces a generator for the strict differential Gerstenhaber structure is referred to in [1] as the *Tian-Todorov lemma*; cf. [43], [44]; in [15] (p. 357) the (somewhat weaker) statement [44] (1.2.4) is called *Tian-Todorov lemma*. In our approach, the fact that the holomorphic volume form induces a generator for the strict differential Gerstenhaber algebra structure is just a special case of our generalization of Koszul's theorem to the bigraded setting, and this generalization indeed provides a conceptual proof thereof. We already mentioned in the introduction that the Tian-Todorov lemma implies Bogomolov's observation that, for a Calabi-Yau manifold M, the deformations of the complex structure are unobstructed, that is to say, there is an open subset of $\mathrm{H}^1(M, \tau_M)$—the base of the corresponding Kuranishi family—which parametrizes the deformations of the complex structure; here $\mathrm{H}^1(M, \tau_M)$ is the first cohomology group of M with values in the holomorphic tangent bundle τ_M. Under these circumstances, after a choice of holomorphic volume form Ω has been made, the canonical isomorphism (2.4), combined with the isomorphism

$$\Omega^\flat \colon \mathrm{Alt}_A(L'', \mathrm{Alt}_A(L', \Lambda_A^n L')) \to \mathrm{Alt}_A(L'', \mathrm{Alt}_A(L', A))$$

induced by $\Omega \colon \Lambda_A^n L' \to A$, identifies $(\mathrm{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$ with the de Rham complex of M, written out as the de Rham bicomplex (or $\partial\overline{\partial}$-complex), and hence the cohomology $\mathrm{H}^*(\mathrm{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$ with the ordinary complex valued cohomology of M. This is nowadays well understood. The cohomology

$$\mathrm{H}^*(\mathrm{Alt}_A(L'', \Lambda_A L'), d'', \partial_\Omega)$$

is referred to in the literature as the *extended moduli space of complex structures* [45]; it underlies what is called the B-*model* in mirror symmetry.

3. Twilled Lie-Rinehart algebras and differential homological algebra. We now spell out interpretations of some of the above results in terms of differential homological algebra.

Let (A, L', L'') be a twilled Lie-Rinehart algebra having L' and L'' finitely generated and projective as A-modules. Let $(\mathcal{A}'', \mathcal{L}'; d'')$ be the differential graded crossed product Lie-Rinehart algebra $(\mathrm{Alt}_A(L'', A), \mathrm{Alt}_A(L'', L'); d'')$ mentioned before. Let $L = L' \bowtie L''$ be the twilled sum of L' and L'', and consider the differential graded Lie-Rinehart cohomology $\mathrm{H}^*(\mathcal{L}', \mathcal{A}'') = \mathrm{Ext}^*_{\mathcal{U}(\mathcal{A}'', \mathcal{L}')}(\mathcal{A}'', \mathcal{A}'')$; here $\mathcal{U}(\mathcal{A}'', \mathcal{L}')$ is the universal differential graded algebra for $(\mathcal{A}'', \mathcal{L}')$, and 'Ext' means the differential graded Ext-functor.. See Section 6 in [21] for details where also a proof of the following result may be found.

THEOREM 3.1. *The differential bigraded algebra* $(\mathrm{Alt}_A(L'', \mathrm{Alt}_A(L', A)), d', d'')$ *computes the differential graded Lie-Rinehart cohomology* $\mathrm{H}^*(\mathcal{L}', \mathcal{A}'')$. *Moreover, this differential graded Lie-Rinehart cohomology is naturally isomorphic to the Lie-Rinehart cohomology* $\mathrm{H}^*(L, A)$.

When L'' is zero, so that $(\mathcal{A}'', \mathcal{L}')$ is an ordinary (ungraded) Lie-Rinehart algebra which we now write (A, L), the differential graded Lie-Rinehart cohomology boils down to the ordinary Lie-Rinehart cohomology $\mathrm{H}^*(L, A)$. Moreover, for the special case when A and L are the algebra of smooth functions and the Lie algebra of smooth vector fields, respectively, on a smooth manifold M, the Lie-Rinehart cohomology $\mathrm{H}^*(L, A)$ amounts to the de Rham cohomology of M; this fact has been established by Rinehart [39]. When the twilled Lie-Rinehart algebra (A, L', L'') arises from the holomorphic and antiholomorphic tangent bundles of a smooth complex manifold M, the differential graded algebra $\mathcal{A}''$ and differential graded Lie algebra $\mathcal{L}'$—the Kodaira-Spencer algebra— compute the sheaf cohomology $\mathrm{H}^*(M, \mathcal{O})$ of M with values in the sheaf $\mathcal{O}$ of germs of holomorphic functions and the sheaf cohomology $\mathrm{H}^*(M, \tau_M)$ of M with values in the holomorphic tangent bundle τ_M (or, what amounts to the same, with values in the sheaf of germs of holomorphic vector fields), respectively. Furthermore, the bicomplex calculating the differential graded Lie-Rinehart cohomology $\mathrm{H}^*(\mathcal{L}', \mathcal{A}'')$ of the differential graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}'; d'') = (\mathrm{Alt}_A(L'', A), \mathrm{Alt}_A(L'', L'); d'')$ then involves the two operators d' and d'' and may be written in the form

$$(\mathrm{Alt}_A(L', \mathrm{Alt}_A(L'', A)), d', d'');$$

with the operator d'' alone, this is just the Dolbeault complex of M, while the entire bicomplex is the de Rham bicomplex of M, and the differential graded Lie-Rinehart cohomology amounts to the de Rham cohomology of M. Thus our approach provides, in particular, an interpretation of the Dolbeault and de Rham complexes in the framework of differential homological algebra. Further, the *Frölicher* spectral sequence (E_r, d_r) [9], sometimes known as the *Hodge-de Rham* spectral sequence, arises in this differential homological algebra context in a natural fashion: When the bicomplex is written out in the ordinary way, with the usual meaning for "base"- and "fiber degree", the spectral sequence arises from the column filtration. Clearly, (E_0, d_0) is just the Dolbeault complex.

Generalizing results in our earlier paper [19], we can now elucidate the concept of generator of a differential bigraded Batalin-Vilkovisky algebra in the framework of homological duality for differential graded Lie-Rinehart algebras in the following way: *An exact generator amounts to the differential in a standard complex computing differential graded Lie-Rinehart homology* (!) *with suitable coefficients*; see Proposition 7.13 in [21] for details. It may then be shown that, when the appropriate additional structure (in terms of Lie-Rinehart differentials and dBV-generators) is taken into account, the above isomorphism (2.4) is essentially just a duality isomorphism in the (co)homology of the differential graded crossed product Lie-Rinehart algebra $(\mathcal{A}'', \mathcal{L}')$; see Proposition 7.14 in [21] for details. In particular, given a Calabi-Yau manifold, the fact that the holomorphic volume form induces a generator for the corresponding differential Gerstenhaber structure comes down to a statement about differential graded (co)homological duality.

4. Twilled Lie-Rinehart algebras and Lie-Rinehart bialgebras. Twilled Lie-Rinehart algebras thus generalize Lie bialgebras, and the twilled sum is an analogue, even a generalization, of the Manin double of a Lie bialgebra. The Lie bialgebroids introduced by Mackenzie and Xu [33] generalize Lie bialgebras as well, and there is a corresponding notion of Lie-Rinehart bialgebra. However, twilled Lie-Rinehart algebras and Lie-Rinehart bialgebras are different, in fact non-equivalent notions which both generalize Lie bialgebras. In a sense, Lie-Rinehart bialgebras generalize Poisson and in particular symplectic structures while twilled Lie-Rinehart algebras generalize complex structures. We now give a characterization of twilled Lie-Rinehart algebras in terms of Lie-Rinehart bialgebras. See Theorem 4.8 in [21] for more details.

Thus, let L and D be (R, A)-Lie algebras which, as A-modules, are finitely generated and projective, in such a way that, as an A-module, D is isomorphic to $L^* = \mathrm{Hom}_A(L, A)$. We then say that L and D *are in duality*. We write d for the differential on $\mathrm{Alt}_A(L, A) \cong \Lambda_A D$ coming from the Lie-Rinehart structure on L and d_* for the differential on $\mathrm{Alt}_A(D, A) \cong \Lambda_A L$ coming from the Lie-Rinehart structure on D. Likewise we denote the Gerstenhaber bracket on $\Lambda_A L$ coming from the Lie-Rinehart structure on L by $[\cdot, \cdot]$ and that on $\Lambda_A D$ coming from the Lie-Rinehart structure on D by $[\cdot, \cdot]_*$. We will say that (A, L, D) constitutes a *Lie-Rinehart* bialgebra if the differential d_* on $\mathrm{Alt}_A(D, A) \cong \Lambda_A L$ and the Gerstenhaber bracket $[\cdot, \cdot]$ on $\Lambda_A L$ are related by

$$d_*[x, y] = [d_* x, y] + [x, d_* y], \quad x, y \in L,$$

or equivalently, if the differential d on $\mathrm{Alt}_A(L, A) \cong \Lambda_A D$ behaves as a derivation for the Gerstenhaber bracket $[\cdot, \cdot]_*$ in all degrees, that is to say,

$$d[x, y]_* = [dx, y]_* - (-1)^{|x|}[x, dy]_*, \quad x, y \in \Lambda_A D.$$

Thus, for a Lie-Rinehart bialgebra (A, L, D),

$$(\Lambda_A L, [\cdot, \cdot], d_*) = (\mathrm{Alt}_A(D, A), [\cdot, \cdot], d_*)$$

is a strict *differential Gerstenhaber algebra*, and the same is true of

$$(\Lambda_A D, [\cdot, \cdot]_*, d) = (\mathrm{Alt}_A(L, A), [\cdot, \cdot]_*, d);$$

see [25] (3.5) for details. In fact, a straightforward extension of an observation of Y.

Kosmann-Schwarzbach [25] shows that Lie-Rinehart bialgebra structures on (A, L, D) and strict differential Gerstenhaber algebra structures on $(\Lambda_A L, [\cdot,\cdot], d_*)$ or, what amounts to the same, on $(\Lambda_A D, [\cdot,\cdot]_*, d)$, are equivalent notions. This parallels the well known fact that Lie-Rinehart structures on (A, L) are in bijective correspondence with differential graded R-algebra structures on $\mathrm{Alt}_A(L, A)$.

Let (A, L', L'') be an almost twilled Lie-Rinehart algebra having L' and L'' finitely generated and projective as A-modules. The (A, L')-module structure on L'' induces an (A, L')-module structure on the A-dual L''^* which, in turn, L''^* being viewed as an abelian Lie algebra and hence abelian (R, A)-Lie algebra, gives rise to the semi-direct product (R, A)-Lie algebra $L' \ltimes L''^*$. Likewise the (A, L'')-module structure on L' determines the corresponding semi-direct product (R, A)-Lie algebra $L'' \ltimes L'^*$. Plainly $L = L' \ltimes L''^*$ and $D = L'' \ltimes L'^*$ are in duality. Consider the obvious adjointness isomorphisms

$$\mathrm{Alt}_A(L'', \Lambda_A L') \to \mathrm{Alt}_A(L'' \ltimes L'^*, A) = \mathrm{Alt}_A(D, A) \tag{4.1.1}$$

and

$$\Lambda_A L = \Lambda_A(L' \ltimes L''^*) \to \mathrm{Alt}_A(L'', \Lambda_A L') \tag{4.1.2}$$

of bigraded A-algebras; these isomorphisms are independent of the Lie-Rinehart semi-direct product constructions and instead of $L' \ltimes L''^*$ and $L'' \ltimes L'^*$, we could as well have written $L' \oplus L''^*$ and $L'' \oplus L'^*$, respectively. However, incorporating these semi-direct product structures, we see that, under (4.1.1), the Lie-Rinehart differential d'' on $\mathrm{Alt}_A(L'', \Lambda_A L')$ passes to the Lie-Rinehart differential d_* on $\mathrm{Alt}_A(D, A)$ and that under (4.1.2) the (bigraded) Gerstenhaber bracket $[\cdot,\cdot]$ on $\Lambda_A L$ passes to the bigraded Gerstenhaber bracket (2.2.1) on $\mathrm{Alt}_A(L'', \Lambda_A L')$ which we now denote by $[\cdot,\cdot]'$. Moreover, by construction, the differentials on both sides of (4.1.1) are derivations with respect to the multiplicative structures.

THEOREM 4.1. *For an almost twilled Lie-Rinehart algebra (A, L', L'') having L' and L'' finitely generated and projective as A-modules, $(\mathrm{Alt}_A(L'', \Lambda_A L'), [\cdot,\cdot]', d'')$ is a differential bigraded Gerstenhaber algebra if and only if (A, L, D) is a Lie-Rinehart bialgebra.*

PROOF. In fact, the first property spelled out above characterizing (A, L, D) to be a Lie-Rinehart bialgebra is plainly equivalent to $(\mathrm{Alt}_A(L'', \Lambda_A L'), [\cdot,\cdot]', d'')$ being a differential bigraded Gerstenhaber algebra. ∎

The following is now immediate, cf. Corollary 4.9 in [21].

COROLLARY 4.2. *An almost twilled Lie-Rinehart algebra (A, L', L'') having L' and L'' finitely generated and projective as A-modules is a true twilled Lie-Rinehart algebra if and only if $(A, L, D) = (A, L' \ltimes L''^*, L'' \ltimes L'^*)$ is a Lie-Rinehart bialgebra.* ∎

This result may be proved directly, i.e. without the intermediate differential bigraded Gerstenhaber algebra in (4.1). The reasoning is formally the same, though. For the special case where L' and L'' arise from Lie algebroids, the statement of Corollary 4.2 is a consequence of (4.7) in [32] since the notion of double Lie algebroid defined there in (2.3) imposes a Lie bialgebroid structure on the pair of duals, cf. [32] (4.4), which, in our case, is (L, D).

Remark 4.3. When A is just the ground field and $\mathfrak{g}$ an ordinary (finite dimensional) Lie algebra, Corollary 4.2 comes down to the statement that, in the terminology of [32], [35], [37], the pair $(\mathfrak{g}', \mathfrak{g}'')$ (with the requisite additional structure) constitutes a matched pair of Lie algebras (which now amounts to $(\mathfrak{g}', \mathfrak{g}'')$ being a Lie bialgebra) if and only if, with the obvious structure, $(\mathfrak{g}' \ltimes \mathfrak{g}''^*, \mathfrak{g}'' \ltimes \mathfrak{g}'^*)$ is a Lie bialgebra. This fact (or a version thereof) was apparently known to S. Zakrzewski [47]. It has been spelled out explicitly as Proposition 1 in [41].

References

[1] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, alg-geom/9710032, Internat. Math. Res. Notices 4 (1998), 201–215.

[2] I. A. Batalin and G. S. Vilkovisky, *Quantization of gauge theories with linearly dependent generators*, Phys. Rev. D 28 (1983) 2567–2582.

[3] I. A. Batalin and G. S. Vilkovisky, *Closure of the gauge algebra, generalized Lie equations and Feynman rules*, Nucl. Phys. B 234 (1984), 106–124.

[4] I. A. Batalin and G. S. Vilkovisky, *Existence theorem for gauge algebra*, Jour. Math. Phys. 26 (1985), 172–184.

[5] F. A. Bogomolov, *Hamiltonian Kähler varieties*, Sov. Math. Dokl. 19 (1978), 1462–1465; translated from: Dokl. Akad. Nauk SSSR 243 (1978), 1101–1104.

[6] F. A. Bogomolov, *Kähler manifolds with trivial canonical class*, preprint, Institut des Hautes Etudes Scientifiques 1981, pp. 1–32.

[7] A. Cannas de Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lecture Notes, Volume 10, Amer. Math. Soc. 1999.

[8] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. 63 (1948), 85–124.

[9] A. Frölicher, *Relations between the cohomology groups of Dolbeault and topological invariants*, Proc. Nat. Acad. Sci. USA 41 (1955), 641–644.

[10] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms, Part I: Derivations in the graded ring of differential forms*, Proc. Kon. Ned. Akad. Wet. Amsterdam 59 (1956), 338–359.

[11] A. Frölicher and A. Nijenhuis, *Some new cohomological invariants for complex manifolds, I*, Proc. Kon. Ned. Akad. Wet. Amsterdam 59 (1956), 540–564.

[12] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. 78 (1963), 267–288.

[13] M. Gerstenhaber and S. D. Schack, *Algebras, bialgebras, quantum groups and algebraic deformations*, in: Deformation theory and quantum groups with applications to mathematical physics, M. Gerstenhaber and J. Stasheff (eds.), Cont. Math. 134, American Mathematical Society, Providence, (1992), 51–92.

[14] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. in Math. Phys. 195 (1994), 265–285.

[15] W. M. Goldman and J. J. Millson, *The homotopy invariance of the Kuranishi space*, Illinois J. of Math. 34 (1990), 337–367.

[16] J. Herz, *Pseudo-algèbres de Lie*, C. R. Acad. Sci. Paris 236 (1953), 1935–1937.

[17] J. Huebschmann, *Poisson cohomology and quantization*, J. für die Reine und Angew. Math. 408 (1990), 57–113.

[18] J.. Huebschmann, *Duality for Lie-Rinehart algebras and the modular class*, J. für die Reine und Angew. Math. 510 (1999), 103–159.

[19] J. Huebschmann, *Lie-Rinehart algebras, Gerstenhaber algebras, and Batalin- Vilkovisky algebras*, Annales de l'Institut Fourier 48 (1998), 425–440.

[20] J. Huebschmann, *Extensions of Lie-Rinehart algebras and the Chern-Weil construction*, in: Festschrift in honor of J. Stasheff's 60th birthday, Cont. Math. 227 (1999), 145–176, Amer. Math. Soc., Providence R. I.

[21] J. Huebschmann, *Twilled Lie-Rinehart algebras and differential Batalin-Vilkovisky algebras*, math.DG/9811069.

[22] J. Huebschmann, *The modular class and master equation for Lie-Rinehart bialgebras*, in preparation.

[23] J. Huebschmann and J. D. Stasheff, *Formal solution of the master equation via HPT and deformation theory*, math.AG/9906036.

[24] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures. I. II.*, Ann. of Math. 67 (1958), 328–401, 403–466.

[25] Y. Kosmann-Schwarzbach, *Exact Gerstenhaber algebras and Lie bialgebroids*, Acta Applicandae Mathematicae 41 (1995), 153–165.

[26] Y. Kosmann-Schwarzbach, *From Poisson algebras to Gerstenhaber algebras*, Annales de l'Institut Fourier 46 (1996), 1243–1274.

[27] Y. Kosmann-Schwarzbach, *The Lie bialgebroid of a Poisson-Nijenhuis manifold*, Letters in Math. Physics 38 (1996), 421–428.

[28] Y. Kosmann-Schwarzbach and F. Magri, *Poisson-Lie groups and complete integrability. I. Drinfeld bigebras, dual extensions and their canonical representations*, Annales Inst. H. Poincaré Série A (Physique théorique) 49 (1988), 433–460.

[29] J. L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, in: E. Cartan et les Mathématiciens d'aujourd'hui, Lyon, 25–29 Juin, 1984, Astérisque, hors-série, (1985), 251–271.

[30] B. H. Lian and G. J. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory*, Comm. in Math. Phys. 154 (1993), 613–646.

[31] J.-H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. of Diff. Geom. 31 (1990), 501–526.

[32] K. Mackenzie, *Double Lie algebroids and the double of a Lie bialgebroid*, preprint 1998; math.DG/9808081.

[33] K. C. Mackenzie and P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. 73 (1994), 415–452.

[34] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften No. 114, Springer, Berlin, 1963.

[35] S. Majid, *Matched pairs of Lie groups associated to solutions of the Yang-Baxter equation*, Pac. J. of Math. 141 (1990), 311–332.

[36] Yu. I. Manin, *Three constructions of Frobenius manifolds*, Atiyah-Festschrift (to appear), math.QA/9801006.

[37] T. Mokri, *Matched pairs of Lie algebroids* Glasgow Math. J. 39 (1997), 167–181.

[38] R. S. Palais, *The cohomology of Lie rings*, Proc. Symp. Pure Math. III (1961), 130–137.

[39] G. Rinehart, *Differential forms for general commutative algebras*, Trans. Amer. Math. Soc. 108 (1963), 195–222.

[40] V. Schechtman, *Remarks on formal deformations and Batalin-Vilkovisky algebras*, math. AG/9802006.

[41] P. Stachura, *Double Lie algebras and Manin triples*, q-alg/9712040.

[42] J. D. Stasheff, *Deformation theory and the Batalin-Vilkovisky master equation*, in: Deformation Theory and Symplectic Geometry, Proceedings of the Ascona meeting, June 1996, D. Sternheimer, J. Rawnsley, S. Gutt (eds.), Mathematical Physics Studies Vol. 20, Kluwer Academic Publishers, Dordrecht/Boston/London, 1997, 271–284.

[43] G. Tian, *A note on Kaehler manifolds with* $c_1 = 0$, preprint.

[44] A. N. Todorov, *The Weil-Petersson geometry of the moduli space of* $\mathfrak{su}(n)$ $(n \geq 3)$ *(Calabi-Yau) manifolds, I.*, Comm. Math. Phys. 126 (1989), 325–346.

[45] E. Witten, *Mirror manifolds and topological field theory*, in: Essays on mirror manifolds, S. T. Yau (ed.), International Press Co. Hong Kong, 1992, 230–310.

[46] P. Xu, *Gerstenhaber algebras and BV-algebras in Poisson geometry*, Comm. Math. Phys. 200 (1999), 545–560.

[47] S. Zakrzewski, *Poisson structures on the Poincaré groups*, Comm. Math. Phys. 185 (1997), 285–311.