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On the distribution of squares of integral Cayley numbers

by

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1. Introduction and statement of the main results. Let $\mathbb{O}$ denote the real division algebra of Cayley's octaves and let Γ denote the smallest (non-associative) subring of $\mathbb{O}$ which contains the eight basal units of $\mathbb{O}$, so that we may identify $\mathbb{O}$ with $\mathbb{R}^8$ and Γ with $\mathbb{Z}^8$. (See [7] for some motivating comments on the choice of Γ as an integral domain.) In a previous paper [7] we developed the following two asymptotic formulas, generalizing a result of H. Müller and W. G. Nowak [9] on the distribution of squares of Gaussian integers, and our results [6] on the distribution of squares of integral quaternions.

As $X \to \infty$,

$$\#\{\theta^2 \mid \theta \in \Gamma \wedge \theta^2 \in [-X, X]^8\} = C_8 X^4 - \frac{8\pi^3}{105} X^{7/2} + X^3 \Delta_8(X), \tag{1.1}$$

where $C_8 = 6.747289\ldots$ *is a numerical constant*, *and the remainder term* $\Delta_8(X)$ *can be estimated by* $\Delta_8(X) \ll X^{23/73} (\log X)^{461/146}$.

$$\#\{\theta^2 \mid \theta \in \Gamma \wedge |\mathrm{Re}(\theta^2)|, |\mathrm{Im}(\theta^2)| \le X\} = \frac{\pi^3}{9} X^4 - \frac{8\pi^3}{105} X^{7/2} + O(X^3), \tag{1.2}$$

where $\mathrm{Re}(a) = a_0$ *is the real part and* $\mathrm{Im}(a) := (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ *is the imaginary part of the octave* $a = (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$, *and* $|\cdot|$ *is the Euclidean norm.*

Referring to the proof of [7, Theorem 1], the order of magnitude of the remainder term $\Delta_8(X)$ is not greater than the bound

$$\max\left\{X^{1/4}, \max_{N \le \sqrt{2X}} \Big|\sum_{n=1}^{N} \psi\Big(\frac{X}{n}\Big)\Big|, \max_{N \ge \sqrt{X/5}} \Big|\sum_{n=N}^{[\sqrt{X}]} \psi(\sqrt{X - n^2})\Big|\right\},$$

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where $\psi(z)=z-[z]-1/2$ is the rounding error function, so that by applying the new, but yet unpublished version [3] of Huxley's method (cf. [2]) the estimate in (1.1) can be improved to

$$\Delta_8(X) \ll X^{131/416}(\log X)^{26947/8320} \qquad (X\to\infty).$$

In the present paper we investigate two further distribution questions which naturally arise from introducing the Cayley numbers by taking the Zorn extension of the complex numbers on the one hand, and by doubling the quaternions on the other. (The second question is also motivated by [8], where the quaternions are introduced by doubling the complex numbers.)

First, consider the set $\mathbb{C}\times\mathbb{C}^3$ and define addition componentwise and multiplication via

$$(x,\mathbf{x})(y,\mathbf{y}) = (xy-\langle \mathbf{y},\mathbf{x}\rangle, \overline{x}\mathbf{y}+y\mathbf{x}+\overline{\mathbf{x}}\times\overline{\mathbf{y}}),$$

where $\overline{(x_1,x_2,x_3)}=(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ for $x_i\in\mathbb{C}$, $\langle\cdot,\cdot\rangle$ is the complex inner product, and $\times$ is the vector product in $\mathbb{C}^3$ overtaken from the standard vector product in $\mathbb{R}^3$. Then, if we identify every real r with $(r,\mathbf{o})$, $\mathbb{C}\times\mathbb{C}^3$ also equals the Cayley algebra $\mathbb{O}$ and $\mathbb{Z}[i]\times\mathbb{Z}[i]^3$ equals the ring Γ.

Secondly, if $\mathbb{H}$ is the division ring of Hamilton's quaternions, then fix a "hyper-quaternion" unit $p\notin\mathbb{H}$ and create $\mathbb{H}+\mathbb{H}p$, defining addition and multiplication formally with respect to $(q_1p)q_2=(q_1\overline{q}_2)p$, $q_1(q_2p)=(q_2q_1)p$, and $(q_1p)(q_2p)=-\overline{q}_2q_1$ for all $q_1,q_2\in\mathbb{H}$. Then $\mathbb{H}+\mathbb{H}p$ equals the Cayley algebra $\mathbb{O}$. Consequently, $\Gamma=\mathbb{J}_0+\mathbb{J}_0p$, where $\mathbb{J}_0=\mathbb{Z}^4$ is the Lipschitz ring of integral quaternions.

If $a\in\mathbb{O}$ then we call $\mathrm{CP}(a):=z$ the *complex part* and $\mathrm{HCP}(a):=\mathbf{z}$ the *hypercomplex part* of the Cayley number $a=(z,\mathbf{z})$ $(z\in\mathbb{C},\mathbf{z}\in\mathbb{C}^3)$.

If $a\in\mathbb{O}$ then we call $\mathrm{QP}(a):=\alpha$ the *quaternion part* and $\mathrm{HQP}(a):=\beta$ the *hyperquaternion part* of the Cayley number $a=\alpha+\beta p$ $(\alpha,\beta\in\mathbb{H})$.

Now the objective of the present paper is to prove the following two theorems.

THEOREM 1. *For $X\ge 1$ let*

$$\mathcal{A}_1(X) := \#\{\theta^2 \mid \theta\in\Gamma \wedge |\mathrm{CP}(\theta^2)|, |\mathrm{HCP}(\theta^2)|\le X\}.$$

Then, as $X\to\infty$,

$$\mathcal{A}_1(X) = C_1X^4 - \frac{8\pi^3}{105}X^{7/2} + O(X^3\Delta(X)),$$

where $C_1=3.500550\ldots$ is a numerical constant so that $2C_1$ equals the eight-dimensional volume of the basic domain $\{a\in\mathbb{O}\mid |\mathrm{CP}(a^2)|,|\mathrm{HCP}(a^2)|\le 1\}$, and

$$\Delta(X) := \max\{X^{1/4}, \Delta_1(X), \Delta_2(X), \Delta_3(X)\},$$

with

$$\Delta_1(X) := \max_{N\le\sqrt{2X}} \Big|\sum_{n=1}^{N}\psi\Big(\frac{X}{n}\Big)\Big|, \qquad \Delta_2(X) := \max_{N\ge 0}\Big|\sum_{n=N}^{[\sqrt{X}]}\psi(\sqrt{X-n^2})\Big|,$$

and

$$\Delta_3(X) := \max_{N\le c_4\sqrt{X}}\Big|\sum_{c_1\sqrt{X}\le n\le N}\psi\Big(\sqrt{\sqrt{2}X-n^2-\frac{X^2}{4n^2}}\Big)\Big|,$$

$$c_1 := \sqrt{(\sqrt{2}-1)/2}, \qquad c_4 := \sqrt{(\sqrt{2}+1)/2}.$$

Numerically,

(1.3) $$\Delta(X) \ll X^{131/416}(\log X)^{26947/8320} \ll X^{0.315} \qquad (X\to\infty).$$

THEOREM 2. *For* $X \ge 1$ *let*

$$\mathcal{A}_2(X) := \#\{\theta^2 \mid \theta\in\Gamma \wedge |\mathrm{QP}(\theta^2)|, |\mathrm{HQP}(\theta^2)| \le X\}.$$

Then, as $X\to\infty$,

$$\mathcal{A}_2(X) = C_2X^4 - \frac{8\pi^3}{105}X^{7/2} + O(X^{101/32+\varepsilon}),$$

where $C_2 = 3.284604\ldots$ *is a numerical constant so that* $2C_2$ *equals the eight-dimensional volume of the basic domain* $\{a\in\mathbb{O} \mid |\mathrm{QP}(a^2)|, |\mathrm{HQP}(a^2)|\le 1\}$. (*The O-constant depends on* ε.)

2. Preparation for the proof. If $a=(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7)\in\mathbb{O}$ let $\overline{a}=(a_0,-a_1,-a_2,-a_3,-a_4,-a_5,-a_6,-a_7)$ be the conjugate of a and $N(a)=a\overline{a}=a_0^2+a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2$ the norm of a. Further let $\mathrm{Im}\,\mathbb{O}=\{0\}\times\mathbb{R}^7$ denote the imaginary space of the algebra $\mathbb{O}$. Then

$$\begin{aligned}a^2 &= a((2a_0,0,0,0,0,0,0,0)-\overline{a}) = 2a_0a-(N(a),0,0,0,0,0,0,0)\\ &= (a_0^2-a_1^2-a_2^2-a_3^2-a_4^2-a_5^2-a_6^2-a_7^2, 2a_0a_1, 2a_0a_2, 2a_0a_3,\\ &\qquad 2a_0a_4, 2a_0a_5, 2a_0a_6, 2a_0a_7).\end{aligned}$$

Consequently, for $a\in\mathbb{O}$ we have

$$\begin{aligned}|\mathrm{CP}(a^2)|, |\mathrm{HCP}(a^2)| \le X \quad &\text{iff} \quad a\in K_1(X),\\ |\mathrm{QP}(a^2)|, |\mathrm{HQP}(a^2)| \le X \quad &\text{iff} \quad a\in K_2(X),\end{aligned}$$

where

$$\begin{aligned}K_1(X) = \{&(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7)\in\mathbb{R}^8 \mid\\ &(a_0^2-(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2))^2+4a_0^2a_1^2\le X^2\\ &\wedge 4a_0^2(a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2)\le X^2\}\end{aligned}$$

and

$$K_2(X) = \{(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7) \in \mathbb{R}^8 \mid \\ (a_0^2-(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2))^2+4a_0^2(a_1^2+a_2^2+a_3^2) \le X^2 \\ \wedge\, 4a_0^2(a_4^2+a_5^2+a_6^2+a_7^2) \le X^2\}.$$

Now define an equivalence relation $\sim$ on $\mathbb{O}$ by $a \sim b$ iff $a^2 = b^2$. Concerning the equivalence classes we observe that (as in the world of Hamilton's quaternions) $[a]_\sim = \{a,-a\}$ if $a \in \mathbb{O}\setminus \operatorname{Im}\mathbb{O}$, and $[a]_\sim = \{b \in \operatorname{Im}\mathbb{O} \mid N(b) = N(a)\}$ if $a \in \operatorname{Im}\mathbb{O}$. Hence, for $i = 1, 2$ we can write

$$\mathcal{A}_i(X) = \#(K_i(X) \cap (\mathbb{N}\times\mathbb{Z}^7)) + O(X) \qquad (X \to \infty),$$

and our two distribution questions become eight-dimensional lattice point problems. Note that the (bounded) domains $K_i(X)$ are obtained by a homothetic dilatation, i.e. $K_i(X) = \sqrt{X}K_i(1)$, but neither is a convex body.

Now, for abbreviation throughout the paper, define constants

$$c_1 := \sqrt{\frac{\sqrt{2}-1}{2}}, \quad c_2 := \sqrt{\frac{1}{2}}, \quad c_3 := \sqrt[4]{\frac{1}{2}}, \quad c_4 := \sqrt{\frac{\sqrt{2}+1}{2}},$$

so that $0 < c_1 < c_2 < c_3 < 1 < c_4 < \sqrt[4]{2}$.

Further (see also [8]), define functions α, β, η, and σ depending on our parameter $X \to \infty$ by

$$\alpha(X;u) := \sqrt{X-u^2} \quad (0 \le u \le \sqrt{X}), \qquad \beta(X;u) := \frac{X}{2u} \quad (u > 0),$$

$$\eta(X;u) := \begin{cases} \alpha(X;u) & (0 \le u \le c_2\sqrt{X}), \\ \beta(X;u) & (u \ge c_2\sqrt{X}) \end{cases}$$

and

$$\sigma(X;u) := \sqrt{\sqrt{2}X - u^2 - X^2/(4u^2)} \quad (c_1\sqrt{X} \le u \le c_4\sqrt{X}).$$

Note that $\alpha(X;u) \le \beta(X;u)$ (with equality iff $u = c_2\sqrt{X}$), $\alpha(X;\sqrt{X}) = \sigma(X;c_1\sqrt{X}) = \sigma(X;c_4\sqrt{X}) = 0$, $\beta(X;u) \ge \sigma(X;u)$ (with equality iff $u = c_3\sqrt{X}$), and $\alpha(X;u) = \sigma(X;u)$ iff $u = c_2c_4\sqrt{X}$, with $\alpha(X;u)\,\{{}^{<}_{>}\}\,\sigma(X;u)$ when $u\,\{{}^{>}_{<}\}\,c_2c_4\sqrt{X}$.

Next we introduce functions F, G, and H depending on our parameter X by

$$G(X;u,v) := u^2 - v^2 - \sqrt{X^2 - 4u^2v^2} \quad (|uv| \le X/2),$$
$$H(X;u,v) := u^2 - v^2 + \sqrt{X^2 - 4u^2v^2} \quad (|uv| \le X/2),$$
$$F(X;u) := \frac{X^2}{4u^2} \quad (u \ne 0).$$

Note that (see also [8]) for $0 < u \le c_4\sqrt{X}$ and $0 < v \le \eta(X;u)$,

$$G(X;u,v) \le 0 \;\Leftrightarrow\; u^2+v^2 \le X \;\Leftrightarrow\; v \le \alpha(X;u),$$

$$\begin{aligned} &H(X;u,v) \le F(X;u) \\ &\qquad\Leftrightarrow\; (u \le c_1\sqrt{X}) \vee (c_1\sqrt{X} \le u \le c_3\sqrt{X} \wedge v \ge \sigma(X;u)), \end{aligned}$$

which immediately implies

$$\min\{F(X;u), H(X;u,v)\} \ll X \qquad (0 < u \le c_4\sqrt{X},\ 0 < v \le \eta(X;u)), \tag{2.1}$$

and

$$\begin{aligned} &0 \le G(X;u,v) \le F(X;u) \\ &\quad\Leftrightarrow\; (c_2\sqrt{X} \le u \le c_3\sqrt{X} \wedge \alpha(X;u) \le v \le \beta(X;u)) \\ &\qquad\vee\; (c_3\sqrt{X} \le u \le \sqrt{X} \wedge \alpha(X;u) \le v \le \sigma(X;u)) \\ &\qquad\vee\; (u \ge \sqrt{X} \wedge v \le \sigma(X;u)). \end{aligned}$$

3. Lattice points in n-dimensional balls. For $n \in \mathbb{N}$ and $R \ge 1$ let $B_n(R)$ denote the closed n-dimensional ball with radius R and center at the origin,

$$B_n(R) = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_1^2+\ldots+x_n^2 \le R^2\}.$$

Note that

$$\operatorname{vol} B_n(R) = \nu_n R^n \quad \text{with} \quad \nu_n := \operatorname{vol} B_n(1) = \frac{\pi^{n/2}}{\Gamma(1+n/2)}, \tag{3.1}$$

specifically, $\nu_6 = \pi^3/6$ and $\nu_4 = \pi^2/2$. Further,

$$\#(B_n(R) \cap \mathbb{Z}^n) = \sum_{0 \le k \le R^2} r_n(k) \tag{3.2}$$

where $r_n(k)$ equals the number of ways to write the integer k as a sum of n squares.

Now, let $P_n(R) := \#(B_n(R) \cap \mathbb{Z}^n) - \operatorname{vol} B_n(R)$ denote the lattice rest of the n-dimensional ball. Then it is well known (cf. Krätzel [5]) that, as $R \to \infty$,

$$P_4(R) \ll R^2(\log R)^{2/3} \quad \text{and} \quad P_n(R) \ll R^{n-2} \quad (n \ge 5), \tag{3.3}$$

and that the latter estimate is best possible, while the first one may be improved only concerning the logarithmic factor.

In dimensions two and three the sharpest-known bounds are given by

$$P_2(R) \ll R^{131/208}(\log R)^{18627/8320} \tag{3.4}$$

due to Huxley, and by

$$P_3(R) \ll R^{21/16+\varepsilon} \tag{3.5}$$

due to Chamizo, Iwaniec and Heath-Brown. Concerning the situation in dimension two, nobody believes that (3.4) is the end of the story, not even the limit of the method, while in dimension three there are absolutely convincing arguments (see [1]) that (3.5) is indeed the limit of the method.

For the number of lattice points on the *surface*

$$\partial B_n(R) := \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_1^2+\dots+x_n^2=R^2\}$$

of the ball $B_n(R)$ we have

$$\#(\partial B_n(R)\cap\mathbb{Z}^n)\ll R^{n-2+\varepsilon}\qquad(n\ge 2),\tag{3.6}$$

with $\varepsilon=0$ for $n\ge 5$, since $\partial B_n(R)\subset B_n(R)\setminus B_n(\sqrt{R^2-R^{-n}})$ and $\operatorname{vol}B_n(R)-\operatorname{vol}B_n(\sqrt{R^2-R^{-n}})\ll R^{-2}$ and $r_n(k)\ll k^{\varepsilon+(n-2)/2}$.

4. Proof of Theorem 1. In order to count the lattice points in the body $K_1(X)$ we define domains $E_6(X;u,v)$ for $X\ge 1$ and $u,v\in\mathbb{R}$, $u\ne 0$, by

$$E_6(X;u,v) := \left\{(x_1,\dots,x_6)\in\mathbb{R}^6 \,\middle|\, x_1^2+\dots+x_6^2\le\frac{X^2}{4u^2} \wedge (u^2-v^2-(x_1^2+\dots+x_6^2))^2+4u^2v^2\le X^2\right\},$$

so that

$$\#(K_1(X)\cap(\mathbb{N}\times\mathbb{Z}^7))=\sum_{u\in\mathbb{N}}\sum_{v\in\mathbb{Z}}\#(E_6(X;u,v)\cap\mathbb{Z}^6).$$

It is plain that

$$E_6(X;u,v)=\{(x_1,\dots,x_6)\in\mathbb{R}^6 \mid G(X;u,v)\le x_1^2+\dots+x_6^2\le\min\{F(X;u),H(X;u,v)\}\}$$

when $|u|\le c_4\sqrt{X}$ and $|v|\le\eta(X;|u|)$, and that $E_6(X;u,v)=\emptyset$ otherwise.

By applying (3.3) and (3.6) for $n=6$, and (2.1) we derive

$$\#(K_1(X)\cap(\mathbb{N}\times\mathbb{Z}^7))=\sum_{0<u\le c_4\sqrt{X}}\ \sum_{|v|\le\eta(X;u)}\operatorname{vol}E_6(X;u,v)+O(X^3),$$

so that with

$$\sum_{0<u\le c_4\sqrt{X}}\operatorname{vol}E_6(X;u,0)=:T(X)$$

and by symmetry we can write

$$\mathcal{A}_1(X)=2\sum_{0<u\le c_4\sqrt{X}}\ \sum_{0<v\le\eta(X;u)}\operatorname{vol}E_6(X;u,v)+T(X)+O(X^3).\tag{4.1}$$

Now, making use of (3.1) for $n = 6$, we split the last double sum in the following way:

$$\sum_u \sum_v \operatorname{vol} E_6(X; u, v) = \frac{\pi^3}{6} \sum_{i=1}^{3} S_i(X) - \frac{\pi^3}{6} \sum_{i=4}^{6} S_i(X) + \frac{\pi^3}{6} S_7(X),$$

where the terms $S_i(X)$ $(i = 1, \ldots, 7)$ are double sums of the form

$$S_i(X) := \sum_{y_i < u \le z_i} \; \sum_{\gamma_i(u) < v \le \delta_i(u)} f_i(u, v)^3,$$

with the summation limits $y_i, z_i, \gamma_i(u), \delta_i(u)$ and the functions f_i all depending on our parameter X and given by the following table:

i	y_i	z_i	$\gamma_i(u)$	$\delta_i(u)$	$f_i(u,v)$
1	0	$c_1\sqrt{X}$	0	$\alpha(X;u)$	$H(X;u,v)$
2	$c_1\sqrt{X}$	$c_2\sqrt{X}$	$\sigma(X;u)$	$\alpha(X;u)$	$H(X;u,v)$
3	$c_2\sqrt{X}$	$c_3\sqrt{X}$	$\sigma(X;u)$	$\beta(X;u)$	$H(X;u,v)$
4	$c_2\sqrt{X}$	$c_3\sqrt{X}$	$\alpha(X;u)$	$\beta(X;u)$	$G(X;u,v)$
5	$c_3\sqrt{X}$	$\sqrt{X}$	$\alpha(X;u)$	$\sigma(X;u)$	$G(X;u,v)$
6	$\sqrt{X}$	$c_4\sqrt{X}$	0	$\sigma(X;u)$	$G(X;u,v)$
7	$c_1\sqrt{X}$	$c_4\sqrt{X}$	0	$\sigma(X;u)$	$F(X;u)$

In order to compute these seven double sums we apply the Euler summation formula (cf. [4]) twice, which yields

$$S_i(X) = V_i(X) + R_i(X) + T_i(X) + U_i(X) + W_i(X) \qquad (i = 1, \ldots, 7),$$

where for $i = 1, \ldots, 7$ and abbreviating $y := y_i$, $z := z_i$, $\gamma(u) := \gamma_i(u)$, $\delta(u) := \delta_i(u)$, and $g(u,v) := f_i(u,v)^3$, we set

$$V_i(X) := \int_y^z \int_{\gamma(u)}^{\delta(u)} g(u,v)\, dv\, du,$$

$$R_i(X) := \psi(y) \int_{\gamma(y)}^{\delta(y)} g(y,v)\, dv - \psi(z) \int_{\gamma(z)}^{\delta(z)} g(z,v)\, dv,$$

$$T_i(X) := \sum_{y < u \le z} \psi(\gamma(u)) g(u, \gamma(u)) - \sum_{y < u \le z} \psi(\delta(u)) g(u, \delta(u)),$$

$$U_i(X) := \sum_{y < u \le z} \int_{\gamma(u)}^{\delta(u)} \frac{\partial g}{\partial v}(u,v) \psi(v)\, dv,$$

$$W_i(X) := \int_y^z \left(\frac{\partial}{\partial u} \int_{\gamma(u)}^{\delta(u)} g(u,v)\, dv \right) \psi(u)\, du.$$

Of course, the terms $V_i(X)$ $(i = 1, \ldots, 7)$ contribute to the main term in Theorem 1 and we obviously have

$$\frac{\pi^3}{6}\Big(\sum_{i=1}^{3} V_i(X) - \sum_{i=4}^{6} V_i(X) + V_7(X)\Big)$$
$$= \int_0^{c_4\sqrt{X}} \int_0^{\eta(X;u)} \operatorname{vol} E_6(X;u,v)\,dv\,du$$
$$= \frac{1}{4}\operatorname{vol} K_1(X) = 64X^4 \operatorname{vol}(K_1(1) \cap \mathbb{R}_+^8).$$

With the help of the software package MATHEMATICA we derive

$$C_1 := 128 \operatorname{vol}(K_1(1) \cap \mathbb{R}_+^8) = 3.500550\ldots$$

Further, $R_7(X) = 0$ and

$$\sum_{i=1}^{3} R_i(X) - \sum_{i=4}^{6} R_i(X) = -\frac{1}{2}\int_0^{\sqrt{X}} H(X;0,v)^3\,dv = -\frac{8}{35}X^{7/2}$$

contributes to the second main term in Theorem 1. Note that the first terms of $T_1(X)$, $T_6(X)$, and $T_7(X)$ together annihilate the term $T(X)$ in (4.1), so that there is no further contribution to the second main term, while the remaining terms of the $T_i(X)$'s are all weighted ψ-sums and thus $\ll X^3\Delta(X)$ by Abel summation. Consequently,

$$\mathcal{A}_1(X) = C_1X^4 - \frac{8\pi^3}{105}X^{7/2} + O(X^3\Delta(X)) + O(\Delta_0(X)),$$

where $\Delta_0(X) = \max\{|U_1(X)|, \ldots, |U_7(X)|, |W_1(X)|, \ldots, |W_7(X)|\}$.

By applying [8, Lemmata 1–3] we obtain

$$\Delta(X) \ll X^{23/73}(\log X)^{461/146},$$

and there is no problem to adapt the three lemmata in [8] according to the new version [3] of Huxley's method, so that the better estimate (1.3) follows as well.

Finally, after a rather long but straightforward argument involving the second mean value theorem and certain routine tricks (see [8]), we obtain $\Delta_0(X) \ll X^{13/4}$. This finishes the proof of Theorem 1.

Remark. As in (1.1) and (1.2), the main term in Theorem 1 (and also, as we claim, in Theorem 2) equals half the volume of the basic body. The second main term reflects the influence of the imaginary space $\operatorname{Im}\mathbb{O}$. Actually, the term counterbalances the surplus of an ordinary counting of half of the lattice points in the particular body which ignores the clotting effect of the equivalence relation $\sim$ on the space $\operatorname{Im}\mathbb{O}$. Therefore, the second main term in all four theorems is the same. In this connection it seems strange that concerning the quaternion problem, in [8, Theorem 1] there occurs a second main term different from the ones in [6, Theorems 1 and 2]. The reason for this is simple. The constant C_2 in [8] has unfortunately been miscalculated! Indeed, this constant has to be replaced by $-2\pi/3$ and knowing this, the reader will not find it hard to track down the error.

5. Proof of Theorem 2. In order to count the lattice points in the body $K_2(X)$ we define domains $E_4(X;u,v)$ for $X \geq 1$ and $u, v \in \mathbb{R}, u \neq 0$, by

$$E_4(X;u,v) := \left\{ (x_1, \ldots, x_4) \in \mathbb{R}^4 \,\middle|\, x_1^2 + \ldots + x_4^2 \leq \frac{X^2}{4u^2} \right. \\ \left. \wedge\, (u^2 - v^2 - (x_1^2 + \ldots + x_4^2))^2 + 4u^2v^2 \leq X^2 \right\},$$

so that the sets $E_4(X;u,v)$ have the same construction as the sets $E_6(X;u,v)$ and hence

$$E_4(X;u,v) = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid \\ G(X;u,v) \leq x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq \min\{F(X;u), H(X;u,v)\}\} \tag{5.1}$$

when $|u| \leq c_4\sqrt{X}$ and $|v| \leq \eta(X;|u|)$, and $E_4(X;u,v) = \emptyset$ otherwise. Additionally, for technical reasons set

$$E_4(X;0,v) = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq H(X;0,v)\}$$

for $|v| \leq \sqrt{X}$. Then

$$\#(K_2(X) \cap (\mathbb{N} \times \mathbb{Z}^7)) = \sum_{u\in\mathbb{N}} \sum_{m\in\mathbb{N}_0} r_3(m)\#(E_4(X;u,\sqrt{m}) \cap \mathbb{Z}^4).$$

Since the contribution coming from the summand with $m = 0$ is trivially $\ll X^{5/2}$ we can rewrite

$$\#(K_2(X) \cap (\mathbb{N} \times \mathbb{Z}^7)) \\ = \sum_{0<u\leq c_4\sqrt{X}} \sum_{0<m\leq\eta(X;u)^2} r_3(m)\#(E_4(X;u,\sqrt{m}) \cap \mathbb{Z}^4) + O(X^{5/2}).$$

By definition, $\eta(X;u) \leq \sqrt{X}$, hence $r_3(m) \ll X^{1/2+\varepsilon}$ uniformly in $1 \leq m \leq \eta(X;u)^2$. Thus, by (3.3) and (3.6) for $n = 4$ we obtain

$$\#(K_2(X) \cap (\mathbb{N} \times \mathbb{Z}^7)) \\ = \sum_{0<u\leq c_4\sqrt{X}} \sum_{0<m\leq\eta(X;u)^2} r_3(m) \operatorname{vol} E_4(X;u,\sqrt{m}) + O(X^{3+\varepsilon}).$$

Next we are going to apply an integral version of Abel summation given by the following lemma. (Cf. [4, Theorem 1.2].)

Lemma. *Let $z : \mathbb{N} \to \mathbb{R}$ be a number-theoretical function and, for $0 \le a < b$, let $\varphi : [a,b] \to \mathbb{R}$, a continuous function whose derivative φ' exists on $]a,b[$ and is (improperly) integrable on $[a,b]$. Further let $Z(t) := \sum_{1\le k\le t} z(k)$ $(t \in \mathbb{R})$. Then*

$$\sum_{a<k\le b} \varphi(k)z(k) = \varphi(b)Z(b) - \varphi(a)Z(a) - \int_a^b \varphi'(t)Z(t)\,dt.$$

Now let $Q(t) := \sum_{1\le k\le t} r_3(k)$, so that by (3.2),

$$1 + Q(t) = \frac{4\pi}{3}\, t^{3/2} + P_3(\sqrt{t}) \quad \text{ for } t \ge 1.$$

Then, by the Lemma, for every $u \in\,]0, c_4\sqrt{X}]$,

$$\sum_{0<m\le\eta(X;u)^2} r_3(m) \operatorname{vol} E_4(X;u,\sqrt{m}) = -\int_0^{\eta(X;u)^2} \Big(\frac{d}{dt} \operatorname{vol} E_4(X;u,\sqrt{t})\Big) Q(t)\,dt,$$

because $Q(0) = 0$ by definition, and obviously $\operatorname{vol} E_4(X;u,\eta(X;u)) = 0$ for $0 < u \le c_4\sqrt{X}$.

Since $Q(t)$ is constant on every interval $k \le t < k+1$ $(k \in \mathbb{Z})$, by (3.5) there exists a function $\Phi : [0,\infty[\, \to \mathbb{R}$ integrable on every compact subinterval of $[0,\infty[$ and such that for every $\varepsilon > 0$ there is a positive constant C_ε such that for all $t \ge 0$,

$$Q(t) = \frac{4\pi}{3}\, t^{3/2} + \Phi(t) \quad \text{and} \quad |\Phi(t)| \le C_\varepsilon(1 + t^{21/32+\varepsilon}).$$

Consequently, uniformly in $0 < u \le c_4\sqrt{X}$ and $X \ge 1$,

$$\sum_{0<m\le\eta(X;u)^2} r_3(m) \operatorname{vol} E_4(X;u,\sqrt{m}) = \frac{4\pi}{3} \int_0^{\eta(X;u)^2} L(t)t^{3/2}\,dt + O\Big(\max_{0\le t\le X} |\Phi(t)| \int_0^{\eta(X;u)^2} |L(t)|\,dt\Big),$$

where (depending on X and u)

$$L(t) := -\frac{d}{dt} \operatorname{vol} E_4(X;u,\sqrt{t}).$$

Now, since $L(t)$ is always algebraic, whence there is a fixed number N such that for every X and every u the function L changes its sign at most N times on $0 \le t \le \eta(X;u)^2$, and since $E_4(X;u,\sqrt{t}) \subset B_4(\sqrt{H(X;c_4\sqrt{X},0)})$, whence $\operatorname{vol} E_4(X;u,\sqrt{t}) < 25X^2$ by (3.1), we get

$$\int\limits_0^{\eta(X;u)^2} |L(t)|\,dt < (N+1)25X^2 \qquad (0<u\le c_4\sqrt{X},\, X\ge 1)$$

and thus arrive at

$$\mathcal{A}_2(X) = \frac{4\pi}{3}\sum_{0<u\le c_4\sqrt{X}}\ \int\limits_0^{\eta(X;u)^2} L(t)t^{3/2}\,dt + O_\varepsilon(X^{101/32+\varepsilon}). \tag{5.2}$$

Applying partial integration we can write

$$\int\limits_0^{\eta(X;u)^2} L(t)t^{3/2}\,dt = \int\limits_0^{\eta(X;u)^2} \operatorname{vol} E_4(X;u,\sqrt{t})\frac{3}{2}\sqrt{t}\,dt,$$

since $\operatorname{vol} E_4(X;u,\eta(X;u)) = 0$, so that after a substitution,

$$\frac{4\pi}{3}\int\limits_0^{\eta(X;u)^2} L(t)t^{3/2}\,dt = 4\pi\int\limits_0^{\eta(X;u)} \operatorname{vol} E_4(X;u,v)v^2\,dv.$$

Thus, by the Euler summation formula the main term in (5.2) equals the sum of

$$4\pi\int\limits_0^{c_4\sqrt{X}}\int\limits_0^{\eta(X;u)} \operatorname{vol} E_4(X;u,v)v^2\,dv\,du, \tag{5.3}$$

$$4\pi\psi(0)\int\limits_0^{\eta(X;0)} \operatorname{vol} E_4(X;0,v)v^2\,dv, \tag{5.4}$$

$$-4\pi\psi(c_4\sqrt{X})\int\limits_0^{\eta(X;c_4\sqrt{X})} \operatorname{vol} E_4(X;c_4\sqrt{X},v)v^2\,dv, \tag{5.5}$$

and

$$4\pi\int\limits_0^{c_4\sqrt{X}}\left(\frac{d}{du}\int\limits_0^{\eta(X;u)} \operatorname{vol} E_4(X;u,v)v^2\,dv\right)\psi(u)\,du. \tag{5.6}$$

Clearly, (5.3) yields the main term in Theorem 2. It is plain that it equals

$$\int\limits_0^{c_4\sqrt{X}}\ \iiint\limits_{x^2+y^2+z^2\le\eta(X;u)^2} \operatorname{vol} E_4(X;u,\sqrt{x^2+y^2+z^2})d(x,y,z)\,du$$
$$= \frac{1}{2}\operatorname{vol} K_2(X),$$

so that, again with the help of MATHEMATICA, (5.3) equals

$$128X^4 \operatorname{vol}(K_2(1) \cap \mathbb{R}_+^8) = C_2 X^4, \qquad C_2 = 3.284604\ldots$$

Further, (5.5) vanishes since $E_4(X; c_4\sqrt{X}, v) = \emptyset$ for $v > 0$, whilst (5.4) yields the second main term and equals

$$-2\pi \int_0^X \frac{\pi^2}{2} H(X;0,v)^2 v^2\,dv = -\pi^3 X^{7/2} \int_0^1 (1-t^2)^2 t^2\,dt = -\frac{8\pi^3}{105} X^{7/2}.$$

Finally, we claim that (5.6) is $O(X^3)$, which finishes the proof of Theorem 2. In order to verify this estimate we set

$$I(X;u) := \frac{d}{du} \int_0^{\eta(X;u)} \operatorname{vol} E_4(X;u,v) v^2\,dv \qquad (0 \le u \le c_4\sqrt{X}),$$

whence by differentiation of a parameter integral

$$I(X;u) = \int_0^{\eta(X;u)} \left(\frac{\partial}{\partial u} \operatorname{vol} E_4(X;u,v)\right) v^2\,dv \qquad (0 \le u \le c_4\sqrt{X}),$$

since $\operatorname{vol} E_4(X;u,\eta(X;u)) = 0$ on $0 \le u \le c_4\sqrt{X}$.

By (5.1) it is clear that (for fixed X) the function $I(X;u)$ is piecewise monotonic in u (with an absolutely bounded number of pieces). Consequently, by making use of the oscillation of the rounding error function ψ, it suffices to show that $I(X;u) \ll X^3$ uniformly in u.

Now, since $\frac{\partial}{\partial u}(F(X;u)^2) \ll X^{3/2}$ uniformly in $u \ge c_1\sqrt{X}$, it suffices to look carefully at $\frac{\partial}{\partial u}(H(X;u,v)^2)$ and $\frac{\partial}{\partial u}(G(X;u,v)^2)$. We compute

$$\frac{\partial}{\partial u}((u^2 - v^2 \pm \sqrt{X^2 - 4u^2v^2})^2)$$
$$= 4u(u^2 - 3v^2 \pm \sqrt{X^2 - 4u^2v^2} \mp 2v^2 K(X;u,v)),$$

where

$$K(X;u,v) := \frac{u^2 - v^2}{\sqrt{X^2 - 4u^2v^2}} \qquad (|uv| < X/2).$$

Hence, by trivial estimation, we certainly have

$$I(X;u) \ll X^3 + X^{5/2} \int_0^{\eta(X;u)} |K(X;u,v)|\,dv$$

uniformly in $0 \le u \le c_4\sqrt{X}$. Finally, when $c_2\sqrt{X} \le u \le c_4\sqrt{X}$,

$$\int_0^{\eta(X;u)} |K(X;u,v)|\,dv = \int_0^{\beta(X;u)} K(X;u,v)\,dv = \pi\,\frac{8u^4 - X^2}{32u^3} \le \sqrt{X},$$

whilst when $0 \le u \le c_2\sqrt{X}$,

$$\int\limits_0^{\eta(X;u)} |K(X;u,v)|\,dv = \int\limits_0^{u} K(X;u,v)\,dv - \int\limits_u^{\alpha(X;u)} K(X;u,v)\,dv$$
$$= \sqrt{X}\cdot\lambda\left(\frac{u}{\sqrt{X}}\right),$$

where $t \mapsto \lambda(t)$ is a function which is continuous on the compact interval $0 \le t \le c_2$. This concludes the proof of Theorem 2.

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