ACTA ARITHMETICA
104.3 (2002)

Solution of a generalised version of a problem of Rankin on sums of powers of cusp-form coefficients

by

R. W. K. Odoni (Glasgow)

In memory of Robert Alexander Rankin, 27.X.1915–27.I.2001

0. Introduction. In [5] we gave a partial solution to a problem of Rankin [8] on the behaviour of the Fourier coefficients of a special family of cusp-forms introduced by Hecke. Our purpose in this paper is to extend and refine the methods of [5]. One consequence of this will be a complete solution of Rankin's problem. To explain our new results we adhere to the notation and terminology of [5]. Let K be a number field, $[K:\mathbb{Q}]=\kappa$, $2\le\kappa<\infty$, and let χ be a normalised Grössencharakter of K. For $n\in\mathbb{N}$ let $T(n,\chi)=\sum\chi(\mathfrak{a})$, the sum being over all integral ideals $\mathfrak{a}$ in K with absolute norm $N\mathfrak{a}=n$. In the present context, the most important result in [5] is the asymptotic formula

$$\sum_{n\le x}|T(n,\chi)|^{2\beta}=A(\chi,\beta)x(\log x)^{c(\chi,\beta)-1}\{1+O(\log x)^{-\gamma(\chi,\beta)}\}\tag{0.1}$$

as $x\to\infty$. Here β is any fixed positive number, while $A(\chi,\beta),c(\chi,\beta)$ and $\gamma(\chi,\beta)$ are positive constants depending only on χ and β. In fact, we showed that (0.1) holds with $\gamma(\chi,\beta)\ge\min\{1,\beta/2\}$; in §1 this will be improved to $\gamma(\chi,\beta)\ge\min\{1,\beta\}$. Unfortunately we were not able in [5] to give an explicit expression for $c(\chi,\beta)$, although we were able to prove the existence of a unique upper-semicontinuous probability distribution function $M(t)=M(t,\chi)$ such that

$$c(\chi,\beta)=\int_{t\in\mathbb{R}}t^{\beta}\,dM(t,\chi)\qquad(\forall\beta>0).\tag{0.2}$$

Here $M(0-,\chi)=0$ and $M(\kappa^2,\chi)=1$. In particular $M(t,\chi)$ is compactly supported, and so, by classical Hausdorff moment theory [2], it is uniquely determined if, for example, we know the values $c(\chi,k)$ for all $k\in\mathbb{N}$, or even just for a subset of $k\in\mathbb{N}$ with $\sum k^{-1}=\infty$.

2000 *Mathematics Subject Classification*: 11F12, 11F30, 11M41, 11R37, 11S40.

One of our main aims in this paper is the determination of $M(t,\chi)$ and $c(\chi,\beta)$ when χ is "generic". The precise definition of genericity will be given in §4, where we also show that, in an appropriate sense, "almost all" χ are generic. Our precise result for $M(t,\chi),\chi$ generic, is given in Theorem 2, stated below.

If we adopt the convention that $y^0 = 1$ for all $y \geq 0$ in $\mathbb{R}$, then (0.1) still holds (trivially) if we put $c(\chi,\beta) = 0$, and $\gamma(\chi,0) > 0$ is arbitrary. Our other main result (Theorem 1) relates to the following generalisation of (0.1). Let $m \in \mathbb{N}$, let $K_1,\ldots,K_m$ be number fields (of respective finite degrees $\kappa_1,\ldots,\kappa_m \geq 2$), and, for $j \leq m$, let χ_j be a normalised Grössencharakter of K_j. In a certain sense, Theorem 1 is a direct "vector" analogue of (0.1). Formally, we have:

THEOREM 1. *Let $\boldsymbol{\chi} = (\chi_1,\ldots,\chi_m)$ be as above, and let $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_m)$ be any vector of non-negative real numbers. Then we have an asympototic formula*

$$(0.3)\quad \sum_{n\leq x}\prod_{j\leq m}|T(n,\chi_j)|^{2\beta_j} = A(\boldsymbol{\chi},\boldsymbol{\beta})x(\log x)^{c(\boldsymbol{\chi},\boldsymbol{\beta})-1}\{1+O(\log x)^{-\gamma(\boldsymbol{\chi},\boldsymbol{\beta})}\}$$

as $x\to\infty$. Here $A(\boldsymbol{\chi},\boldsymbol{\beta}), c(\boldsymbol{\chi},\boldsymbol{\beta})$ and $\gamma(\boldsymbol{\chi},\boldsymbol{\beta})$ are positive and depend only on $\boldsymbol{\chi},\boldsymbol{\beta}$. If $\boldsymbol{\beta} = \mathbf{0}$ then $A(\boldsymbol{\chi},\mathbf{0}) = 1, c(\boldsymbol{\chi},\mathbf{0}) = 1$ and $\gamma(\boldsymbol{\chi},\mathbf{0})$ can be chosen to be arbitrarily large positive. If $\boldsymbol{\beta} \neq \mathbf{0}$, we may choose $\gamma(\boldsymbol{\chi},\boldsymbol{\beta}) = \min\{1,\beta_j > 0\}$.

Moreover the following formula holds for all $\boldsymbol{\beta}$:

$$(0.4)\quad c(\boldsymbol{\chi},\boldsymbol{\beta}) = \int_{\mathbf{t}\in\mathbb{R}^m} t_1^{\beta_1}\ldots t_m^{\beta_m}\,dM(\boldsymbol{\chi},\mathbf{t}).$$

Here, given $\boldsymbol{\chi}$, $\mathbf{t} \mapsto M(\boldsymbol{\chi},\mathbf{t})$ is a unique compactly supported probability distribution function on $\mathbb{R}^m$, upper-semicontinuous in the sense that $M(\boldsymbol{\chi},\mathbf{t}) = \inf_{\mathbf{s}>\mathbf{t}}\{M(\boldsymbol{\chi},\mathbf{s})\}$, where $\mathbf{s} > \mathbf{t}$ means that $s_1 \geq t_1,\ldots,s_m \geq t_m$, while $\mathbf{t} \geq \mathbf{0}$ and $\mathbf{s} \neq \mathbf{t}$.

The proof of Theorem 1 occupies §1, and relies on a refinement of the classical Wirsing method [15, 16] for certain non-negative multiplicative functions, due to H. Halberstam. In §1 we axiomatise the processes involved in proving Theorem 1, since the method may turn out to be applicable to other problems.

To explain Theorem 2 (stated below), we introduce a sequence D_r $(r\geq 0)$ of upper-semicontinuous probability distribution functions on $\mathbb{R}$, by the following procedure. We have $D_r(t) = \mathrm{Prob}\{\mathbf{X}_r \leq t\}$ for all $r \geq 0$ and all $t \in \mathbb{R}$, where the random variables $\mathbf{X}_0,\mathbf{X}_1,\ldots,\mathbf{X}_r,\ldots$ are as follows:

(a) $\mathbf{X}_0$ is identically $0, \mathbf{X}_1$ is identically 1;

(b) For $r \geq 2$, let $\Theta_1, \ldots, \Theta_r$ be independent random variables, each uniformly distributed in the closed unit interval $[0,1]$. Then

$$\mathbf{X}_r := \Big| \sum_{j=1}^{r} \exp(2\pi i \Theta_j) \Big|^2 \qquad (i = \sqrt{-1}). \tag{0.5}$$

Now suppose that $[K : \mathbb{Q}] = \kappa$, $2 \leq \kappa < \infty$. For $r = 0, \ldots, \kappa$ let ∂_r be the Dirichlet density of the set of primes $p \in \mathbb{N}$ having, in K, exactly r distinct prime ideal factors $\mathfrak{p}$ of residual degree 1 (i.e. $N\mathfrak{p} = p$). By Chebotarev's density theorem [4, pp. 379–389] the ∂_r exist, and are non-negative rational numbers summing to 1. Consequently, $\sum_{r=0}^{\kappa} \partial_r D_r(t)$ is another upper-semicontinuous probability distribution on $\mathbb{R}$, supported by $[0, \kappa^2]$. Then Theorem 2 is as follows.

THEOREM 2. *Let* $[K : \mathbb{Q}] = \kappa$, $2 \leq \kappa < \infty$, *and let* χ *be a generic normalised Grössencharakter of* K (see §4). *Then* $M(\chi, t)$ *of* (0.2) *satisfies*

$$M(\chi, t) = \sum_{r=0}^{\kappa} \partial_r D_r(t) \qquad (\forall t \in \mathbb{R}) \tag{0.6}$$

while for $\beta \geq 0$ *in* $\mathbb{R}$ *we have*

$$c(\chi, \beta) = \sum_{r=0}^{\kappa} \partial_r \int_{t \in \mathbb{R}} t^{\beta} \, dD_r(t), \tag{0.7}$$

with $c(\chi, \beta)$ *as in* (0.1).

Whereas in [5] we obtained (0.1) without recourse to global relative Weil groups, the latter seem to be essential for the proof of Theorem 2, and we devote §3 to an exposition of the relevant properties of Weil groups, their representations and the associated Weil L-functions. On the other hand, these methods are not needed for Theorem 1. The crucial "replication formula" (1.14) for products $T(p, \chi)T(p, \psi)$ can be proved by entirely "elementary" methods, as we showed in [6].

Our paper is organised as follows. In §1 we review the Halberstam refinement of Wirsing's method, and then prove an "interpolation theorem" of some independent interest; by combining the latter with Halberstam–Wirsing method we then obtain Theorem 1.

In §2 we summarise the (classical) theory of the distribution functions $D_r(t)$ which figure in Theorem 2, concentrating only on those aspects relevant for the proof of the latter.

In §3 we give a detailed description of those properties of Weil groups, their representations and associated Weil L-functions which are needed for the proof of Theorem 2.

In §4 we study the action of Galois groups on normalised Grössencharaktere χ, leading to the formal definition of *generic* χ, and a proof that "almost all" χ are generic.

In §5 we prove Theorem 2, while in §6 we discuss possible further extensions of our results and methods.

I am greatly indebted to Professor R. Rankin and Professor K. A. Brown of Glasgow University; the former inspired the work leading to [5] and the present paper, while discussions with the latter have been very helpful in developing the ring and module theory used in §4. I am also grateful to Professor H. Halberstam (University of Illinois, Urbana) for his letter of July 1990, pointing out how to sharpen the error term of (0.1).

1. Halberstam's refinement of Wirsing's method, and an interpolation theorem

1A. For $n \in \mathbb{N}$ let $d(n) = \sum_{d|n} 1$ be the classical divisor function. We introduce the special class $\mathcal{M}$ of non-negative multiplicative functions f: $\mathbb{N} \to \mathbb{R}$ by assigning f to $\mathcal{M}$ if and only if there are positive constants $A = A_f, B = B_f$ such that

$$0 \le f(n) \le Ad(n)^B \quad (\forall n \in \mathbb{N}). \tag{1.1}$$

Note that if $f, g \in \mathcal{M}$ so does $f * g$ (Dirichlet convolution); also, for $\beta \ge 0$, $m \mapsto f(m)^\beta$ is in $\mathcal{M}$ when $f \in \mathcal{M}$.

The original method of Wirsing [15, 16] starts from an $f \in \mathcal{M}$ with the additional property

$$\sum_{p \le x} f(p) \sim c \frac{x}{\log x} \quad (x \to \infty), \tag{1.2}$$

where $c = c_f > 0$, and p denotes a prime in $\mathbb{N}$. Wirsing shows that (1.2) implies

$$\sum_{n \le x} f(n) \sim Ex(\log x)^{c-1} \quad (x \to \infty), \tag{1.3}$$

where $E = E_f > 0$. Wirsing's techniques are "elementary", but very ingenious.

First, using summation by parts, (1.2) is converted to

$$\sum_{p \le x} f(p) \frac{\log p}{p} \sim c \log x \quad (x \to \infty). \tag{1.4}$$

Next, using $\log n = \sum_{p^k \| n} \log(p^k)$ and another summation by parts, (1.4) is converted to

$$\sum_{n \le x} \frac{f(n)}{n} \sim F(\log x)^{c-1} \quad (x \to \infty), \tag{1.5}$$

with $F = F_f > 0$. The next, and most difficult step, is the transition from (1.5) to (1.3). First, to avoid problems caused by high powers of small primes,

f is replaced by f_0, where $f_0(n) = f(n)\delta_0(n)$ and $\delta_0(n) = 1$ unless n is divisible by some prime $p \le p_0$ (suitably chosen), when $\delta_0(n) = 0$. It is then shown that (1.5) (with f_0 in place of f) yields (1.3) with f_0 in place of f, and some E_0 in place of E. For this a study of $g(n) = \sum_{d|n} f_0(d)\widehat{f_0}(n/d)\log d$ is required, where $\widehat{f_0}$ is the inverse of f_0 under Dirichlet convolution. Finally, further summation by parts allows one to obtain (1.3) with f and E in place of f_0 and E_0.

For various purposes one requires a quantitative version of Wirsing's method. Specifically, suppose that $f \in \mathcal{M}$ and that (1.2) is replaced by the stronger condition

$$(1.2)^* \qquad \sum_{p\le x} f(p) \sim c\,\frac{x}{\log x} + O(x(\log x)^{-1-\gamma}) \qquad (x\to\infty).$$

(Here $c, \gamma > 0$.) We then ask whether a corresponding stronger version of (1.3) holds, in particular, whether we have

$$(1.3)^* \qquad \sum_{n\le x} f(n) = Ex(\log x)^{c-1}\{1 + O(\log x)^{-\delta}\} \qquad (x\to\infty)$$

for some $\delta > 0$.

In [5] we showed that $(1.3)^*$ does indeed hold, with $\delta = \min\{1, \gamma/2\}$; our method used a subtle Tauberian theorem of Subhankulov, depending on the behaviour of Laplace transforms with complex argument. In a letter to the author H. Halberstam (July 1990) showed that $(1.3)^*$ holds, even with $\delta = \min\{1, \gamma\}$; his method involved a careful re-examination of Wirsing's arguments, and avoids Subhankulov's theorem. Moreover his main argument is very similar to that used in [1]; since this is readily available, there is no point in reproducing Halberstam's argument here; we merely record that $(1.2)^*$ implies $(1.3)^*$ with $\delta = \min\{1, \gamma\}$, and refer to this result as *Halberstam's refinement* (*of Wirsing's method*).

1B. *An interpolation theorem.* Now suppose that $m \in \mathbb{N}$, and that $f_1, \ldots, f_m \in \mathcal{M}$. We assume a very strong form of $(1.2)^*$, namely that

$$(1.6) \qquad \sum_{p\le x} \prod_j f_j(p)^{k_j} = c(\mathbf{k})\,\mathrm{Li}(x) + O(A_1^{\sum k_j} x \exp(-A_2\sqrt{\log x}))$$

as $x \to \infty$, where $\mathrm{Li}(x) = \int_2^x dt/\log t$.

Here the result holds for all vectors $\mathbf{k} = (k_1, \ldots, k_m)$ of non-negative integers, while $c(\mathbf{k}) > 0$ for all $\mathbf{k}$. Moreover $A_1, A_2 > 0$ are constants (i.e. independent of $\mathbf{k}, x$) and the implied constant in $O(\ldots)$ is also independent of $\mathbf{k}, x$. Our aim here is to show that a version of $(1.3)^*$ holds for $f(n) = f_1(n)^{\alpha_1} \ldots f_m(n)^{\alpha_m}$, where $\boldsymbol{\alpha}$ is an arbitrary vector of non-negative numbers. (Again we emphasise the convention that $y^0 = 1$ for all $y \ge 0$ in

$\mathbb{R}$.) Our task thus involves the determination of $c(\boldsymbol{\alpha}) > 0$ and $\gamma(\boldsymbol{\alpha}) > 0$ such that

$$\sum_{p \le x} \prod_{j} f_j(p)^{\alpha_j} = c(\boldsymbol{\alpha}) \frac{x}{\log x} \{1 + O((\log x)^{-\gamma(\boldsymbol{\alpha})})\} \tag{1.7}$$

as $x \to \infty$.

Our procedure is, in essence, an induction on m. (If any α_j is 0, the result (1.7) will follow from the induction hypothesis, using fewer than m functions.) The formal result is as follows.

THEOREM 0. *Suppose that* (1.6) *holds. Then for all* $m, \boldsymbol{\alpha}$ *we have*

$$\sum_{p \le x} \prod_{j \le m} f_j(p)^{\alpha_j} = c(\boldsymbol{\alpha}) \frac{x}{\log x} \{1 + O((\log x)^{-\gamma(\boldsymbol{\alpha})})\}. \tag{1.8}$$

Here $c(\boldsymbol{\alpha}) > 0$ *for all* $\boldsymbol{\alpha} \ge \mathbf{0}$. *Moreover* $c(\mathbf{0}) = 1$ *and* $\gamma(\mathbf{0}) = 1$. *If* $\boldsymbol{\alpha} \ne \mathbf{0}$ *we have* $\gamma(\boldsymbol{\alpha}) = \frac{1}{2} \min\{\alpha_j > 0\}$. *Further there is a unique, compactly supported upper-semicontinuous probability distribution function* Ω *on* $[0, \infty)^m$ *such that*

$$c(\boldsymbol{\alpha}) = \int t_1^{\alpha_1} \ldots t_m^{\alpha_m} \, d\Omega(\mathbf{t}) \tag{1.9}$$

for all $\boldsymbol{\alpha} \ge \mathbf{0}$.

REMARK. The case $m = 1$ of Theorem 0 was proved in [5]. There we used a Fourier-series argument. Here we use the binomial expansion of $(1 + x)^\theta$ $(\theta > 0)$, which turns out to yield a rather simpler proof of the general case of Theorem 0.

The following elementary result is very useful in the proof of Theorem 0.

LEMMA 1.1. *Let* $\theta \in \mathbb{R}$, $\theta > 0$, *and for* $n \ge 0$ *in* $\mathbb{Z}$ *let* $\binom{\theta}{n}$ *be the classical binomial coefficient. Thus* $\binom{\theta}{0} = 1$, $\binom{\theta}{1} = \theta$ *and* $\binom{\theta}{n} = O(n^{-1-\theta})$ *as* $n \to \infty$; *moreover, the power series* $\sum_{n=0}^{\infty} \binom{\theta}{n} x^n$ *converges absolutely and uniformly for* $-1 \le x \le 1$ *to* $(1 + x)^\theta$.

We omit the proof, which is quite simple (although the last part requires Abel's continuity theorem for convergent power series).

As a corollary we have the estimate

$$\left| y^\theta - \sum_{n=0}^{N} \binom{\theta}{n} (y - 1)^n \right| \le A(\theta) N^{-\theta}, \tag{1.10}$$

where $A(\theta)$ depends only on θ, and $N \in \mathbb{N}$, $0 \le y \le 1$ are arbitrary.

More generally, if $\alpha_1, \ldots, \alpha_m > 0$ we have an analogous formula:

$$\left| \prod_{j \le n} y_j^{\alpha_j} - \sum_{0 \le n_1, \ldots, n_m \le N} \prod_{j \le m} \binom{\alpha_j}{n_j} (y_j - 1)^{n_j} \right| \le B(\boldsymbol{\alpha}) N^{-\mu} \tag{1.11}$$

whenever $N \in \mathbb{N}$, and $0 \le \max_j\{y_j\} \le 1$. Here $B(\boldsymbol{\alpha}) > 0$ depends only on $\boldsymbol{\alpha}$, while $\mu = \min\{\alpha_1, \dots, \alpha_m\} > 0$.

We first prove (1.7) (given (1.6)) when $\mu = \min\{\alpha_1, \dots, \alpha_m\} > 0$. Since $f_1, \dots, f_m$ are in $\mathcal{M}$, we see from (1.1) that there is a $C > 0$ such that $0 \le f_j(p) \le C$ holds for all $j \le m$ and all primes p.

We substitute $y_j = C^{-1} f_j(p)$ in (1.11) and sum over all $p \le x$, using (1.6) with $0 \le k_j \le N$ for all j. If we then choose $N = \lambda\sqrt{\log x}$, λ fixed but suitably small > 0, we obtain (1.7) for some (as yet unspecified) $c(\boldsymbol{\alpha}) > 0$, with $\gamma(\boldsymbol{\alpha}) = \frac{1}{2}\mu = \frac{1}{2}\min\{\alpha_1, \dots, \alpha_m\}$.

If some, but not all, the α_j are zero, then the f_j with $\alpha_j = 0$ are effectively absent, and we obtain (1.7) with $\gamma(\boldsymbol{\alpha}) = \frac{1}{2}\min_{j \le m}\{\alpha_j : \alpha_j > 0\}$. Finally if $\boldsymbol{\alpha} = \mathbf{0}$, the left-hand side of (1.6) reduces to $\sum_{p \le x} 1$; we may then take $c(\mathbf{0}) = 1$ and $\gamma(\mathbf{0}) = 1$. To complete the proof of Theorem 0 we need only prove the existence and uniqueness of Ω of (1.9). To do this we choose a large x and let S_x be the set of all primes p in $\mathbb{N}$ with $p \le x$. We make the set of all subsets of S_x into a finite probability space by assigning to each $A \subseteq S_x$ the measure $\#A/\pi(x)$, where $\pi(x) = \sum_{p \le x} 1 = \#S_x$. Now choose a prime $\mathcal{P}_x$ "at random" in S_x. Then the functions $f_j(\mathcal{P}_x)$ $(j \le m)$ become random variables on $[0, C]$, for any $C \ge \sup_{j,p}\{f_j(p)\}$. By (1.6), we have

$$\mathcal{E}\Big(\prod_j f_j(\mathcal{P}_x)^{k_j}\Big) = c(\mathbf{k}) + o(1) \quad (x \to \infty) \tag{1.12}$$

for each fixed integral $\mathbf{k} \ge \mathbf{0}$, $\mathcal{E}$ denoting expectation. Let

$$\Omega_x(\mathbf{t}) = \mathrm{Prob}\{f_j(\mathcal{P}_x) \le t_j,\ \forall j\}. \tag{1.13}$$

Using the obvious multivariable analogue of Hausdorff's moment theory [2], and noting the uniform boundedness of the $f_j(\mathcal{P}_x)$, we let $x \to \infty$.

Then there is a unique upper-semicontinuous probability distribution function $\Omega(\mathbf{t})$, supported by $[0, C]^m$, such that

$$c(\mathbf{k}) = \int_{\mathbb{R}^m} t_1^{k_1} \dots t_m^{k_m}\, d\Omega(\mathbf{t}) \quad (\mathbf{k} \text{ integral non-negative}),$$

while $\Omega(\mathbf{t})$ is the weak limit of the $\Omega_x(\mathbf{t})$. Hence, for every fixed $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$ $(\alpha_j \ge 0$ in $\mathbb{R})$ we have

$$\int_{\mathbb{R}^m} t_1^{\alpha_1} \dots t_m^{\alpha_m}\, d\Omega(\mathbf{t}) = \lim_{x \to \infty} \mathcal{E}\Big(\prod_j f_j(\mathcal{P}_x)^{\alpha_j}\Big),$$

while the latter is the $c(\boldsymbol{\alpha})$ of (1.9), whose existence we proved earlier. This completes the proof of Theorem 0.

1C. *Proof of Theorem 1.* This will follow easily from Theorem 0, once we note the following result.

LEMMA 1.2. *Let $\chi_1, \ldots, \chi_m$ be normalised Grössencharaktere of $K_1, \ldots$ $\ldots, K_m$ respectively, let $L/\mathbb{Q}$ be the Galois hull of the compositum $K_1 \ldots K_m$ over $\mathbb{Q}$, and let $g \in \mathbb{N}$ be divisible by the finite parts of the conductors of $\chi_1, \ldots, \chi_m$. We denote $[K_j : \mathbb{Q}]$ by $\kappa_j, 1 \le j \le m$. Then there exist normalised Grössencharaktere $\theta_1, \ldots, \theta_t$ of subfields of L ($1 \le t \le \prod_{j \le m} \kappa_j$) such that*

(a) *the finite part of the conductor of each θ_r ($r \le t$) divides g;*
(b) *the equation*

$$\prod_{j \le m} T(p, \chi_j) = \sum_{r \le t} T(p, \theta_r) \tag{1.14}$$

holds for all primes $p \in \mathbb{N}$ such that $p \nmid g \operatorname{dis}(L/\mathbb{Q})$, where $\operatorname{dis}(L/\mathbb{Q})$ is the discriminant of $L/\mathbb{Q}$.

This was proved by Moroz in [3, p. 24], using Weil groups; the author supplied an "elementary" proof in [6]. We apply Lemma 1.2 in the following way to obtain Theorem 1. We fix $\chi_1, \ldots, \chi_m$, choose $\mathbf{k} = (k_1, \ldots, k_m)$ ($k_j \in \mathbb{Z}$, $k_j \ge 0$) and use the identity

$$|T(p, \chi_j)|^{2k_j} = T(p, \chi_j)^{k_j} T(p, \overline{\chi}_j)^{k_j}, \quad \text{in conjunction with (1.14).}$$

We then have

$$\prod_j |T(p, \chi_j)|^{2k_j} = \sum_{r \le t} T(p, \theta_r) \quad (p \nmid g \operatorname{dis} L/\mathbb{Q}) \tag{1.15}$$

where t now satisfies $1 \le t \le \prod_{j \le m} \kappa_j^{2k_j}$.

Again the θ_r are normalised Grössencharaktere of subfields of L, the finite parts of whose conductors still divide g. Because of this, there are asymptotic formulae [3, p. 48] of the type

$$\sum_{p \le x} T(p, \theta_r) = \delta_r \operatorname{Li}(x) + O(x \exp(-A\sqrt{\log x})), \tag{1.16}$$

where A depends only on g and L, the implied constant in $O(\ldots)$ similarly depends only on g and L, while $\delta_r = 1$ if θ_r is the trivial character and is 0 otherwise. Moreover, from (1.15), at least one of the δ_r is 1, since $|T(p, \chi_j)|^2 \ge 0$ for all p, χ_j.

If we now put $f_j(n) = |T(n, \chi_j)|^2$ ($j = 1, \ldots, m$), we see that $f_j \in \mathcal{M}$ for all j and that (1.6) holds for suitable $A_1, A_2 > 0$. Theorem 0 can now be applied, giving Theorem 1.

2. The distribution function $D_r(t)$. We refer to the function $D_r(t)$ occurring in Theorem 2.

It seems that the problem of determining all the $D_r(t)$ ($r \ge 0$) was originally posed by the English statistician Karl Pearson in the 1880's. For

$r \geq 3$ there is no description of $D_r(t)$ in terms of elementary functions. However, in a virtuoso performance in the manipulation of Bessel functions [9], Lord Rayleigh was able to express $D_r(t)$ in terms of such functions for all $r \geq 3$. Rayleigh worked with Fourier transforms. Here we obtain quick proof of the main relevant results of Rayleigh by means of Hankel's transformation [13]. We summarise the pertinent results here.

For $r = 0, 1$, there is nothing to discuss. Also, an elementary geometrical argument (not given here) shows that

$$D_2(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ 1 & \text{for } t \geq 4, \\ 1 - \pi^{-1}\cos^{-1}(2^{-1}(t-2)) & \text{for } 0 \leq t \leq 4. \end{cases} \tag{2.1}$$

Here, for $y \in [-1, 1], \cos^{-1}(y) \in [0, \pi]$.

For $r \geq 3$ there is no elementary analogue of (2.1). However, for $r \geq 2$ we can use the following general method.

Let $\mathbf{X}_r$ have D_r as its distribution function, so that $D_r(t) = \mathrm{Prob}\{\mathbf{X}_r \leq t\}$ for $t \in \mathbb{R}$, while $\mathbf{X}_r = |\sum_{j=1}^{r} \exp(2\pi i \Theta_j)|^2$ (as defined above the statement of Theorem 2 in §0). Denoting expectations by $\mathcal{E}(\ldots)$, we have, for $k \in \mathbb{N}$,

$$\mathcal{E}(\mathbf{X}_r^k) = \int\limits_{0 \leq \theta_j \leq 1} \Big| \sum_{j=1}^{r} \exp(2\pi i \theta_j) \Big|^{2k} d\theta_1 \ldots d\theta_r. \tag{2.2}$$

Now the multinomial theorem gives

$$S := \Big(\sum_{j \leq r} \exp(2\pi i \theta_j) \Big)^k = \sum_{\mathbf{m}} \binom{k}{\mathbf{m}}_r (\exp 2\pi i (\mathbf{m}.\boldsymbol{\theta})), \tag{2.3}$$

where $\mathbf{m}$ runs over all ordered r-tuples $\mathbf{m} = (m_1, \ldots, m_r)$ of non-negative integers summing to k, $\binom{k}{\mathbf{m}}_r$ is $k!/\prod_{j \leq r}(m_j!)$ and $\mathbf{m}.\boldsymbol{\theta} = \sum_{j \leq r} m_j \theta_j$. Multiplying S by $\overline{S}$, and integrating, we see from (2.2) and (2.3) that

$$\mathcal{E}(\mathbf{X}_r^k) = \sum_{\mathbf{m}} \binom{k}{\mathbf{m}}_r^2 \quad (\forall k \in \mathbb{N}), \tag{2.4}$$

since the functions $\theta \mapsto \exp(2\pi i n \theta)$ $(n \in \mathbb{Z})$ are orthonormal over $[0, 1]$.

In fact, (2.4) also holds (vacuously) when $k = 0$.

(2.4) is important in its own right, and will be used in §5.

For completeness we push the analysis further when $r \geq 3$. Let

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(k!)^2} \quad (z \in \mathbb{C}) \tag{2.5}$$

so that J_0 is the standard Bessel function of order 0; it is entire, the power series (2.5) converging uniformly on each compact subset of $\mathbb{C}$.

We pick $y \in \mathbb{R}$, multiply (2.4) by $(-1)^k (k!)^{-2} (y/2)^{2k}$ and sum over $k, 0 \le k \le \infty$, obtaining

$$\mathcal{E}(J_0(y\sqrt{\mathbf{X}_r})) = J_0(y)^r \quad (y \ge 0 \text{ in } \mathbb{R}). \tag{2.6}$$

(The rearrangements of series involved in this calculation are justified by the absolute convergence of (2.5) at each $z \in \mathbb{C}$.) For $r \ge 3$ we use the theory of Hankel's transformation [13], applied to (2.6). This shows that $\frac{d}{dt} D_r(t) = f_r(t)$ exists for $0 < t < r^2$, and is, in fact, continuous, while

$$f_r(t) = \frac{1}{2} \int_0^\infty y J_0(y\sqrt{t}) J_0(y)^r \, dy \quad (0 < t < r^2), \tag{2.7}$$

which is one of Rayleigh's formulae. In fact (2.7) is of only limited interest; for example, it is not directly clear from (2.7) that $D_r(t) = 1$ for $t \ge r^2$, although the latter is immediate from $\mathbf{X}_r = |\sum_{j \le r} \exp(2\pi i \Theta_j)|^2$. Moreover, (2.7) is not well-adapted for obtaining numerical approximations to $f_r(t)$, because the integrand in (2.7) is oscillatory. Fortunately, (2.4) will suffice to prove Theorem 2.

3. Weil groups, representations and L-functions. For this section convenient general references are [3, 10, 11, 12, 14]; for the most part we use [3].

3A. *Global relative Weil groups.* Suppose that $[K : \mathbb{Q}] < \infty$. We denote by J_K the group of idèles of K, given, as is standard, the restricted direct-product topology; thus J_K is a locally compact abelian group. As usual we embed K^* diagonally into J_K; then K^* becomes a discrete subgroup. For each place v of K let $|\cdot|_v$ be the usual normalised absolute value on the completion K_v, and for $\mathbf{x} = (xv)_v \in J_K$ let $\mathrm{vol}(\mathbf{x}) = \prod_v |x_v|_v$. The map $\mathbf{x} \mapsto \log(\mathrm{vol}(\mathbf{x}))$ is a continuous epimorphism $J_K \twoheadrightarrow \mathbb{R}$ (under $+$), the kernel being J_K^0, the closed subgroup of J_K consisting of those $\mathbf{x} \in J_K$ with $\mathrm{vol}(\mathbf{x}) = 1$. The group J_K^0 contains K^* and J_K^0/K^* is compact; we denote it by $C_0(K)$.

For $r \in \mathbb{R}$ let $j_K(r)$ be the idèle $\mathbf{x}$ having $x_v = 1$ for all non-archimedean v and $x_v = e^r = \exp(r)$ for all archimedean v. Then $j_K(\mathbb{R})$ is a closed subgroup of J_K, while $j_K(\mathbb{R}) \cap J_K^0 = \langle 1 \rangle$, and $r \mapsto J_K(r)$ is a bicontinuous isomorphism between $\mathbb{R}$ and $j_K(\mathbb{R})$. Putting $C(K) = J_K/K^*$ we see by means of j_K that

$$C(K) \cong \mathbb{R} \oplus C_0(K) \tag{3.1}$$

as topological groups.

Now let $[L : \mathbb{Q}] < \infty$ with L/K Galois, $\mathrm{Gal}\, L/K = H$. Then H acts on $J_L, C(L)$, and $C_0(L)$, fixing $j_L(\mathbb{R})$ pointwise, and we have

$$C(L) \cong \mathbb{R} \oplus C_0(L) \tag{3.2}$$

as $\mathbb{Z}H$-modules, H acting trivially on $\mathbb{R}$. (The H-action of $\sigma \in H$ on $x \in J_L$ is denoted by $x \mapsto x^\sigma$; thus we are treating all terms in (3.2) as *right* $\mathbb{Z}H$-modules.)

Now $\mathbb{R}$ is uniquely divisible as a $\mathbb{Z}$-module; consequently, for $q \in \mathbb{N}$, the cohomology groups $H^q(H, \mathbb{R})$ vanish, and so (3.2) implies that

$$H^q(H, C(L)) \cong H^q(H, C_0(L)) \quad (\forall q \in \mathbb{N}). \tag{3.3}$$

In particular, each class in $H^q(H, C(L))$ can be represented by a q-cocycle with values in $C_0(L)$. This is useful in our definition of reduced relative global Weil groups, which we shall give shortly.

First, taking $q = 1$ in (3.3), we have $H^1(H, C_0(L)) = 0$ since $H^1(H, C(L)) = 0$. We may thus safely identify $C(L)^H = \{H\text{-fixed points of } C(L)\}$ with $C(K)$ and $C_0(L)^H$ with $C_0(K)$.

Next let $q = 2$, and let u be the fundamental class in $H^2(H, C(L))$, in the sense of class field theory [12].

By (3.3) we may represent u by a 2-cocycle $a(\varrho, \sigma)$ of H in $C_0(L)$, normalised to satisfy the conditions:

$$\begin{aligned} &\text{(i) } a(\varrho, 1) = 1 = a(1, \sigma), && \forall \varrho, \sigma \in H, \\ &\text{(ii) } a(\varrho\sigma, \tau) a(\varrho, \sigma)^\tau = a(\varrho, \sigma\tau) a(\sigma, \tau), && \forall \varrho, \sigma, \tau \in H. \end{aligned} \tag{3.4}$$

We now define the (relative global) *Weil group* $W(L/K)$ as follows. As a *set*, $W(L/K)$ consists of all ordered pairs (ϱ, b), $\varrho \in H$, $b \in C(L)$. We now give H the discrete topology, $C(L)$ its usual topology, and $W(L/K)$ the corresponding product topology, making $W(L/K)$ into a locally compact space. Finally, we impose a group law on $W(L/K)$ via

$$(\varrho, b) \cdot (\sigma, c) = (\varrho\sigma, b^\sigma c a(\varrho, \sigma)) \tag{3.5}$$

for all $\varrho, \sigma \in H$, $b, c \in C(L)$, with $a(\varrho, \sigma)$ as in (3.4). Then $W(L/K)$ becomes a locally compact topological group. In general $W(L/K)$ is not abelian.

The *reduced* Weil group $W_0(L/K)$ is defined analogously by using only the ordered pairs (ϱ, b) with $\varrho \in H, b \in C_0(L)$. Using the natural embedding $C_0(L) \subset C(L)$ we see that $W_0(L/K)$ is a *compact* normal subgroup of $W(L/K)$.

(This compactness is useful since it guarantees the existence of a unique Haar measure μ on $W_0(L/K)$ which is (2-sided) translation-invariant and assigns measure 1 to $W_0(L/K)$ itself; also the continuous representations of $W_0(L/K)$ (over $\mathbb{C}$) are easy to classify. This will simplify some of our later calculations.)

Next we note that the subgroup $(1, C(L))$ of $W(L/K)$ is closed, normal and abelian, of finite index, and fits into an exact sequence

$$1 \to (1, C(L)) \to W(L/K) \to H \to 1 \tag{3.6}$$

of topological groups. In (3.6) we may also replace $C(L)$ by $C_0(L)$ and $W(L/K)$ by $W_0(L/K)$.

The fact that $u \in H^2(H, C(L))$ is the fundamental class leads to a number of important functorial properties of the groups $W(L/K)$ under changes of L and K.

For us, the most important of these properties is the (topological) isomorphism

$$W(L/K)^{\mathrm{ab}} \cong C(K), \tag{3.7}$$

where $W(L/K)^{\mathrm{ab}}$ is $W(L/K)/W(L/K)^{\mathrm{c}}$, and $W(L/K)^{\mathrm{c}}$ is the *closure* of the commutator subgroup of $W(L/K)$. The isomorphism (3.7) arises from the reciprocity law isomorphism [12]:

$$C(K)/N_{L/K}(C(L)) \cong H^{\mathrm{ab}}, \tag{3.8}$$

and the transfer map $W(L/K)^{\mathrm{ab}} \to (1, C(L))$ (well-defined since $(1, C(L))$ has finite index in $W(L/K)$ and is abelian).

It is not difficult to see that we may replace $C(L), C(K)$ and $W(L/K)$ by $C_0(L), C_0(K)$ and $W_0(L/K)$ in (3.7) and (3.8). For later use it is important to make (3.7), and its analogue $(3.7)_0$ (for W_0, C_0), quite explicit.

It is clear that

$$W_0(L/K) = \bigcup_{\sigma \in H} (\sigma, 1) \cdot (1, C_0(L)) \tag{3.9}$$

(disjoint union). Let $(\varrho, b) \in W_0(L/K)$. Then, for $\sigma \in H$, we have, using (3.4) and (3.5),

$$(\varrho, b) \cdot (\sigma, 1) = (\varrho\sigma, b^{\sigma} a(\varrho, \sigma)) = (\varrho\sigma, 1) \cdot (1, b^{\sigma} a(\varrho, \sigma)). \tag{3.10}$$

Let Ver be the transfer map $W_0(L/K) \to (1, C_0(L))$. Then, by (3.10), we have

$$\mathrm{Ver}((\varrho, b)) = \Big(1, \prod_{\sigma \in H} b^{\sigma} a(\varrho, \sigma)\Big) = (1, N_{L/K}(b))\Big(1, \prod_{\sigma \in H} a(\varrho, \sigma)\Big). \tag{3.11}$$

Now let $\ell(\varrho) = \prod_{\sigma \in H} a(\varrho, \sigma) \in C_0(L)$.

From (3.4)(ii), we easily see that $\ell(\varrho)$ is H-invariant (i.e. $\ell(\varrho)^{\sigma} = \ell(\varrho)$ for all $\varrho, \sigma \in H$). From this we see that the isomorphism (3.7) arises by mapping $(\varrho, b) \in W(L/K)$ to $N_{L/K}(b)\ell(\varrho) \in C(K)$ for $b \in C(K)$, and that $(3.7)_0$ arises in the same way with $b \in C_0(L)$.

Now let χ be a continuous character of $C_0(K)$. By $(3.7)_0$, χ yields a continuous character of $W_0(L/K)^{\mathrm{ab}}$, lifting to a continuous character χ_* on $W_0(L/K)$ via the formula

$$\chi_*((\varrho, b)) = \chi(N_{L/K}(b))\chi(\ell(\varrho)) \quad (\varrho \in H,\ b \in C_0(L)). \tag{3.12}$$

Viewed another way, χ lifts via (3.1) to a continuous character of $C(K)$, lifting to a continuous character χ_* of $W(L/K)$. The analogue of (3.12) then holds with $b \in C(L)$ and χ_* in place of χ^*.

3B. *Weil L-functions.* Let $\mathit{\Gamma}$ be a (Hausdorff) topological group. In this section we mean by a *representation* of $\mathit{\Gamma}$ a continuous homomorphism $\mathcal{R} : \mathit{\Gamma} \to \mathrm{GL}(V)$, V a finite-dimensional $\mathbb{C}$-vector space; $\mathrm{GL}(V)$ is topologised by choosing some fixed (abstract group) isomorphism $\theta : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V)$ ($n = \dim V$) and then transporting the standard topology on $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{GL}(V)$ via θ. (Clearly, the choice of θ is irrelevant.) Obviously, $\ker \mathcal{R}$ is a closed normal subgroup of $\mathit{\Gamma}$. The degree $\deg \mathcal{R}$ is $\dim V$. Let L/K be a finite Galois extension of number fields ($[L : \mathbb{Q}] < \infty$). In [14] Weil introduced an L-function $L(s, \mathcal{R}, W(L/K))$ for each representation $\mathcal{R}$ of $W(L/K)$. We do not need to consider the general $\mathcal{R}$ here, merely those which arise as follows. From (3.1), and our construction of $W(L/K), W_0(L/K)$, we have a bicontinuous isomorphism

$$W(L/K) \cong \mathbb{R} \oplus W_0(L/K). \tag{3.13}$$

Thus a representation $\mathcal{R}^0$ of $W_0(L/K)$ can be lifted to a representation $\mathcal{R}$ of $W(L/K)$, of the same degree, via $\mathcal{R}(r \oplus w_0) = \mathcal{R}^0(w_0)$ for all $r \in \mathbb{R}, w_0 \in W_0(L/K)$.

Representations $\mathcal{R}$ of $W(L/K)$ arising in this way are called *liftable*, and only these will be used here. The corresponding Weil L-functions have particularly pleasant properties; we quote from [3].

(a) For liftable $\mathcal{R}, L(s, \mathcal{R}, W(L/K))$ is a meromorphic function of $s \in \mathbb{C}$.

It has no zeros for $\operatorname{Re} s \geq 1$, and the only singularity for $\operatorname{Re} s \geq 1$ is a pole at $s = 1$ of order $\langle \mathcal{R}^0, \mathbf{1} \rangle$, where the latter is the multiplicity of the trivial representation $\mathbf{1}$ of $W_0(L/K)$ in $\mathcal{R}^0$. If $\langle \mathcal{R}^0, \mathbf{1} \rangle = 0$, the apparent pole at $s = 1$ is a removable singularity, and we can choose a suitable non-zero value for $L(1, \mathcal{R}, W(L/K))$ to make $L(s, \mathcal{R}, W(L/K))$ analytic for $\operatorname{Re} s \geq 1$.

(b) If $\mathcal{R}, \mathcal{T}$ are liftable, so is $\mathcal{R} \oplus \mathcal{T}$ and we have

$$L(s, \mathcal{R} \oplus \mathcal{T}, W(L/K)) = L(s, \mathcal{R}, W(L/K)) L(s, \mathcal{T}, W(L/K))$$

for all $s \in \mathbb{C}$.

(c) Now suppose (temporarily) that $L/\mathbb{Q}$ is a finite Galois, K a subfield of L, $\mathcal{R}$ a representation of $W(L/K)$, lifted from $\mathcal{R}^0$ in $W_0(L/K)$. Then $\mathcal{R}$ (resp. $\mathcal{R}^0$) induces a representation $\mathcal{T}$ (resp. $\mathcal{T}^0$) of $W(L/\mathbb{Q})$ (resp. $W_0(L/\mathbb{Q})$), $\mathcal{T}$ is lifted from $\mathcal{T}^0$, and we have

$$L(s, \mathcal{T}, W(L/\mathbb{Q})) = L(s, \mathcal{R}, W(L/K)) \quad (\forall s \in \mathbb{C}). \tag{3.14}$$

(Clearly $\deg \mathcal{T} = [L : K] \deg \mathcal{R}$ here.)

(d) Let χ be a normalised Grössencharakter of K; by the standard procedure, χ can be regarded as a continuous character of $C_0(K)$, lifting via (3.12) to a continuous character χ^* of $W_0(L/K)$; i.e. χ^* is really a representation $\mathcal{R}^0$ of $W_0(L/K)$ with $\deg \mathcal{R}^0 = 1$, lifting to an $\mathcal{R}$ on $W(L/K)$. Then we have

$$L(s,\mathcal{R},W(L/K)) = \mathcal{L}(s,\chi) \quad (s \in \mathbb{C}) \tag{3.15}$$

where $\mathcal{L}(s,\chi)$ is the classical Hecke L-function associated with χ. It is entire unless χ is the trivial Grössencharakter, when the only singularity of $\mathcal{L}(s,\chi)$ is a simple pole at $s=1$.

(e) We now suppose once more that $L/\mathbb{Q}$ is finite Galois, with $K_1,\dots,K_m$ subfields of L, and that χ_j is a normalised Grössencharakter of K_j ($j=1,\dots,m$). Imitating the procedure outlined in (d), we construct the corresponding degree-one representation $\mathcal{R}_j^0$ on $W_0(L/K_j)$, lifting to $\mathcal{R}_j$ on $W(L/K_j)$. Then the $\mathcal{R}_j^0$ induce representations $\mathcal{T}_j^0$ on $W_0(L/\mathbb{Q})$, lifting to $\mathcal{T}_j$ on $W(L/\mathbb{Q})$. Clearly the tensor product $\mathcal{T}_1\otimes\ldots\otimes\mathcal{T}_m$ of any representations $\mathcal{T}_j$ of $W(L/\mathbb{Q})$ lifted from $\mathcal{T}_j^0$ on $W_0(L/\mathbb{Q})$ is lifted from $\mathcal{T}_1^0\otimes\ldots\otimes\mathcal{T}_m^0$ on $W_0(L/\mathbb{Q})$. Hence $\mathcal{T}_1\otimes\ldots\otimes\mathcal{T}_m$ is lifted from $\mathcal{T}_1^0\otimes\ldots\otimes\mathcal{T}_m^0$. We consider the Weil L-function $L(s,\mathcal{T}_1\otimes\ldots\otimes\mathcal{T}_m,W(L/\mathbb{Q}))$. On the one hand, by (a), this L-function has a pole of order $\langle\mathcal{T}_1^0\otimes\ldots\otimes\mathcal{T}_m^0,\mathbf{1}\rangle$ (calculated over $W_0(L/\mathbb{Q})$) at $s=1$, and no other zero of pole for $\operatorname{Re}s\ge 1$. On the other hand, with the branch of log such that $\log 1=0$, we have [3, p. 24] the relation

$$\log L(s,\mathcal{T}_1\otimes\ldots\otimes\mathcal{T}_m,W(L/\mathbb{Q})) = \sum_{\substack{p\in\mathbb{N}\\ \text{prime}}} p^{-s}\prod_{j\le m} T(p,\chi_j) + E(s) \tag{3.16}$$

for $\operatorname{Re}s>1$, where $E(s)$ is analytic, non-zero and bounded, say for $\operatorname{Re}s\ge 3/4$. Here $T(n,\chi)$ is as in §0.

(f) Returning to (a), and taking $K=\mathbb{Q}$, $L/\mathbb{Q}$ finite Galois, the function $L(s,\mathcal{R},W(L/\mathbb{Q}))$ has the following properties. There is an $a(\mathcal{R})>0$ such that

(i) $L(s,\mathcal{R},W(L/\mathbb{Q}))$ has no zero in the s-region

$$\triangle_{\mathcal{R}}:\quad \operatorname{Re}s \ge 1-\frac{a(\mathcal{R})}{\log(2+(\operatorname{Im}s)^2)},$$

(ii) there is a $b(\mathcal{R})>0$ such that on the left edge of $\triangle_{\mathcal{R}}$ we have

$$|L(s,\mathcal{R},W(L/\mathbb{Q}))|^{\pm 1} \le (2+(\operatorname{Im}s)^2)b(\mathcal{R}). \tag{3.17}$$

By the standard Mellin-transform method of analytic number theory, (3.16) and (3.17), together with (e), can be made to yield the estimate (1.6) with $f_j(n)=|T(n,\chi_j)|^2$ ($j=1,\dots,m$) needed for the proof of Theorem 1. Moreover if we fix χ, K, and choose $m=2k$, $\chi_1=\ldots=\chi_k=\chi$, $\chi_{k+1}=\ldots=\chi_{2k}=\overline{\chi}$, we obtain the formula

$$c(\chi,k) = \langle\mathcal{T}^0\underbrace{\otimes\ldots\otimes}_{k}\mathcal{T}^0,(\text{self})\rangle \quad (k\in\mathbb{N}) \tag{3.18}$$

for $c(\chi,k)$ as in (0.1), (0.2); here $\mathcal{T}^0$ is the representation of $W_0(L/\mathbb{Q})$ arising from the normalised Grössencharakter χ of K by the procedure of (e). Theorem 2 will be proved by evaluating the right-hand side of (3.18) for "generic" χ.

4. Generic normalised Grössencharaktere. Our aim here is to give a suitable definition of generic normalised Grössencharaktere of K $(2 \le \kappa = [K:\mathbb{Q}] < \infty)$.

Our procedure is to embed K in a finite Galois extension $L/\mathbb{Q}$ and then to study the action of $\operatorname{Gal} L/\mathbb{Q}$ on normalised Grössencharaktere of L. Our main references here are [3, 4]. It is acceptable to take $L/\mathbb{Q}$ to be the Galois hull of $K/\mathbb{Q}$.

Let $[K:\mathbb{Q}] = \kappa$, $2 \le \kappa < \infty$, and let $f \in \mathbb{N}$. We consider $\mathrm{Gr}_0(K,f)$, the (abelian) group of all those normalised Grössencharaktere χ of K such that the finite part of the conductor of χ divides $f! = 1\cdot 2\cdot \ldots \cdot f$. It is well known [6] that, as an abstract group, $\mathrm{Gr}_0(K,f)$ has the structure

$$\mathrm{Gr}_0(K,f) \cong \mathbb{Z}^{\kappa-1} \oplus R_f(K), \tag{4.1}$$

where $R_f(K)$ is a finite abelian group, isomorphic to the ray class group $(\operatorname{mod}^X f!)$ of K.

If $g \ge f$ we have $\mathrm{Gr}_0(K,f) \triangleleft \mathrm{Gr}_0(K,g)$, while $\bigcup_{f\in\mathbb{N}} \mathrm{Gr}_0(K,f)$ is the group of all normalised Grössencharaktere of K.

If we regard $\mathrm{Gr}_0(K,f)$ as a group of homomorphisms $I(K,f) \to \mathbb{R}/\mathbb{Z} \cong \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, where $I(K,f)$ is the group of fractional ideals of K generated by the prime ideals $\mathfrak{p} \ne 0$ (in the ring $\mathbb{Z}_K$ of integers of K) with $\mathfrak{p} \nmid f!$, then the intersection of the kernels of the $\chi \in \mathrm{Gr}_0(K,f)$ is precisely the set of fractional ideals of the type $q\mathbb{Z}_K$, where $q > 0$ lies in $\mathbb{Q}$ and $q \equiv 1 \ (\operatorname{mod}^X f!)$; these q have the form $q = r/s$, $r,s \in \mathbb{N}$, with $r \equiv 1 \equiv s \ (\operatorname{mod}(f!\mathbb{Z}))$.

By a standard procedure [3, pp. 12–14], normalised Grössencharaktere of K correspond $1:1$ to continuous characters of the compact abelian group $C_0(K)$ of §3. Let $\widehat{C}_0(K)$ be the dual of $C_0(K)$, i.e. the group of continuous characters of $C_0(K)$. We may then identify $\mathrm{Gr}_0(K,f)$ $(f \in \mathbb{N})$, with a subgroup of $\widehat{C}_0(K)$, and we have

$$\widehat{C}_0(K) = \bigcup_{f\in\mathbb{N}} \mathrm{Gr}_0(K,f). \tag{4.2}$$

Since $C_0(K)$ is compact, $\widehat{C}_0(K)$ is discrete, so we do not need to use topological arguments when discussing $\widehat{C}_0(K)$.

Now suppose that $L/\mathbb{Q}$ is finite Galois, and that K is a subfield of L. If $\chi \in \mathrm{Gr}_0(K,f)$, the map

$$y \mapsto \chi(N_{L/K}(y)) \quad (y \in C_0(L))$$

is clearly in $\mathrm{Gr}_0(L,f)$; we denote it by $E_{L/K}(\chi)$. It is then clear that $\chi \mapsto E_{L/K}(\chi)$ is a homomorphism $\mathrm{Gr}_0(K,f) \mapsto \mathrm{Gr}_0(L,f)$; moreover, $E_{L/K}(\chi)$ has finite order if and only if χ has finite order.

Now let $G = \mathrm{Gal}\, L/\mathbb{Q}, H = \mathrm{Gal}\, L/K$. We denote the action of $\sigma \in G$ on $C_0(L)$ by $x \mapsto x^\sigma$, i.e. we treat $C_0(L)$ as a right $\mathbb{Z}G$-module.

Let $\chi \in \widehat{C}_0(L), \sigma \in G$ and $y \in C_0(L)$. We denote the map $y \mapsto \chi(y^\sigma)$ by $\sigma.\chi$. It is immediately clear that $\sigma.\chi \in \widehat{C}_0(L)$, and in this way we make $\widehat{C}_0(L)$ into a left $\mathbb{Z}G$-module. Moreover, for each $f \in \mathbb{N}$, $\mathrm{Gr}_0(L,f)$ is a left $\mathbb{Z}G$-submodule of $\widehat{C}_0(L)$.

Using the classical interpretation of $\mathrm{Gr}_0(L,f)$ (i.e., taking $L = K$ in the discussion following (4.1), it is easy to see that the left $\mathbb{Z}G$-annihilator of $\mathrm{Gr}_0(L,f)$ $(f \in \mathbb{N})$, is exactly $\mathbb{Z} \cdot (\sum_{g\in G} g)$. Here, for $\lambda \in \mathbb{Z}G, \mathbb{Z} \cdot \lambda$ is the $\mathbb{Z}$-span of λ.

Let U_L be the group of units of $\mathbb{Z}_L$. The map $u \mapsto u^\sigma$ $(u \in U_L, \sigma \in G)$ makes U_L into a right $\mathbb{Z}G$-module. By a classical result of Minkowski [4, p. 113] we can find a *cyclic* right $\mathbb{Z}G$-submodule V_L of U_L such that

(i) V_L has finite index in U_L,
(ii) V_L is a free $\mathbb{Z}$-submodule of U_L.

By classical results [4, p. 138] there is an $f_0 = f_0(L) \in \mathbb{N}$ such that V_L contains all totally positive units $\varepsilon \in U_L$ such that $\varepsilon \equiv 1 \pmod{f_0!}$.

If we now examine the compatibility relations [3, p. 13] on the local components of a character $\chi \in \mathrm{Gr}_0(L,f)$, we see that, for $f \geq f_0(L), \mathrm{Gr}(L,f)$ contains (many) left $\mathbb{Z}G$-submodules $\mathrm{Gr}_*(L,f)$ with the following properties:

$$\text{(4.3)}\qquad \begin{aligned} &\text{(i)}\ \ \mathrm{Gr}_*(L,f) \text{ has finite index in } \mathrm{Gr}_0(L,f);\\ &\text{(ii)}\ \ \mathrm{Gr}_*(L,f) \text{ is a free } \mathbb{Z}\text{-module of rank } \#G-1;\\ &\text{(iii)}\ \ \mathrm{Gr}_*(L,f) \text{ is a \textit{cyclic} left } \mathbb{Z}G\text{-module.}\end{aligned}$$

For each $f \geq f_0(L)$ we choose some arbitrary but fixed $\mathrm{Gr}_*(L,f)$ satisfying (4.3). We claim that, as a left $\mathbb{Z}G$-module, $\mathrm{Gr}_*(L,f)$ is isomorphic to $\mathbb{Z}/\mathbb{Z}\cdot\sum_{g\in G} g$. (It is clear that $\mathbb{Z}\cdot\sum_{g\in G} g$ is actually a 2-sided ideal in $\mathbb{Z}G$.)

Indeed, since $\mathrm{Gr}_*(L,f) = \mathbb{Z}G \cdot \theta$ for some θ, we have $\mathrm{Gr}_*(L,f) \cong_{\mathbb{Z}G} \mathbb{Z}G/\mathrm{ann}(\theta)$, where $\mathrm{ann}(S)$ is the left $\mathbb{Z}G$-annihilator of any subset S of any left $\mathbb{Z}G$-module. Now $\mathrm{ann}(\theta) \supseteq \mathrm{ann}(\mathrm{Gr}_*(L,f)) = \mathrm{ann}(\mathrm{Gr}_0(L,f))$ since $\mathrm{Gr}_*(L,f)$ has finite index in $\mathrm{Gr}_0(L,f)$, while it is clear that $\mathrm{ann}(\mathrm{Gr}_0(L,f))$ is $\mathbb{Z}\cdot\sum_{g\in G} g$, again from the discussion below (4.1). In particular, $\mathrm{Gr}_*(L,f)$ is left $\mathbb{Z}G$-isomorphic to a quotient of $\mathbb{Z}G/\mathbb{Z}\cdot\sum_{g\in G} g$; but both of these modules are free $\mathbb{Z}$-modules of rank $\#G-1$. This forces $\mathrm{ann}(\theta) = \mathbb{Z}\cdot\sum_{g\in G} g$, as required.

For commutative superrings R of $\mathbb{Z}$, let

$$I_G(R) = \Big\{ \sum x_g g \in RG : \sum x_g = 0 \Big\}. \tag{4.4}$$

Thus $I_G(R)$ is the usual augmentation ideal of RG; it is a 2-sided ideal in RG, and is also a free R-submodule of RG on $\#G - 1$ generators.

It is clear that $I_G(\mathbb{Z}) \cap \mathbb{Z} \cdot \sum_{g\in G} g = 0$. Thus the natural left $\mathbb{Z}G$-epimorphism $\mathbb{Z}G \twoheadrightarrow \mathbb{Z}G/\mathbb{Z} \cdot \sum_{g\in G} g \cong \mathrm{Gr}_*(L, f)$ yields a monomorphism $I_G(\mathbb{Z}) \hookrightarrow \mathrm{Gr}_*(L, f)$ of free $\mathbb{Z}$-modules; both of these are of rank $\#G - 1$; hence there exist $b, c \in \mathbb{N}$ allowing us to assume that

$$cI_G(\mathbb{Z}) \subseteq b\,\mathrm{Gr}_*(L, f) \subseteq I_G(\mathbb{Z}). \tag{4.5}$$

It is clear that $b\,\mathrm{Gr}_*(L, f)$ shares the important properties (4.3) with $\mathrm{Gr}_*(L, f)$. (Incidentally, a useful check that our arguments yielding (4.5) are valid is to note that $I_G(\mathbb{Z})$ also has left $\mathbb{Z}G$-annihilator $\mathbb{Z} \cdot \sum_{g\in G} g$; this is trivial to verify.)

Now let K $(\neq \mathbb{Q})$ be a subfield of $L, \mathrm{Gal}\, L/K = H \subsetneq G$.

We consider $\mathrm{Gr}_*(L, f)^H = \{\chi \in \mathrm{Gr}_*(L, f) : h\chi = \chi, \forall h \in H\}$, i.e. the set of H-fixed points of $\mathrm{Gr}_*(L, f)$. It is clear from (4.5) that $\mathrm{Gr}_*(L, f)^H$ is a free $\mathbb{Z}$-submodule of $\mathrm{Gr}_*(L, f)$, of rank $\kappa - 1, \kappa = [K : \mathbb{Q}] = (G : H) \geq 2$.

Now let $\mathrm{Gr}_*(K, f) = \{\chi \in \mathrm{Gr}_0(K, f) : E_{L/K}(\chi) \in \mathrm{Gr}_*(L, f)\}$. Then $\mathrm{Gr}_*(K, f)$ is a free $\mathbb{Z}$-module of rank $\kappa - 1$, and maps monomorphically into $\mathrm{Gr}_*(L, f)^H$ under $E_{L/K}$. Thus $E_{L/K}(\mathrm{Gr}_*(K, f))$ and $\mathrm{Gr}_*(L, f)^H$ are free $\mathbb{Z}$-modules of the same rank $\kappa - 1$; in particular, $E_{L/K}(\mathrm{Gr}_*(K, f))$ has finite index in $\mathrm{Gr}_*(L, f)^H$. We recall that we wish to define the notion of a generic member of $\mathrm{Gr}_0(K, f)$. We do this by considering the left $\mathbb{Z}G$-annihilators of "typical" members of $E_{L/K}(\mathrm{Gr}_*(K, f))$. To describe our results, let $S \subseteq G$ be such that $G = \bigcup_{s\in S}(sH)$ (disjoint union), and let

$$J = \sum_{s\in S} \mathbb{Z}s + \mathbb{Z}G \cdot I_H(\mathbb{Z}) \subseteq \mathbb{Z}G. \tag{4.6}$$

We shall prove that "almost all" members of $E_{L/K}(\mathrm{Gr}_*(K, f))$ have J as left $\mathbb{Z}G$-annihilator. To make the notion "almost all" precise, let $\psi_1, \ldots, \psi_{\kappa-1}$ be any $\mathbb{Z}$-basis for $E_{L/K}(\mathrm{Gr}_*(K, f))$. Then the number of $\mathbf{n} \in \mathbb{Z}^{\kappa-1}$ such that $\max_{j<\kappa} |n_j| \leq N$ and $\prod_j \psi_j^{n_j}$ fails to have J as its left $\mathbb{Z}G$-annihilator is $O(N^{\kappa-2})$ as $N \to \infty$.

To prove this we first form $\Gamma = \mathrm{Gr}_*(L, f) \otimes \mathbb{Q}$ (where $\otimes$ means $\otimes_{\mathbb{Z}}$), making Γ into a left $\mathbb{Q}G$-module by "extension of scalars". By (4.5) we have

$$\Gamma = I_G(\mathbb{Z} \otimes \mathbb{Q}) = I_G(\mathbb{Q}) \tag{4.7}$$

and $\Gamma^H = \mathrm{Gr}_*(L, f)^H \otimes \mathbb{Q} = E_{L/K}(\mathrm{Gr}_*(K, f)) \otimes \mathbb{Q}$, where Γ^H is $\{\gamma \in \Gamma : h\gamma = \gamma \text{ for all } h \in H\}$. Moreover we have $\Gamma^H = e_H I_G(\mathbb{Q})$, where $e_H = (\#H)^{-1} \sum_{h\in H} h \in \mathbb{Q}H$. Since $I_G(\mathbb{Q})$ is a 2-sided ideal in $\mathbb{Q}G$, Γ^H is a *right*

$\mathbb{Q}G$-ideal. We proceed to find the left $\mathbb{Q}G$-annihilator of a "typical" member of Γ^H. We obtain this as a special case of the following general result.

LEMMA 4.1. *Let F be any field, and let A be a finite-dimensional (associative) semisimple F-algebra. Let $R \neq 0$ be a right ideal of A, and let $v_1, \ldots, v_n$ be an F-basis for R. Then there is a non-zero polynomial $p(\mathbf{X}) = p(X_1, \ldots, X_n)$ in n independent (commuting) variables, with coefficients in F, such that, if $\mathbf{f} \in F^n$ and $p(\mathbf{f}) \neq 0$, we have $(\sum_{j=1}^n f_j v_j)R = R$.*

Proof. Let $\mu \in R$; then $\mu R \subseteq R$. For $\mu \in R$ let $\mathbf{L}(\mu)$ be the F-linear map $x \mapsto \mu x$ on R. We define

$$p(X_1, \ldots, X_n) = \det\Big(\sum_{j=1}^n X_j \mathbf{L}(v_j)\Big) \in F[\mathbf{X}].$$

Then for $\mathbf{f} \in F^n$ we have

$$p(\mathbf{f}) = \det\Big(\mathbf{L}\Big(\sum_{j=1}^n f_j v_j\Big)\Big).$$

In particular, if $p(\mathbf{f}) \neq 0$, then $\mathbf{L}(\sum_{j=1}^n f_j v_j)$ is an F-linear bijection on R and so $(\sum_{j=1}^n f_j v_j)R = R$. Conversely if $p(\mathbf{f}) = 0$, then $\mathbf{L}(\sum_{j=1}^n f_j v_j)$ is singular and so is neither injective nor surjective. (Recall that $\dim_F(R) = n \in \mathbb{N}$.)

It remains to show that $p(\mathbf{X}) \neq \mathbf{0}$ in $F[\mathbf{X}]$. To prove this, we simply note that, since A is semisimple, R is a (right) direct summand of the right A-module A. Consequently, there is a $\pi \in R$ with $\pi = \pi^2 \neq 0$ such that $R = \pi A$. But then $R = \pi R$. From the above arguments we see that $\det \mathbf{L}(\pi) \neq 0$; hence $p(\mathbf{X})$ cannot be $\mathbf{0}$. ∎

We note that if $R = \mu R$ ($\mu \in R$) then μ and R have the same *left* A-annihilator. For $\mathrm{ann}(\mu) \supseteq \mathrm{ann}(R) = \mathrm{ann}(\mu R) \supseteq \mathrm{ann}(\mu)$.

We apply Lemma 4.1 as follows. We take $F = \mathbb{Q}$, $A = \mathbb{Q}G$ and $R = e_H I_G(\mathbb{Q}) = \Gamma^H$. (Since $H \neq G$, $R \neq 0$.)

Now let $\psi_1, \ldots, \psi_{\kappa-1}$ be a $\mathbb{Z}$ basis for $E_{L/K}(\mathrm{Gr}_*(K, f))$. Then $\psi_1 \otimes 1, \ldots, \psi_{\kappa-1} \otimes 1$ is a $\mathbb{Q}$-basis $v_1, \ldots, v_{\kappa-1}$ for Γ^H. By Lemma 4.1, there is a non-zero $p(\mathbf{X}) \in \mathbb{Q}[\mathbf{X}] = \mathbb{Q}[X_1, \ldots, X_{\kappa-1}]$ such that $\mathrm{ann}_{\mathbb{Q}G}\{\sum_{j<\kappa} q_j(\psi_j \otimes 1)\} = \mathrm{ann}_{\mathbb{Q}G}(\Gamma^H)$ (left $\mathbb{Q}G$-annihilators) unless $p(q_1, \ldots, q_{\kappa-1}) = 0$; here $\mathbf{q} \in \mathbb{Q}^{\kappa-1}$. In particular if $\mathbf{q} = \mathbf{n} \in \mathbb{Z}^{\kappa-1}$ then the left $\mathbb{Q}G$-annihilator of $\prod_j \psi_j^{n_j}$ is $\mathrm{ann}_{\mathbb{Q}G}(\Gamma^H)$ unless $p(\mathbf{n}) = 0$. Since $p(\mathbf{X}) \neq \mathbf{0}$, it is clear that the number of $\mathbf{n} \in \mathbb{Z}^{\kappa-1}$ with $\max_{j<\kappa} |n_j| \leq N$ and $p(\mathbf{n}) = 0$ is $O(N^{\kappa-2})$ as $N \to \infty$. (A simple induction on $\kappa \geq 2$ will give this.)

Finally, we note that the left $\mathbb{Q}G$-annihilator of $e_H I_G(\mathbb{Q})$ is

$$\mathrm{ann}_{\mathbb{Q}G}(\Gamma^H) = \mathbb{Q}G \cdot (1 - e_H) + \mathbb{Q} \cdot e_G. \tag{4.8}$$

A simple calculation shows that the intersection of (4.8) with $\mathbb{Z}G$ is exactly J of (4.6). To summarise, "almost all" $\psi \in E_{L/K}(\mathrm{Gr}_*(k,f))$ have left $\mathbb{Z}G$-annihilator equal to J of (4.6).

We are now ready to define generic $\chi \in \mathrm{Gr}_0(K,f)$ ($K \neq \mathbb{Q}$). We choose $L/\mathbb{Q}$ to be the Galois hull of $K/\mathbb{Q}$ and take $f \geq f(L)$.

Let $a = a(f) \in \mathbb{N}$ be the index of $\mathrm{Gr}_*(L,f)$ in $\mathrm{Gr}_0(L,f)$. Then for $\chi \in \mathrm{Gr}_0(K,f)$ we have $\chi^a \in \mathrm{Gr}_*(K,f)$ and $E_{L/K}(\chi^a) \in E_{L/K}(\mathrm{Gr}_*(K,f))$.

We say that χ is *generic* if and only if the left $\mathbb{Z}G$-annihilator of $E_{L/K}(\chi^a)$ is J of (4.6). From the foregoing "almost all" $\chi \in \mathrm{Gr}_0(K,f)$ are generic.

In our proof of Theorem 2 it is important to note the relation

$$J \cap \sum_{s\in S} \mathbb{Z}\cdot s = \mathbb{Z}\cdot \Big\{\sum_{s\in S} s\Big\} \tag{4.9}$$

with J as in (4.6); the proof of (4.9) is a simple exercise.

It is interesting to consider some further properties of generic χ. First of all, such χ have infinite order, since $J \cap \mathbb{Z}\cdot \mathbf{1} = 0$. Secondly, if G is cyclic of prime order p and $K = L$, then $H = \{1\}$ and $\Gamma = \mathrm{Gr}_*(L,f) \otimes \mathbb{Q}$ is isomorphic to $I_G(\mathbb{Q})$; the latter is actually a field $\mathbb{Q}$-isomorphic to $\mathbb{Q}(\exp(2\pi i/p))$. Thus every non-trivial member of $\mathrm{Gr}_*(L,f)$ has left $\mathbb{Z}G$-annihilator exactly $\mathbb{Z}\cdot\{\sum_{g\in G} g\}$. This shows that every χ of infinite order in $\mathrm{Gr}_0(K,f)$ is generic, when $K/\mathbb{Q}$ is cyclic of prime order p. The special case $K/\mathbb{Q}$ quadratic is important since the cusp-forms of Hecke mentioned in §0 derive from Grössencharaktere of imaginary quadratic fields. In particular, because the original problem of Rankin [8] involves only normalised χ from imaginary quadratic fields, the complete solution of his problem is a very special case of Theorem 2.

5. Proof of Theorem 2. Let $[K:\mathbb{Q}] = \kappa \geq 2$ and let $\chi \in \mathrm{Gr}_0(K,f)$ be generic. We assume that $f \geq f_0(L)$ of §4, where $L/\mathbb{Q}$ is the Galois hull of $K/\mathbb{Q}$.

Theorem 2 will follow if we can show that, for each $k \in \mathbb{N}$, the constants $c(\chi,k)$ of (0.1) satisfy

$$c(\chi,k) = \sum_{r=0}^{\kappa} \partial_r \int_0^\infty t^k \, dD_r(t), \tag{5.1}$$

with $\partial_r, D_r(t)$ as in the statement of Theorem 2 in §0.

By the procedure described in §3, we first regard χ as a continuous character of $C_0(K)$, lifting to a character χ_* on $W_0(L/K)$, given by (3.12). Then χ_* is a degree-one representation of $W_0(L/K)$, inducing a representation $\mathcal{R}^0$ of degree $(G:H) = \kappa$ on $W_0(L/\mathbb{Q})$, which lifts to an $\mathcal{R}$ of degree κ on $W(L/\mathbb{Q})$. In view of (3.18) the constant $c(\chi,k)$ equals the $W_0(L/\mathbb{Q})$-inner product $\langle \mathcal{T}^0, \mathcal{T}^0\rangle$, where $\mathcal{T}^0 = \mathcal{R}^0 \underbrace{\otimes \ldots \otimes}_{k} \mathcal{R}^0$.

Clearly we need a formula for $\operatorname{tr}\mathcal{T}^0 = (\operatorname{tr}\mathcal{R}^0)^k$, which we proceed to obtain.

Let $G = \bigcup_{s\in S}(sH)$ be our irredundant decomposition of G into right H-cosets. For short we put $B = C_0(L)$, $\Gamma = W_0(L/K)$ and $\Delta = W_0(L/\mathbb{Q})$. Then we have parallel decompositions

$$\Delta = \bigcup_{s\in S}(s,1)\Gamma, \qquad \Delta = \bigcup_{\sigma\in G}(\sigma, B). \tag{5.2}$$

Of course, $\#S = (G : H) = \kappa$ here. The representation $\mathcal{R}^0$ can be described in matrix terms (with rows and columns labelled by members s, t of S) via

$$\mathcal{R}^0((\sigma, b)) = (\dot{\chi}_*((s,1)^{-1}(\sigma, b)(t,1))). \tag{5.3}$$

Here $\dot{\chi}_*$ is χ_* on Γ and 0 off Γ, while $\sigma \in G$ and $b \in B$. Hence we have

$$\operatorname{tr}\mathcal{R}^0((\sigma, b)) = \sum_{s\in S}\dot{\chi}_*((s,1)^{-1}(\sigma, b)(s,1)). \tag{5.4}$$

If $(s,1)^{-1}(\sigma, b)(s,1) \in \Gamma$ then we can deduce from (3.4), (3.5), (3.12) that $\chi_*((s,1)^{-1}(\sigma, b)(s,1))$ has the form $\chi(N_{L/K}(b^s))\chi(\theta)$, where $\theta \in C_0(K)$ depends only on s, σ. Hence we may rewrite (5.4) in the form

$$\operatorname{tr}\mathcal{R}^0((\sigma, b)) = \sum_{s\in S}\omega(s,\sigma)\chi(N_{L/K}(b^s)), \tag{5.5}$$

where $\omega(s,\sigma) \in \mathbb{C}$ is independent of $b \in B$, and $|\omega(s,\sigma)| = \delta(s^{-1}\sigma s)$, δ being the characteristic function of H. The values $c(\chi, k)$ are then given by

$$\int_{\mathbf{x}\in\Delta} |\operatorname{tr}\mathcal{R}^0(x)|^{2k}\,dx = c(\chi, k) \qquad (k \in \mathbb{N}) \tag{5.6}$$

where dx is normalised Haar measure on Δ, i.e. $\int_\Delta dx = 1$.

Now, for each $\sigma \in G$, we have $\int_{x\in(\sigma,B)} dx = (\#G)^{-1}$. Thus (5.6) gives

$$\#G \cdot c(\chi, k) = \sum_{\sigma\in G}\int_{b\in B} |\operatorname{tr}\mathcal{R}^0((\sigma, b))|^{2k}\,\widehat{db}, \tag{5.7}$$

where $\widehat{db}$ is normalised Haar measure on $B = C_0(L)$. It remains to evaluate, for each fixed $\sigma \in G$, the integral on the right of (5.7).

Let $k, r \in \mathbb{N}$. We use the multinomial identity

$$(X_1 + \ldots + X_r)^k = \sum_{\mathbf{m}}\binom{k}{\mathbf{m}}_r X_1^{m_1}\ldots X_r^{m_r} \tag{5.8}$$

$(X_j^0 = 1)$, where $\mathbf{m}$ runs over all $\mathbf{m} \in \mathbb{Z}^r$ with $m_1, \ldots, m_r \geq 0$ and $\sum_{j\leq r} m_j = k$, while

$$\binom{k}{\mathbf{m}}_r = k!/(m_1!\ldots m_r!).$$

Let $\sigma \in G$, $b \in B$ and $r = \kappa$; we take the kth power of (5.5), multiply by its complex conjugate, and integrate over B (with respect to $\widehat{db}$). This gives

$$\int\limits_{b\in B} |\mathrm{tr}\,\mathcal{R}^0((\sigma,b))|^{2k}\,\widehat{db} = \sum_{\mathbf{m}}\sum_{\mathbf{n}} \binom{k}{\mathbf{m}}_\kappa \binom{k}{\mathbf{n}}_\kappa \prod_{s\in S}\omega(s,\sigma)^{m(s)}\overline{\omega}(s,\sigma)^{n(s)} \times \int\limits_{b\in B} \chi(N_{L/K}(b^{\sum_{s\in S}(m(s)-n(s))s}))\,\widehat{db}. \tag{5.9}$$

(Here, since $\#S = \kappa$ we may label the components of $\mathbf{m},\mathbf{n}$ via members of S.)

Consider, for given $\mathbf{m},\mathbf{n}$, the integral $I(\mathbf{m},\mathbf{n})$ on the right of (5.9).

As χ is generic, $E_{L/K}(\chi^a)$ has left $\mathbb{Z}G$-annihilator J of (4.6). Hence $\mathrm{ann}_{\mathbb{Z}G}(E_{L/K}(\chi)) \subseteq J$. Since, by (4.9), $J \cap \sum_{s\in S}\mathbb{Z}\cdot s = \mathbb{Z}\cdot\{\sum_{s\in S} s\}$, the integral $I(\mathbf{m},\mathbf{n})$ is 0 unless $\sum_{s\in S}(m(s)-n(s))\underline{s} \in \mathbb{Z}\cdot(\sum_{s\in S} s)$.

For the latter to happen, there must be some $d \in \mathbb{Z}$ such that $m(s) - n(s) = d$ for all $s \in S$. But then $\kappa d = \sum_s m(s) - \sum_s n(s) = k - k = 0$, so that $d = 0$ and $\mathbf{m} = \mathbf{n}$. Obviously $I(\mathbf{m},\mathbf{m}) = 1$. Hence (5.9) reduces to

$$\int\limits_{b\in B} |\mathrm{tr}\,\mathcal{R}^0((\sigma,b))|^{2k}\,\widehat{db} = \sum_{\mathbf{m}} \binom{k}{\mathbf{m}}_\kappa^2 \prod_{s\in S}\delta(s^{-1}\sigma s)^{m(s)}. \tag{5.10}$$

Let $r(\sigma)$ be the number of $s \in S$ with $\delta(s^{-1}\sigma s) = 1$. The terms on the right of (5.10) vanish unless $m(s) = 0$ whenever $\delta(s^{-1}\sigma s) = 0$, so that (5.10) collapses down to

$$\int\limits_{b\in B} |\mathrm{tr}\,\mathcal{R}^0((\sigma,b))|^{2k}\,\widehat{db} = \sum_{\underline{\mathbf{m}}} \binom{k}{\underline{\mathbf{m}}}_{r(\sigma)}^2. \tag{5.11}$$

By (2.4) the right-hand side here is $\mathcal{E}(\mathbf{X}^k_{r(\sigma)})$. Thus (5.7) reduces to

$$\#G\cdot c(\chi,k) = \sum_{\sigma\in G}\mathcal{E}(\mathbf{X}^k_{r(\sigma)}). \tag{5.12}$$

Finally, for $r = 0,\dots,\kappa$, let a_r be the number of $\sigma \in G$ with $r(\sigma) = r$. Then we have

$$c(\chi,k) = \sum_{r=0}^{\kappa}\mathcal{E}(\mathbf{X}^k_r)\cdot(\#G)^{-1}a_r \qquad (\forall k \in \mathbb{N}). \tag{5.13}$$

But, by a simple application of Chebotarev's density theorem [4, pp. 379–389], $a_r(\#G)^{-1}$ is exactly ∂_r in the statement of Theorem 2, which is thus proved.

6. Concluding remarks. There is some interest in deciding what happens to Theorem 2 when χ is not generic. If χ has finite order, there is no real difficulty involved; the author's method of "Frobenian functions" [7] can be used, not only to obtain (0.1), but even a complete asymptotic expansion in terms of the asymptotic sequence

$$x(\log x)^{c(\chi,\beta)-n} \quad (n \in \mathbb{Z}).$$

For other types of non-generic χ, a diversity of behaviour of $c(\chi,\beta)$ can be seen by considering various special K,χ, but there is no obvious general pattern as yet. Next, if χ is not normalised, we have

$$|T(n,\chi)| = n^r |T(n,\widehat{\chi})| \quad (\forall n \in \mathbb{N})$$

for some fixed $r = r(\chi) \in \mathbb{R}$ and *normalised* $\widehat{\chi}$. The analogue of (0.1) for χ can then be obtained by using summation by parts on the result of $\widehat{\chi}$. A similar remark covers Theorem 1 with non-normalised $\chi_1,\ldots,\chi_m$.

Finally, we consider the exponents $c(\chi,\beta)$ in Theorem 1. To a certain extent, we may "vectorise" the processes which led to Theorem 2. If $\boldsymbol{\chi} = (\chi_1,\ldots,\chi_m)$, χ_j a normalised Grössencharakter of K_j, we choose $L/\mathbb{Q}$ to be the Galois hull of the composition $K_1\ldots K_m$ over $\mathbb{Q}$. It is clear how to formulate the notion of a generic "vector" $\boldsymbol{\chi} \in \mathrm{Gr}_0(K_1,f)\times\ldots\times\mathrm{Gr}_0(K_m,f)$.

Let $[K_j:\mathbb{Q}] = \kappa_j \ge 2$, $j = 1,\ldots,m$, and let $\mathbf{r} = (r_1,\ldots,r_m)$, $r_j \in \mathbb{Z}$, $0 \le r_j \le \kappa_j$. We define $\partial(\mathbf{r})$ to be the Dirichlet density of the set of primes $p \in \mathbb{N}$ having, for each j, exactly r_j prime ideal factors in K_j of residual degree 1. Then for $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_m) > \mathbf{0}$ in $\mathbb{R}^m$ and χ generic, one can show that

$$c(\boldsymbol{\chi},\boldsymbol{\beta}) = \sum_{\mathbf{r}} \partial(\mathbf{r}) \int_{t_j\in\mathbb{R}} \prod_{j=1}^{m} t_j^{\beta_j}\, dD_{r_j}(t_j), \tag{6.1}$$

representing an exact analogue of Theorem 2.

References

[1] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974; esp. Ch. 5.

[2] G. H. Hardy, *Divergent Series*, Oxford Univ. Press, 1949; esp. Ch. XI.

[3] B. Moroz, *Vistas in Analytic Number Theory*, Bonner Math. Schriften 156, Bonn, 1984.

[4] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, 2nd ed., Springer, 1990.

[5] R. W. K. Odoni, *A problem of Rankin on sums of powers of cusp-form coefficients*, J. London Math. Soc. 44 (1991), 203–217.

[6] —, *The spatial distribution of algebraic integers on constant-norm hypersurfaces*, Mathematika 39 (1992), 350–366.

[7] —, *Notes on the method of Frobenian functions, with applications to Fourier coefficients of modular forms*, in: Elementary and Analytic Theory of Numbers, Banach Center Publ. 17, PWN, Warszawa, 1985, 371–403.

[8] R. A. Rankin, *A family of newforms*, Ann. Acad. Sci. Fenn. 10 (1985), 461–467.

[9] Lord Rayleigh, *Collected Papers*, Vol. 6, Cambridge Univ. Press, 1920, 604–626.

[10] J. Tate, *Number theoretic background*, in: Automorphic Forms, Representation and L-functions, Proc. Sympos. Pure Math. 33, Part 2, Amer. Math. Soc., Providence, RI, 1979, 3–26.

[11] — (with E. Artin (*op. posth.*)), *Class Field Theory*, Benjamin, New York, 1968.

[12] —, *Global class field theory*, in: Algebraic Number Theory, J. W. S. Cassels and A. Fröhlich (eds.), Academic Press, 1967, 162–203.

[13] E. C. Titchmarsh, *The Theory of Fourier Integrals*, 2nd ed., Oxford Univ. Press, 1948, p. 240 et seq.

[14] A. Weil, *Sur la théorie des corps de classes*, J. Math. Soc. Japan 3 (1951), 1–35.

[15] E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann. 143 (1961), 75–102.

[16] —, *Das Asymptotische Verhalten von Summen über multiplikative Funktionen. II*, Acta Math. Acad. Sci. Hungar. 18 (1967), 411–467.

Department of Mathematics
University of Glasgow
Glasgow G12 8QW, U.K.
E-mail: rwko@maths.gla.ac.uk

Received on 16.9.1999 (3670)