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The Jacobi–Perron Algorithm and Pisot numbers

by

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The Jacobi–Perron Algorithm introduced by Jacobi [7] and O. Perron [9] is a generalization of the continued fraction algorithm. Applied to an n-uple of real numbers, it gives simultaneous approximations. In case of periodicity it yields a unit of a number field which commands the quality of simultaneous approximations.

We prove that for $n = 2$ this unit is a Pisot number (positive algebraic integer with each conjugate in $|z| < 1$) and that this is not necessarily the case for $n \geq 3$.

The problem of characterizing the periodicity of JPA (Jacobi–Perron Algorithm) is still open for $n \geq 2$. Many families of sets of n real numbers for which the JPA is periodic were found by L. Bernstein [2], E. Dubois & R. Paysant-Le Roux [5], C. Levesque & G. Rhin [8]. M. Bouhamza for $n = 3$ and $n = 4$ [3, 4], and then E. Dubois and R. Paysant-Le Roux for every n [6] proved that there exists, in any real number field of degree $n + 1$, an n-uple of real numbers with periodic JPA.

Recently B. Adam & G. Rhin [1] found a method yielding all pairs of real numbers with periodic JPA which produce a given unit in a real cubic field. In many examples they get no set with periodic JPA when the given unit is not a Pisot number. So they ask if this is always true. In this paper we give a positive answer in case $n = 2$ and we prove that this is not always true for $n \geq 3$.

I. The Jacobi–Perron Algorithm. The continued fraction algorithm, applied to an irrational real number α, yields a sequence $(\alpha_k)_{k\geq 0}$ of real numbers, a sequence $(a_k)_{k\geq 0}$ of integers and a sequence $(p_k/q_k)_{k\geq -2}$ of

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rational approximations of α by the well known formula

$$\text{(1)} \qquad \alpha_0 = \alpha, \quad \alpha_k = a_k + \frac{1}{\alpha_{k+1}} \quad \text{with } a_k = [\alpha_k] \quad (k \geq 0)$$

where $[x]$ denotes the integer part of x,

$$\text{(2)} \qquad \begin{array}{lll} p_{-2} = 0, & p_{-1} = 1, & p_k = a_k p_{k-1} + p_{k-2} \quad (k \geq 0), \\ q_{-2} = 1, & q_{-1} = 0, & q_k = a_k a_{k-1} + q_{k-2} \quad (k \geq 0). \end{array}$$

When $\alpha_{k_0+l} = \alpha_{k_0}$, the continued fraction is periodic and the product $\varrho = \alpha_{k_0}\alpha_{k_0+1}\ldots\alpha_{k_0+l-1}$ is a unit in the quadratic field $\mathbb{Q}(\alpha)$. Moreover this unit is clearly a Pisot number.

The JPA applied to an n-uple $(\alpha_1, \ldots, \alpha_n)$ determines three sequences: $(\alpha_1^{(\nu)}, \ldots, \alpha_n^{(\nu)})_{\nu \geq 0}$ of real n-uples, $(a_1^{(\nu)}, \ldots, a_n^{(\nu)})_{\nu \geq 0}$ of integer n-uples and $(A_1^{(\nu)}/A_0^{(\nu)}, \ldots, A_n^{(\nu)}/A_0^{(\nu)})_{\nu \geq 0}$ of simultaneous approximations of $(\alpha_1, \ldots, \alpha_n)$ by the formulae

$$\text{(3)} \qquad \begin{gathered} \alpha_i^{(0)} = \alpha_i \quad (1 \leq i \leq n); \\ \alpha_1^{(\nu)} = a_1^{(\nu)} + \frac{1}{\alpha_n^{(n+1)}}, \quad \alpha_i^{(\nu)} = a_i^{(\nu)} + \frac{\alpha_{i-1}^{(\nu+1)}}{\alpha_n^{(\nu+1)}} \quad (2 \leq i \leq n) \end{gathered}$$

and

$$\text{(4)} \qquad \begin{gathered} A_i^{(j)} = \delta_{ij} \quad (0 \leq i, j \leq n), \\ A_i^{(\nu+n+1)} = A_i^{(\nu)} + a_1^{(\nu)} A_i^{(\nu+1)} + \ldots + a_n^{(\nu)} A_i^{(\nu+n)} \quad (0 \leq i \leq n,\ \nu \geq 0). \end{gathered}$$

We say that the JPA is *periodic* when the sequence $(\alpha_1^{(\nu)}, \ldots, \alpha_n^{(\nu)})_{\nu \geq k_0}$ is periodic, or equivalently when the sequence $(a_1^{(\nu)}, \ldots, a_n^{(\nu)})_{\nu \geq k_0}$ is periodic.

If $\alpha_i^{(0)} = \alpha_i^{(l)}$ $(1 \leq i \leq n)$ we say that the JPA is *purely periodic*. We assume that we are in this particular case. In the general case, it is easy to imagine the formula.

The matrix

$$\text{(5)} \qquad M = \begin{pmatrix} A_0^{(l)} & A_0^{(l+1)} & \ldots & A_0^{(l+n)} \\ \cdots & \cdots & \cdots & \cdots \\ A_n^{(l)} & A_n^{(l+1)} & \ldots & A_n^{(l+n)} \end{pmatrix}$$

characterizes the development and contains much information.

The sequence $(a_1^{(\nu)}, \ldots, a_n^{(\nu)})_{\nu \geq 0}$ of nonnegative integer n-uples is a *JPA development* if and only if

$$\text{(6)} \qquad (a_n^{(\nu)}, a_{n-1}^{(\nu+1)}, \ldots, a_{n-i}^{(\nu+i)}) \geq (a_i^{(\nu)}, a_{i-1}^{(\nu+1)}, \ldots, a_1^{(\nu+i-1)}, 1) \qquad (0 \leq i \leq n-1, \nu \geq 1),$$

where $\geq$ denotes the lexicographical order.

For every JPA development we have

$$\det(A_i^{(\nu+j)})_{0\le i,j\le n} = (-1)^{n\nu} \quad (\nu \ge 0), \tag{7}$$

$$\alpha_i = \frac{A_i^{(\nu)} + \alpha_1^{(\nu)} A_i^{(\nu+1)} + \ldots + \alpha_n^{(\nu)} A_i^{(\nu+n)}}{A_0^{(\nu)} + \alpha_1^{(\nu)} A_0^{(\nu+1)} + \ldots + \alpha_n^{(\nu)} A_0^{(\nu+n)}} \quad (1 \le i \le n, \nu \ge 0), \tag{8}$$

$$\lim_{\nu\to\infty} \frac{A_i^{(\nu)}}{A_0^{(\nu)}} = \alpha_i \quad (1 \le i \le n), \tag{9}$$

$$\alpha_n^{(1)}\alpha_n^{(2)}\ldots\alpha_n^{(\nu)} = A_0^{(\nu)} + \alpha_1^{(\nu)} A_0^{(\nu+1)} + \ldots + \alpha_n^{(\nu)} A_0^{(\nu+n)} \quad (\nu \ge 1). \tag{10}$$

In case of purely periodic JPA with minimal length l, $\varrho_0 = \alpha_n^{(1)}\alpha_n^{(2)}\ldots\alpha_n^{(l)}$ is an eigenvalue of M and $(1, \alpha_1^{(0)}, \ldots, \alpha_n^{(0)})$ is an eigenvector associated to ϱ_0. The real value ϱ_0 is the maximal positive real root of the characteristic polynomial

$$f(X) = \det(M - XI) \tag{11}$$

and ϱ_0 is a simple root of f. But f is not always irreducible.

Starting with a periodic sequence of nonnegative integer n-uples $(a_1^{(\nu)}, \ldots$ $\ldots, a_n^{(\nu)})$ satisfying (6) we get a matrix M and an n-uple $(\alpha_1, \ldots, \alpha_n)$ which have $(a_1^{(\nu)}, \ldots, a_n^{(\nu)})$ as JPA development.

We will use the following result:

THEOREM 1 ([9]). *Let $\varrho_0, \varrho_1, \ldots, \varrho_n$ be the roots of the characteristic polynomial of a periodic JPA, with minimal length l, $\varrho_0 \in \mathbb{R}$ and $\varrho_0 > |\varrho_1| = \max\{|\varrho_i| : 1 \le i \le n\}$. Then*

$\forall \varepsilon > 0, \exists c > 0, \exists \nu_0,\ \forall \nu \ge \nu_0,$

$$\left|\frac{A_i^{(\nu l+\lambda)}}{A_0^{(\nu l+\lambda)}} - \alpha_i\right| < c\left|\frac{\varrho_1(1+\varepsilon)}{\varrho_0}\right|^\nu \quad (0 \le \lambda \le l-1,\ 1 \le i \le n).$$

But there exist (i, λ) and $c' > 0$ such that

$$\left|\frac{A_i^{(\nu l+\lambda)}}{A_0^{(\nu l+\lambda)}} - \alpha_i\right| > c'\left|\frac{\varrho_1}{\varrho_0}\right|^\nu \quad \text{for infinitely many } \nu.$$

This theorem shows that the quality of these simultaneous approximations is given by ϱ_1. So it is important to know if ϱ_0 is a Pisot number or not.

We will also use the following theorem:

THEOREM 2 ([5]). *Let be an n-uple of real numbers $(\alpha_1, \ldots, \alpha_n)$ with periodic JPA. Then $\lim_{\nu\to\infty}(A_i^{(\nu)} - \alpha_i A_0^{(\nu)}) = 0$ if and only if the characteristic polynomial is irreducible with a Pisot number as root. We have*

$\mathbb{Q}(\varrho_0) = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ and the degree of $\mathbb{Q}(\varrho)$ is $n+1$ if and only if $1, \alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent.

II. Main result in the case of two real numbers

THEOREM 3. *For any JPA the sequences $(A_i^\nu - \alpha_i A_0^\nu)_{\nu \geq 0}$ are bounded for $i = 1, 2$. For a periodic JPA of two real numbers (α_1, α_2), we have*

$$\lim_{\nu \to \infty} (A_i^{(\nu)} - \alpha_i A_0^{(\nu)}) = 0 \quad (i = 1, 2).$$

The characteristic polynomial is irreducible and its maximal real root is a Pisot number.

Proof. We consider

$$V_\nu = (A_1^{(\nu)} - \alpha_1 A_0^{(\nu)},\, A_2^{(\nu)} - \alpha_2 A_0^{(\nu)}) \quad (\nu \geq 0).$$

From (8) we have $V_\nu + \alpha_1^{(\nu)} V_{\nu+1} + \alpha_2^{(\nu)} V_{\nu+2} = 0$ and from (4) we have $V_{\nu+3} = V_\nu + a_1^{(\nu)} V_{\nu+1} + a_2^{(\nu)} V_{\nu+2}$. So we get

$$V_{\nu+3} = b_\nu V_\nu + c_\nu V_{\nu+1} \tag{12}$$

with

$$b_\nu = \frac{\alpha_2^{(\nu)} - a_2^{(\nu)}}{\alpha_2^{(\nu)}}, \qquad c_\nu = \frac{a_1^{(\nu)} \alpha_2^{(\nu)} - a_2^{(\nu)} \alpha_1^{(\nu)}}{\alpha_2^{(\nu)}}.$$

We shall prove that $|b_\nu| + |c_\nu| < 1$. Using (3) we have

$$b_\nu + c_\nu = 1 - \frac{1}{\alpha_2^{(\nu)}} \left\{ a_2^{(\nu)} - a_1^{(\nu)} \left(a_2^{(\nu)} + \frac{\alpha_1^{(\nu+1)}}{\alpha_2^{(\nu+1)}} \right) + a_2^{(\nu)} \left(a_1^{(\nu)} + \frac{1}{\alpha_2^{(\nu+1)}} \right) \right\},$$

$$b_\nu + c_\nu = 1 - \frac{a_2^{(\nu)}}{\alpha_2^{(\nu)}} \left\{ 1 - \frac{a_1^{(\nu)} \alpha_1^{(\nu+1)}}{a_2^{(\nu)} \alpha_2^{(\nu+1)}} + \frac{1}{\alpha_2^{(\nu+1)}} \right\}.$$

Then $-1 < b_\nu + c_\nu < 1$ because the expression in brackets is clearly between 0 and 2.

Similarly, we have

$$b_\nu - c_\nu = 1 - \frac{1}{\alpha_2^{(\nu)}} \left\{ a_2^{(\nu)} + a_1^{(\nu)} \left(a_2^{(\nu)} + \frac{\alpha_1^{(\nu+1)}}{\alpha_2^{(\nu+1)}} \right) - a_2^{(\nu)} \left(a_1^{(\nu)} + \frac{1}{\alpha_2^{(\nu+1)}} \right) \right\},$$

$$b_\nu - c_\nu = 1 - \frac{a_2^{(\nu)}}{\alpha_2^{(\nu)}} \left\{ 1 - \frac{1}{\alpha_2^{(\nu+1)}} + \frac{a_1^{(\nu)} \alpha_1^{(\nu+1)}}{a_2^{(\nu)} \alpha_2^{(\nu+1)}} \right\}.$$

Then $-1 < b_\nu - c_\nu < 1$ because the expression in brackets is clearly between 0 and 2.

Now, since $b_\nu > 0$ we deduce that $|b_\nu| + |c_\nu| < 1$. From (12) and $|b_\nu| + |c_\nu| < 1$ we see that the components $A_i^{(\nu)} - \alpha_i A_0^{(\nu)}$ of V_ν are bounded by $\max(|V_0|, |V_1|, |V_2|)$ for any ν and any JPA development.

Assume now that the JPA development is periodic, with $\alpha_i^{(\nu_0)} = \alpha_i^{(\nu_0+l)}$. Consider

$$m = \max\{|b_\nu| + |c_\nu| : \nu_0 \le \nu < \nu_0 + l\}.$$

Since $m < 1$, it is easy to get $\lim_{\nu\to\infty}(A_i^{(\nu)} - \alpha_i A_0^{(\nu)}) = 0$ for $1 \le i \le 2$. From Theorem 2, ϱ_0 is a Pisot number of degree 3 and f is irreducible. ■

COROLLARY. *If the JPA development of* (α_1,α_2) *is periodic, then* $1,\alpha_1,\alpha_2$ *are* $\mathbb{Q}$*-linearly independent and form a basis of a cubic number field.*

Proof. We can assume that the JPA is purely periodic with length l. Since $(1,\alpha_1,\alpha_2)$ is an eigenvector of M associated to ϱ_0 we have $\mathbb{Q}(\alpha_1,\alpha_2) \subseteq \mathbb{Q}(\varrho_0)$. From $\varrho_0 = \alpha_2^{(1)} \ldots \alpha_2^{(l)}$ and from (3) we have $\mathbb{Q}(\varrho_0) \subseteq \mathbb{Q}(\alpha_1,\alpha_2)$ and therefore $\mathbb{Q}(\alpha_1,\alpha_2) = \mathbb{Q}(\varrho_0)$. From Theorem 3, $\mathbb{Q}(\varrho_0)$ is a cubic number field and from Theorem 2, $1,\alpha_1,\alpha_2$ are $\mathbb{Q}$-linearly independent. ■

III.a. *The case* $n = 3$. In this case, we consider the special cases of purely periodic JPA with small length.

Consider first a purely periodic JPA with length $l = 1$:

$$(a_1^{(\nu)}, a_2^{(\nu)}, a_3^{(\nu)}) = (a_1,a_2,a_3) \quad (\nu \ge 0) \tag{13}$$

with $(a_3,a_2,a_1) \ge (a_2,a_1,1)$ and $(a_3,a_2) \ge (a_1,1)$, where $\ge$ denotes the lexicographical order.

PROPOSITION 1. *The characteristic polynomial of a JPA with period* 1 *defined by* (13) *is reducible if and only if* $a_2 = a_3$ *and* $a_1 = 0$. *In this case* ϱ_0 *is a Pisot number of degree* 3 *but the* $A_i^{(\nu)} - \alpha_i A_0^{(\nu)}$ *do not converge to* 0 *and there exists a* $\mathbb{Q}$*-linear relation between* $1,\alpha_1,\alpha_2,\alpha_3$. *In the other case* ($a_3 > a_2$ *or* $a_1 \ne 0$), ϱ_0 *is a Pisot number of degree* 4.

Proof. Each step of the proof is elementary.

The characteristic polynomial is $f(x) = x^4 - a_3x^3 - a_2x^2 - a_1x - 1$ and the different steps are the following:

If $a_2 = a_3$ and $a_1 = 0$, then $f(-1) = 0$ and the root ϱ_0 of $f(x)/(x+1)$ is a Pisot number. If f is reducible we show that f is not the product of two factors of degree two and that $f(1) < 0$. Then $f(-1) = 0$ gives $a_2 = a_3$ and $a_1 = 0$. ■

Consider now a purely periodic JPA with length $l = 2$,

$$\begin{gathered}(a_1^{(2\nu)}, a_2^{(2\nu)}, a_3^{(2\nu)}) = (b_1,b_2,b_3),\\ (a_1^{(2\nu+1)}, a_2^{(2\nu+1)}, a_3^{(2\nu+1)}) = (c_1,c_2,c_3)\end{gathered} \tag{14}$$

with

$$\begin{gathered}(b_3,c_2,b_1) \ge (b_2,c_1,1); \quad (b_3,c_2) \ge (b_1,1);\\ (c_3,b_2,c_1) \ge (c_2,b_1,1); \quad (c_3,b_2) \ge (c_1,1),\end{gathered} \tag{15}$$

where $\ge$ denotes the lexicographical order.

After some computing, we prove that the characteristic polynomial is

$$f(x) = x^4 - (b_3c_3 + b_2 + c_2)x^3 - (b_3c_1 + b_1c_3 + 2 - b_2c_2)x^2 - (b_1c_1 - b_2 - c_2)x + 1. \tag{16}$$

Using (15), we can see that f is not a product of two factors of degree 2.

$f(-1) = 0$ if and only if

$$(c_3 = c_1 \text{ and } c_2 = 0) \quad \text{or} \quad (b_3 = b_1 \text{ and } b_2 = 0), \tag{17}$$

$f(1) = 0$ if and only if

$$b_3 = b_2, \quad c_3 = c_2 \quad \text{and} \quad b_1 = c_1 = 0. \tag{18}$$

In these two cases, it is an easy exercise to prove that ϱ_0 is a Pisot number and with Theorem 1 we get:

PROPOSITION 2. *The characteristic polynomial of a JPA defined by* (14) *and* (15) *is reducible if and only if* (17) *or* (18) *is true. In these cases* ϱ_0 *is a Pisot number of degree* 3 *but the* $A_i^{(\nu)} - \alpha_i A_0^{(\nu)}$ *do not converge to* 0 *and there exists a* $\mathbb{Q}$*-linear relation between* $1, \alpha_1, \alpha_2, \alpha_3$. *The conjugates of* ϱ_0 *are real if* (17) *holds and complex if* (18) *holds.*

For periodic JPA with length 2 when (17) and (18) are not satisfied, the characteristic polynomial, f, is irreducible. From $f(1) < 0$, $f(1/\varrho_0) > 0$, there is at least one root of f, say ϱ_1, which satisfies $0 < 1/\varrho_0 < \varrho_1 < 1$ and then the other roots ϱ_2, ϱ_3 satisfy $|\varrho_2\varrho_3| = 1/(\varrho_0\varrho_1) < 1$. So when ϱ_2, ϱ_3 are complex conjugates, $\max(|\varrho_1|, |\varrho_2|, |\varrho_3|) < 1$ and ϱ_0 is a Pisot number. In the other case we must locate ϱ_2, ϱ_3. To do this we consider the roots $\beta_1, \beta_2, \beta_3$ of f' with $\beta_1 \le \beta_2 \le \beta_3$. From a discussion of the signs of $f'(0)$ and $f(\beta_1)$ it is easy to show that in any case ϱ_2, ϱ_3 belong to $]-1, 1[$ and ϱ_0 is a Pisot number. So we have

PROPOSITION 3. *If the characteristic polynomial of JPA defined by* (14) *and* (15) *is irreducible, then its root* ϱ_0 *is a Pisot number.*

For a periodic JPA with length greater than 3, we can consider numerical examples. For $l = 3$, we consider the JPA with pure period $(1, b, b+1)$; $(b, 1, b)$; (b, b, b). The characteristic polynomial

$$f(x) = x^4 - (b^3 + 3b^2 + 4b + 1)x^3 + (2b^3 - 2b^2 - b - 1)x^2 + (b^2 - 3b)x - 1$$

is irreducible when $b \ge 6$ and has two real roots greater than 1. So the root ϱ_0 is not a Pisot number.

III.b. *The case* $n \ge 4$. For $n = 4$, we consider the purely periodic JPA with length one (a_1, a_2, a_3, a_4) with the lexicographical conditions:

$$(a_4, a_3, a_2, a_1) \ge (a_3, a_2, a_1, 1); \quad (a_4, a_3, a_2) \ge (a_2, a_1, 1); \quad (a_4, a_3) \ge (a_1, 1). \tag{19}$$

We have $f(x) = x^5 - a_4x^4 - a_3x^3 - a_2x^2 - a_1x - 1$.

Since $f(-1) > 0$, that is, $a_1 + a_3 \geq a_2 + a_4 + 3$, f has a root less than -1 and hence ϱ_0, which is greater than 1, is not a Pisot number of degree 5. For example $a_3 = a_4 \geq a_1 \geq a_2 + 3$ give many possibilities.

For every even n we can find examples with $f(-1) > 0$ and the same conclusion.

For every odd n greater than 5 we can consider a purely periodic JPA with length one. The condition $f(-1) < 0$ gives a root less than -1 and a ϱ_0 which is not Pisot of degree $n+1$. For example if $n = 5$ and $l = 1$, $f(x) = x^6 - a_5x^5 - a_4x^4 - a_3x^3 - a_2x^2 - a_1x - 1$ has a root less than -1 if $a_5 + a_3 + a_1 < a_2 + a_4$. This is compatible with the lexicographical condition (6). For example $(1, a, 1, a, a)$ with $a \geq 3$ is suitable.

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