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On irregularities in the graph of generalized divisor functions

by

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1. Introduction. It is partly known [1], partly easy to prove that for the divisor function

(1) $$d(n) := \sum_{d\mid n} 1,$$

it is true that for all $\omega > 0$ there is an $n \in \mathbb{N}$ such that

(2) $$d(n) > \omega + \max(d(n-1), d(n+1))$$

and also there is an $m \in \mathbb{N}$ such that

(3) $$d(m) + \omega < \min(d(m-1), d(m+1)).$$

P. Erdős [1] proved (2) in the following stronger form: for all $k \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ such that

(4) $$d(n) > \prod_{i=1}^{k} d(n-i)d(n+i).$$

We will extend these theorems to generalized divisor functions $d(\mathcal{A}, n)$ defined for any set $\mathcal{A} \subseteq \mathbb{N}$ as

(5) $$d(\mathcal{A}, n) := \sum_{a\in\mathcal{A},\, a\mid n} 1.$$

These functions were introduced by Erdős and Sárközy [2]. Among other results they proved that for any infinite $\mathcal{A}$ the large values of $d(\mathcal{A}, n)$ are much greater than its average:

(6) $$\limsup_{N\to\infty} \frac{\max_{n\le N} d(\mathcal{A}, n)}{\sum_{a\in\mathcal{A}, a\le N} 1/a} = \infty.$$

A. Sárközy posed the following three related problems in [5] (Problems 25–27):

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Problem 1. *Is it true that* $|d(\mathcal{A},n+1)-d(\mathcal{A},n)|$ *cannot be bounded for an infinite set* $\mathcal{A}\subseteq\mathbb{N}$*?*

Problem 2. *Is it true that for any infinite set* $\mathcal{A}\subseteq\mathbb{N}$ *there are infinitely many* n *with*

$$d(\mathcal{A},n)>\max(d(\mathcal{A},n+1),d(\mathcal{A},n-1))?$$

Problem 3. *What assumption is needed to ensure that*

$$d(\mathcal{A},n)<\min(d(\mathcal{A},n-1),d(\mathcal{A},n+1))$$

for infinitely many n*?*

This article solves these problems and also generalizes Erdős's theorem.

2. Notation and the lemma. Following [4], we will use the following notations: Let $\mathcal{B}\subset\mathbb{N}$ be an arbitrary finite sequence, $X:=|\mathcal{B}|$. Let $\mathcal{P}\subset\mathbb{N}$ be an arbitrary set of primes. Set

$$P(z):=\prod_{p\in\mathcal{P},\,p\le z}p. \tag{7}$$

$$S(\mathcal{B},\mathcal{P},z):=|\{b:b\in\mathcal{B},\ (b,P(z))=1\}|. \tag{8}$$

Let ω be a multiplicative arithmetical function such that $\omega(n)=0$ if n is not squarefree and also if n has a prime factor not in $\mathcal{P}$, and $\omega(1):=1$. Let γ be Euler's constant and Γ be the well-known Gamma function, μ be the Möbius function, and $\nu(d)$ be the number of distinct prime divisors of d. We define

$$W(z):=\prod_{p\le z}\left(1-\frac{\omega(p)}{p}\right). \tag{9}$$

$$\sigma_\kappa(u):=2^{-\kappa}\,\frac{e^{-\gamma\kappa}}{\Gamma(\kappa+1)}\,u^\kappa\qquad\text{if } 0\le u\le 2, \tag{10}$$

$$(u^{-\kappa}\sigma_\kappa(u))':=-\kappa u^{-\kappa-1}\sigma_\kappa(u-2)\qquad\text{if } u>2, \tag{11}$$

with σ_κ required to be continuous at $u=2$. We set

$$\eta_\kappa(u):=\kappa u^{-\kappa}\int_u^\infty t^{\kappa-1}\left(\frac{1}{\sigma_\kappa(t-1)}-1\right)dt\qquad(u>1). \tag{12}$$

$$R_d:=|\{b\in\mathcal{B}:d\,|\,b\}|-\frac{\omega(d)}{d}X\qquad\text{if }\mu(d)\ne 0. \tag{13}$$

Let us now define four properties as in [4]:

(Ω_1): There exists A_1 such that $0\le\omega(p)/p\le 1-1/A_1$ for all primes p.
($\Omega_2(\kappa,A_2,A_3)$): There exist $\kappa\ge 0$ and $A_2,A_3\ge 1$ such that

$$(14)\qquad -A_2 \le \sum_{w\le p<z \text{ prime}} \frac{\omega(p)\log p}{p} - \kappa \log \frac{z}{w} \le A_3 \quad \text{if } 2\le w\le z.$$

(R): $|R_d| \le \omega(d)$ if $\mu(d)\ne 0$, and $(d,p)=1$ for all $p\notin\mathcal{P}$.

(R(κ,α)): There exist constants $0<\alpha<1$ and $A_4, A_5 \ge 1$ such that if $X\ge 2$ then

$$(15)\qquad \sum_{\substack{d<X^\alpha/(\log X)^{A_4}\\ \forall p\notin\mathcal{P}\ (d,p)=1}} \mu^2(d)3^{\nu(d)}|R_d| \le A_5 \frac{X}{\log^{\kappa+1} X}.$$

It is not difficult to see that (R(κ,α)) is less restrictive than (R) beside (Ω_1) (see [4]). The strongest lower bound for $S(\mathcal{B},\mathcal{P},z)$ in [4] is the following:

LEMMA 1 (see [4, p. 219]). *If* (Ω_1), $(\Omega_2(\kappa,A_2,A_3))$ *and* (R(κ,α)) *hold and*

$$z^2 \le X^\alpha/(\log X)^{A_4} \qquad (X\ge 2),$$

then

$$S(\mathcal{B},\mathcal{P},z) \ge XW(z)\biggl(1-\eta_\kappa\biggl(\alpha\frac{\log X}{\log z}\biggr) - A_6 \frac{A_2(\log\log 3X)^{3\kappa+2}}{\log X}\biggr)$$

where $A_6\ge 1$ *is a constant which depends only on* $\kappa,\alpha,A_1,A_2,A_3,A_4,A_5$.

3. The results

THEOREM 1. *Let* $\mathcal{A}=\{a_1<a_2<\ldots\}\subseteq\mathbb{N}$ *and* $k\in\mathbb{N}$. *Then there exist infinitely many* $n\in\mathbb{N}$ *such that*

$$d(\mathcal{A},n) > \prod_{i=1}^{k} d(\mathcal{A},n-i)d(\mathcal{A},n+i).$$

Proof. We are going to prove that there exists a constant $C=C(k)>0$ such that there are infinitely many n for which

$$(16)\qquad \prod_{i=1}^{k} d(\mathcal{A},n-i)d(\mathcal{A},n+i) < C$$

and $d(\mathcal{A},n)$ can be arbitrarily large for these n's. Define

$$(17)\qquad X := \prod_{p\le 2k+1 \text{ prime}} p^{1+[\log_p k]} \prod_{j=1}^{N} a_j,$$

$$(18)\qquad \mathcal{B} := \biggl\{\prod_{i=1}^{k}(jX-i)(jX+i) : j\in\{1,\ldots,X\}\biggr\},$$

$$(19)\qquad \mathcal{P} := \{p : (p,X)=1 \text{ prime}\},$$

$$(20)\qquad \omega(p) := 2k \quad \text{if } p\in\mathcal{P},$$

and extend ω multiplicatively to squarefree d's for which $(d,p)=1$ if $p\notin\mathcal{P}$. It is easy to see that $|\mathcal{B}|=X$. Now we should check the conditions we need for the lemma:

(Ω_1): Since $0\le\omega(p)\le 2k$ and $p>2k+1$ if $\omega(p)\neq 0$, we have

$$0\le\frac{\omega(p)}{p}\le 1-\frac{1}{2k+1}. \tag{21}$$

($\Omega_2(\kappa,A_2,A_3)$): This condition is trivial by the following well-known statement:

$$\sum_{w\le p<z\text{ prime}}\frac{\log p}{p}=\log\left(\frac{z}{w}\right)+O(1)\quad\text{ if } 2\le w\le z \tag{22}$$

because $0\le\omega(p)\le 2k$, and $\omega(p)=2k$ if $p>2k+1$.

(R(κ,α)): It is enough to prove (R) because it is more restrictive beside (Ω_1). Suppose that $d=\prod_{r=1}^{l}p_r$ where $p_r\in\mathcal{P}$ are distinct primes. We can get $|\{b\in\mathcal{B}:d\,|\,b\}|$ by counting how many $j\in\{1,\dots,X\}$ there exist such that $p_r\,|\,jX+i_r$ for fixed $i_r\in\{1,\dots,k,-1,-2,\dots,-k\}$ for all $1\le r\le l$. Now $(X,d)=1$ and this condition holds for j if and only if it does for $j+d$, so there are $[X/d]$ or $[X/d]+1$ pieces of such j's. Hence if we take it X/d then the bias is at most 1. There are $(2k)^l=\omega(d)$ choices for the i_r's and therefore $|R_d|\le\omega(d)$.

Now we can use the lemma. Let $z=X^{1/c}$ and choose c such that

$$z^2\le\frac{X^\alpha}{(\log X)^{A_4}}, \tag{23}$$

$$\eta_\kappa\left(\alpha\frac{\log X}{\log z}\right)=\eta_\kappa(\alpha c)<1 \tag{24}$$

for X large enough. Such a c exists because η_κ is a decreasing function with limit 0 at $+\infty$. Now we choose N large enough and

$$N>\Bigl(2^{4kc}\prod_{p\le k\text{ prime}}(2[k/p][\log_p k]+1)\Bigr)^{2k}.$$

Then

$$1-\eta_\kappa\left(\alpha\frac{\log X}{\log z}\right)-A_6\,\frac{A_2(\log\log 3X)^{3\kappa+2}}{\log X}>0. \tag{25}$$

So we can conclude from the lemma that $S(\mathcal{B},\mathcal{P},z)>0$, which means that there exists $b\in\mathcal{B}$ with $(b,p)=1$ if $p\in\mathcal{P}$ and $p\le z$, and $b=\prod_{i=1}^{k}(jX-i)(jX+i)$ for some $j\in\{1,\dots,X\}$. In view of the lemma below, $n=jX$ is a good choice for the theorem.

Lemma 2. *We have*

$$d(\mathcal{A},jX\pm i)\le d(\mathcal{A},b)\le d(b)\le 2^{4kc}\prod_{p\le k\ \textit{prime}}(2[k/p][\log_p k]+1).$$

Proof. The first two inequalities are trivial. For the third one we use the formula $d(\prod_{i=1}^{m} p_i^{\alpha_i}) = \prod_{i=1}^{m}(\alpha_i + 1)$:

1. If $p \le k$ then $p^{1+[\log_p k]} \mid X$ so only $2[k/p]$ factors in

$$b = \prod_{i=1}^{k}(jX - i)(jX + i)$$

are divisible by p and all of them contain at most $[\log_p k]$ factors p because $p^{1+[\log_p k]} > k$.

2. If $k < p$ and $p \mid X$ then $(p, b) = 1$.

3. If $k < p$ and $(p, X) = 1$ then $p \in \mathcal{P}$. So if $p \le z$ then $(p, b) = 1$ else these primes give at most a multiplier of 2^{4kc} in $d(b)$ because $b < X^{4k} = z^{4kc} \le p^{4kc}$. ∎

Now the proof of the theorem can be completed: For $n = jX$,

$$d(\mathcal{A}, n) \ge N > \Big(2^{4kc} \prod_{p \le k \text{ prime}} (2[k/p][\log_p k] + 1)\Big)^{2k} \ge \prod_{i=1}^{k} d(\mathcal{A}, n - i) d(\mathcal{A}, n + i). \quad ∎ \tag{26}$$

From this theorem we know that the generalized divisor functions have isolated large values. One may ask: what about the isolated small values? The set $\mathcal{A} = \{a : a \in \mathbb{N},\ 3 \mid a\}$ shows that it may occur that

$$d(\mathcal{A}, n) < \min(d(\mathcal{A}, n - 1), d(\mathcal{A}, n + 1)) \tag{27}$$

never holds. The following two theorems answer the question by giving a necessary and sufficient condition on $\mathcal{A}$.

THEOREM 2. *There are infinitely many $n \in \mathbb{N}$ such that*

$$d(\mathcal{A}, n) < \min(d(\mathcal{A}, n - 1), d(\mathcal{A}, n + 1))$$

if and only if there exist $a, b \in \mathcal{A}$ (not necessarily distinct) such that $a, b > 1$ and $(a, b) \le 2$.

Proof. One direction is trivial because if there exists an $n \in \mathbb{N}$ such that (27) holds then $n - 1$ and also $n + 1$ must have a divisor in $\mathcal{A}$; the two divisors are greater than 1 and their greatest common divisor is at most 2.

For the other direction assume that $a, b \in \mathcal{A}$ are such that $a, b > 1$ and $(a, b) \le 2$. From the Chinese Remainder Theorem we know that there is a residue-class mod$[a, b]$ which is congruent to 1 $(\bmod\, a)$ and -1 $(\bmod\, b)$. From Dirichlet's theorem we see that there are infinitely many prime numbers in this residue-class. If infinitely many of these primes do not belong to $\mathcal{A}$ then we are done. If all but finitely many of these primes belong to $\mathcal{A}$ then let $p_1 < p_2 < p_3 < p_4$ be such primes from the set $\mathcal{A}$.

Applying again the Chinese Remainder Theorem and Dirichlet's theorem we find that there are infinitely many primes p such that $p \equiv 1 \pmod{p_1 p_2}$ and $p \equiv -1 \pmod{p_3 p_4}$ and for these primes $n = p$ satisfies (27). ∎

THEOREM 3. *For all $\omega > 0$ there are infinitely many $n \in \mathbb{N}$ such that*

$$d(\mathcal{A}, n) + \omega < \min(d(\mathcal{A}, n-1), d(\mathcal{A}, n+1)) \tag{28}$$

if and only if for all $k \in \mathbb{N}$ there exist $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathcal{A}$ so that $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$, $([a_1, \ldots, a_k], [b_1, \ldots, b_k]) \leq 2$ and all $a_i, b_j > 1$.

Proof. One direction is trivial: if (28) holds for all ω with some $n \in \mathbb{N}$ then we choose $k = [\omega] + 1$, the numbers $n+1$ and $n-1$ have at least k divisors (> 1) in $\mathcal{A}$, and these $2k$ elements satisfy the condition.

To prove the other direction we use the Chinese Remainder Theorem and Dirichlet's theorem to deduce that there are infinitely many prime numbers p for which the following two relations hold for all $i, j \in \{1, \ldots, k\}$:

$$a_i \mid p - 1, \tag{29}$$

$$b_j \mid p + 1. \tag{30}$$

Now $n = p$ satisfies (28) with $\omega = k - 1$, and since k was an arbitrary natural number, the proof is complete. ∎

4. Corollaries

COROLLARY 1 (Theorem of Erdős, see [1] and [3, p. 277]). *For the divisor function $d(n)$, for all $k \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with*

$$d(n) > \prod_{i=1}^{k} d(n-i) d(n+i).$$

Proof. Choose $\mathcal{A} = \mathbb{N}$ and apply Theorem 1. ∎

COROLLARY 2. *For all $\omega > 0$ there are infinitely many $n \in \mathbb{N}$ with*

$$d(n) + \omega < \min(d(n-1), d(n+1)).$$

Proof. Choose $\mathcal{A} = \mathbb{N}$ and apply Theorem 3. ∎

COROLLARY 3. *For the number $\nu(n)$ of distinct prime divisors, for all $k \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with*

$$\nu(n) > \prod_{i=1}^{k} \nu(n-i) \nu(n+i).$$

Proof. Choose $\mathcal{A} = \{p \in \mathbb{N} : \text{prime}\}$ and apply Theorem 1. ∎

COROLLARY 4. *For all $\omega > 0$ there are infinitely many $n \in \mathbb{N}$ with*

$$\nu(n) + \omega < \min(\nu(n-1), \nu(n+1)).$$

Proof. Choose $\mathcal{A} = \{p \in \mathbb{N} : \text{prime}\}$ and apply Theorem 3. ∎

Corollary 5. *For the total number $\Omega(n)$ of prime divisors, for all $k \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ with*

$$\Omega(n) > \prod_{i=1}^{k} \Omega(n-i)\Omega(n+i).$$

Proof. Choose $\mathcal{A} = \{q \in \mathbb{N} : \text{prime or power of a prime}\}$ and apply Theorem 1. ■

Corollary 6. *For all $\omega > 0$ there are infinitely many $n \in \mathbb{N}$ with*

$$\Omega(n) + \omega < \min(\Omega(n-1), \Omega(n+1)).$$

Proof. Choose $\mathcal{A} = \{q \in \mathbb{N} : \text{prime or power of a prime}\}$ and apply Theorem 3. ■

Corollary 7 (Problem of Sárközy, see [5, Problem 25]). *For every infinite set $\mathcal{A} \subseteq \mathbb{N}$, the sequence $|d(\mathcal{A}, n+1) - d(\mathcal{A}, n)|$ cannot be bounded.*

Proof. Apply Theorem 1 for the set $\mathcal{A} \cup \{1\}$. ■

Corollary 8. *For every infinite set $\mathcal{A} \subseteq \mathbb{N}$ and any $\omega > 0$ there are infinitely many n with*

$$d(\mathcal{A}, n) > \omega + \max(d(\mathcal{A}, n-1), d(\mathcal{A}, n+1)).$$

Proof. Apply Theorem 1 for the set $\mathcal{A} \cup \{1\}$. ■

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