Zbigniew Jurek
Uniwersytet Wrocławski

O rachunku na losowych odwzorowaniach całkowych

(*A calculus on random integral mappings*)

For an interval $(a, b]$ in the positive half-line, two deterministic functions h and r, and a Lévy process $Y_\nu(t), t \geq 0$, where ν is the law of random variable $Y_\nu(1)$, we consider the following mapping

$$\nu \longmapsto I^{h,r}_{(a,b]}(\nu) := \mathcal{L}\Big(\int_{(a,b]} h(t)\, dY_\nu(r(t))\Big), \quad (\star)$$

where $\mathcal{L}$ denotes the probability distribution of the random integral.

1. Retriving the measure ν.

PROPOSITION 1 *Let $\nu \in ID(R^d)$, r is continuously differentiable and there exists $c \in (a, b)$ such that $h(c) \neq 0$ and $r'(c) \neq 0$. Then*

$$\mathcal{L}\Big(\int_c^x \frac{h(t)}{h(c)}\, dY_\nu(\frac{r(t)}{r'(c)})\Big)^{*\frac{1}{x-c}} \Rightarrow \nu, \quad as\ x \downarrow c.$$

2. Compositions of random integral mappings.

For functions $h_1, ..., h_m$, intervals $(a_1, b_1], ..., (a_m, b_m]$ and measures $\rho_1, ..., \rho_m$ let

$\mathbf{h} := h_1 \otimes ... \otimes h_m$, (tensor product of functions)

i.e. $\mathbf{h}(t_1, t_2, ..., t_m) := h_1(t_1) \cdot h_2(t_2) \cdot ... \cdot h_m(t_m)$, where $a_i < t_i \leq b_i$;

$\mathbf{(a,b]} := (a_1, b_1] \times ... \times (a_m, b_m]$, $\boldsymbol{\rho} := \rho_1 \times ... \times \rho_m$ (product measure)

THEOREM 1 *If the image $\boldsymbol{h((a,b])} = (c, d] \subset \mathbb{R}^+$ and $\nu \in ID(E)$ is from an appropriate domain then we have*

$$I^{h_1,\rho_1}_{(a_1,b_1]}(I^{h_2,\rho_2}_{(a_2,b_2]}(...(I^{h_m,\rho_m}_{(a_m,b_m]}(\nu)))) = I^{t,\boldsymbol{h}\,\boldsymbol{\rho}}_{(c,d]}(\nu)$$

where $\boldsymbol{h}\ \boldsymbol{\rho}$ is the image of the product measure $\boldsymbol{\rho} = \rho_1 \times ... \times \rho_m$ under the mapping $\boldsymbol{h} := h_1 \otimes ... \otimes h_m$.

3. Fixed points of random integral mappings.

We will say that an infinitely divisible measure ρ is *a fixed point of an integral mapping* $I^{h,r}_{(a,b]}$, if

$$I^{h,r}_{(a,b]}(\rho) = \rho^{*c} * \delta_z, \qquad \text{for some } c > 0 \text{ and } z \in E.$$

Equivalently, in terms of Lévy exponents Φ's:

$$\mathcal{I}^{h,r}_{(a,b]}(\Phi)(y) = c\,\Phi(y) + i < y, z > \text{ for all } y \in E'.$$

PROPOSITION 2 *In order that p-stable measure γ_p be a fixed point of the mapping $I^{h,r}_{(a,b]}$ it is necessary and sufficient that $0 < \int_{(a,b]} |h(t)|^p |dr(t)| < \infty$.*

4. Examples of iterated integral mappings and image measures

Example 1 *For $\nu \in ID_{\log^m}$ and $m = 1, 2, ...$*

$$I^{e^{-s},\,s}_{(0,\infty)}(I^{e^{-s},\,s}_{(0,\infty)}(...I^{e^{-s},\,s}_{(0,\infty)}(\nu))) = I^{e^{-t},\,\frac{t^m}{m!}}_{(0,\infty)} \in L_m, \quad (m\text{-times selfdecomposable measures};$$

Example 2 *For $\beta > 0$ we have*

$$I^{t^{1/\beta},\,t}_{(0,1]} \circ I^{s^{1/2\beta},\,s}_{(0,1]} = I^{w,\,2w^\beta(1-(1/2)w^\beta)}_{(0,1]} = I^{(1-\sqrt{t})^{1/\beta},\,t}_{(0,1]}$$

Or equivalently, for Lebesque measure l_1 on $0 < w \leq 1$ we get

$$(t^{1/\beta} \otimes s^{1/(2\beta)})(l_1 \times l_1)(dw) = id^{\otimes 2}(\beta t^{\beta-1}dt \times 2\beta s^{2\beta-1}dt)(dw) = 2\beta w^{\beta-1}(1-w^\beta)\,dw$$

Example 3 *For $\beta > 0$*

$$I^{t^{1/\beta},\,t}_{(0,1]} \circ I^{e^{-s},\,s}_{(0,\infty)} = I^{e^{-s},\,s+\beta^{-1}e^{-\beta s}-\beta^{-1}}_{(0,\infty)} = I^{-w,\,\beta^{-1}w^\beta - \log w - \beta^{-1}}_{(0,1]},$$

or equivalently

$$(t^{1/\beta} \otimes e^{-s})(l_1 \times l)(dw) = (\beta^{-1}w^\beta - \log w - \beta^{-1})dw.$$

Example 4 *For $\alpha \in \mathbb{R}$* $I^{t,\,\Gamma(\alpha;t)}_{(0,\infty)} \circ I^{e^{-s},\,s}_{(0,\infty)} = I^{t,\,\int_t^\infty s^{-1}\Gamma(\alpha;s)ds}_{(0,\infty)}$,

Bibliografia

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