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On the $2k$th power mean of the character sums over short intervals

by

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1. Introduction. Let $q \geq 3$ be an integer, and χ be a Dirichlet character modulo q. Over the several past decades, many authors have investigated various arithmetical properties of the character sums

$$\sum_{a=N+1}^{N+H} \chi(a).$$

Pólya [6] and Vinogradov [8] studied the character sums when the modulus q is equal to a prime p and obtained the inequality

$$\Big|\sum_{a=1}^{x} \chi(a)\Big| \leq c\sqrt{p}\,\ln p,$$

where c is a constant. Actually, one can establish the above inequality with the constant $c=1$. If χ is a primitive character modulo q, A. V. Sokolovskiĭ [7] proved the existence of x with

$$\Big|\sum_{n=x}^{x+[q/2]} \chi(n)\Big| > \sqrt{1 - \frac{8\ln q}{q}} \cdot \frac{1}{2\sqrt{2}} \cdot \sqrt{q},$$

where $[y]$ denotes the greatest integer less than or equal to y. For a general nonprincipal character, D. A. Burgess [2] obtained the mean value estimate of character sums

$$\sum_{n=1}^{k} \Big|\sum_{m=1}^{h} \chi(n+m)\Big|^2 < kh,$$

where h is any positive integer. This was conjectured by Norton [5], who

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obtained the weaker upper bound $\frac{9}{8}kh$. For higher moments and $q = p$, Burgess [3] summed the fourth power mean over all nonprincipal characters and proved that

$$\sum_{\chi\neq\chi_0}\sum_{n=1}^{p}\Big|\sum_{m=1}^{h}\chi(n+m)\Big|^4 \le 6p^2h^2.$$

For general moduli q, he summed the mean value over all primitive characters and obtained (see [4])

$$\sideset{}{^*}\sum_{\chi \bmod q}\sum_{n=1}^{p}\Big|\sum_{m=1}^{h}\chi(n+m)\Big|^4 \le 8\tau^7(q)q^2h^2,$$

where $\sum^*_{\chi \bmod q}$ denotes summation over all primitive characters modulo q and $\tau(n)$ is the Dirichlet divisor function.

The present work deals mainly with the $2k$th power mean of the character sums over the interval $[1, q/4)$. First we transform the character sums to L-functions by an elementary method. Then using the mean value theorems of Dirichlet L-functions, we study the mean value properties of the character sums over short intervals, and obtain a sharper asymptotic formula for it. Namely, we shall prove the following:

THEOREM. *Let $q \ge 5$ be an odd integer. Then we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\chi(-1)=1}\Big|\sum_{a<q/4}\chi(a)\Big|^{2k} = \frac{J(q)q^k}{16}\left(\frac{\pi}{8}\right)^{2k-2}\prod_{p\mid q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)+O(q^{k+\varepsilon}),$$

where $\sum^*_{\chi(-1)=1}$ denotes summation over all primitive characters modulo q such that $\chi(-1) = 1$, ε is any fixed positive number, $J(q)$ denotes the number of primitive characters modulo q and $\prod_{p\mid q}$ denotes the product over all prime divisors p of q, and $C_m^n = m!/n!(m-n)!$.

Noting that $J(q) = \phi^2(q)/q$ if q is a square-full number, we have the following

COROLLARY 1. *Let $q \ge 5$ be a square-full number with $2\nmid q$. Then we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\chi(-1)=1}\Big|\sum_{a<q/4}\chi(a)\Big|^{2k} = \frac{q^{k-1}\phi^2(q)}{16}\left(\frac{\pi}{8}\right)^{2k-2}\prod_{p\mid q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)+O(q^{k+\varepsilon}).$$

Taking $k = 2$ in our Theorem and noting that

$$\prod_{p|q}\left(1-\frac{1}{p^2}\right)^3\prod_{p\nmid 2q}\left(1+\frac{1}{p^2}\right) = \prod_{p}\left(1+\frac{1}{p^2}\right)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^3\prod_{p|2q}\frac{1}{1+1/p^2}$$
$$= \frac{4}{5}\,\frac{\zeta(2)}{\zeta(4)}\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)},$$

$\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, we immediately get

COROLLARY 2. *Let $q \ge 5$ be an odd integer. Then we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\chi(-1)=1}\Big|\sum_{a<q/4}\chi(a)\Big|^4 = \frac{3}{256}\,J(q)q^2\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^{2+\varepsilon}).$$

If $q = p$ is a prime, then $J(p) = p - 2$. So from Corollary 2, we have

COROLLARY 3. *Let $p \ge 5$ be a prime. Then we have the asymptotic formula*

$$\sum_{\chi(-1)=1}\Big|\sum_{a<p/4}\chi(a)\Big|^4 = \frac{3}{256}\,p^3 + O(p^{2+\varepsilon}),$$

where $\sum_{\chi(-1)=1}$ denotes summation over all nonprincipal characters modulo p such that $\chi(-1) = 1$.

2. Some lemmas. To prove the Theorem, we need the following lemmas.

LEMMA 1. *Let χ be a primitive character modulo m with $\chi(-1) = -1$. Then*

$$\frac{1}{m}\sum_{b=1}^{m} b\chi(b) = \frac{i}{\pi}\,\tau(\chi)L(1,\overline{\chi}),$$

where $\tau(\chi) = \sum_{a=1}^{m}\chi(a)e(a/q)$ is the Gauss sum, $e(y) = e^{2\pi iy}$, and $L(s,\chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof. This can be easily deduced from Theorems 12.11 and 12.20 of [1].

LEMMA 2. *Let $q \ge 5$ be an odd integer and χ be a primitive Dirichlet character modulo q such that $\chi(-1) = 1$. Then*

$$\sum_{a=1}^{[q/4]}\chi(a) = -\frac{i\overline{\chi}(4)}{2\pi}\,\tau(\chi\chi_4)L(1,\overline{\chi}\chi_4),$$

where χ_4 is the primitive Dirichlet character modulo 4.

Proof. First, we suppose $q \equiv 1 \pmod 4$. Since χ_4 is the primitive Dirichlet character modulo 4, we have $\chi_4(1) = \chi_4(-3) = 1$ and $\chi_4(3) = \chi_4(-1) = -1$. Then the following identity is obvious:

$$(1)\qquad \sum_{a=1}^{4q} a\chi(a)\chi_4(a) = \sum_{a=0}^{q-1}(4a+1)\chi(4a+1) - \sum_{a=0}^{q-1}(4a+3)\chi(4a+3).$$

Noting that $\sum_{a=0}^{q-1}\chi(a) = 0$, we can write

$$(2)\qquad \sum_{a=0}^{q-1}(4a+1)\chi(4a+1)$$
$$= 4\chi(4)\sum_{a=0}^{q-1} a\chi(a+\overline{4}) = 4\chi(4)\sum_{a=0}^{q-1}(a+\overline{4})\chi(a+\overline{4})$$
$$= 4\chi(4)\sum_{a=0}^{(q-1)/4}(a+\overline{4})\chi(a+\overline{4}) + 4\chi(4)\sum_{a=(q-1)/4+1}^{q-1}(a+\overline{4})\chi(a+\overline{4}),$$

where $4 \cdot \overline{4} \equiv 1 \pmod q$. We know that $(3q+1)/4$ is an integer and $\overline{4} = (3q+1)/4$, so

$$\begin{aligned} 0 \le a+\overline{4} \le q \qquad & \text{if } a \le \frac{q-1}{4}, \\ q < a+\overline{4} \le 2q-1 \qquad & \text{if } a > \frac{q-1}{4}. \end{aligned}$$

Hence, we have

$$(3)\qquad 4\chi(4)\sum_{a=0}^{(q-1)/4}(a+\overline{4})\chi(a+\overline{4}) + 4\chi(4)\sum_{a=(q-1)/4+1}^{q-1}(a+\overline{4})\chi(a+\overline{4})$$
$$= 4\chi(4)\sum_{a=0}^{(q-1)/4}(a+\overline{4})\chi(a+\overline{4}) + 4\chi(4)\sum_{a=(q-1)/4+1}^{q-1}(a+\overline{4}-q)\chi(a+\overline{4}-q)$$
$$+ 4\chi(4)\sum_{a=(q-1)/4+1}^{q-1} q\chi(a+\overline{4})$$
$$= 4\chi(4)\sum_{a=1}^{q-1} a\chi(a) + 4\chi(4)q\sum_{a=(q-1)/4+1}^{q-1}\chi(a+\overline{4}).$$

Noting that $\chi(-1) = 1$, we can get (see Theorem 12.20 of [1])

$$\sum_{a=1}^{q-1} a\chi(a) = 0.$$

Now combining (2) and (3), we have

$$\sum_{a=0}^{q-1}(4a+1)\chi(4a+1) = 4\chi(4)q \sum_{a=(q-1)/4+1}^{q-1} \chi(a+\overline{4}) \tag{4}$$

$$= -4\chi(4)q \sum_{a=0}^{(q-1)/4} \chi\left(a+\frac{3q+1}{4}\right) = -4\chi(4)q \sum_{a=0}^{(q-1)/4} \chi\left(a-\frac{q-1}{4}\right)$$

$$= -4\chi(4)q \sum_{a=0}^{(q-1)/4} \chi\left(\frac{q-1}{4}-a\right) = -4\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a).$$

By using the same method, we can also get

$$\sum_{a=0}^{q-1}(4a+3)\chi(4a+3) = 4\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a). \tag{5}$$

From (1), (4) and (5), we have

$$\sum_{a=1}^{4q} a\chi(a)\chi_4(a) = -8\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a). \tag{6}$$

Since $\chi(a)$ is a primitive character modulo q and χ_4 is a primitive character modulo 4 and $(q,2)=1$, it follows that $\chi\chi_4$ is also a primitive character modulo $4q$. Noting that

$$\chi\chi_4(-1) = \chi(-1)\chi_4(-1) = -1,$$

combining (6) and Lemma 1, we can easily get

$$\sum_{a=1}^{(q-1)/4} \chi(a) = -\frac{i\overline{\chi}(4)}{2\pi}\,\tau(\chi\chi_4)L(1,\overline{\chi}\chi_4). \tag{7}$$

For the case of $q \equiv 3 \pmod 4$, by the same argument we can get

$$\sum_{a=1}^{(q-3)/4} \chi(a) = -\frac{i\overline{\chi}(4)}{2\pi}\,\tau(\chi\chi_4)L(1,\overline{\chi}\chi_4). \tag{8}$$

Combining (7) and (8), we have

$$\sum_{a=1}^{[q/4]} \chi(a) = -\frac{i\overline{\chi}(4)}{2\pi}\,\tau(\chi\chi_4)L(1,\overline{\chi}\chi_4).$$

This proves Lemma 2.

Lemma 3 ([9, Lemma 4]). *Let q and r be integers with $q \geq 2$ and $(r,q)=1$, and χ be a Dirichlet character modulo q. Then we have the identities*

$$\sideset{}{^*}\sum_{\chi \bmod q} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^*_{\chi \bmod q}$ denotes summation over all primitive characters modulo q, and $J(q)$ denotes the number of primitive characters modulo q.

Lemma 4 ([10, Lemma 3]). *Let $q > 2$ be an integer and let $\tau_k(n)$ denote the kth divisor function (i.e., the number of solutions of the equation $n_1 \cdots n_k = n$ in positive integers $n_1, \ldots, n_k$). Then we have the identity*

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\tau_k^2(n)}{n^2} = \zeta^{2k-1}(2) \prod_{p|q}\left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p\nmid q}\left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right),$$

where $\zeta(s)$ is the Riemann zeta function.

Lemma 5. *Let $q > 2$ be an odd integer, χ be a Dirichlet character modulo q and χ_4 be the primitive character modulo 4. Then we have the asymptotic formula*

$$\sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\overline{\chi}\chi_4)|^{2k}$$
$$= \frac{J(q)}{2}\left(\frac{\pi^2}{8}\right)^{2k-1} \prod_{p|q}\left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p\nmid 2q}\left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O(q^{\varepsilon}),$$

where $\sum^*_{\chi(-1)=1}$ denotes summation over all even primitive characters modulo q.

Proof. For convenience, we put

$$A(y,\chi) = \sum_{N<n\leq y} \chi(n)\tau_k(n),$$

where N is a parameter with $q \leq N < q^{2^k+1}$. Then from Abel's identity we have

$$L^k(1,\overline{\chi}\chi_4) = \sum_{n=1}^{\infty} \frac{\overline{\chi}\chi_4(n)\tau_k(n)}{n} = \sum_{1\leq n\leq N} \frac{\overline{\chi}\chi_4(n)\tau_k(n)}{n} + \int_N^{\infty} \frac{A(y,\overline{\chi}\chi_4)}{y^2}\,dy.$$

Hence, we can write

$$
\begin{aligned}
(9)\qquad & \sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\overline{\chi}\chi_4)|^{2k} \\
&= \sideset{}{^*}\sum_{\chi(-1)=1} \biggl(\sum_{1\le n_1\le N} \frac{\overline{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1} + \int_N^\infty \frac{A(y,\overline{\chi}\chi_4)}{y^2}\,dy \biggr) \\
&\quad \times \biggl(\sum_{1\le n_2\le N} \frac{\chi\chi_4(n_2)\tau_k(n_2)}{n_2} + \int_N^\infty \frac{A(y,\chi\chi_4)}{y^2}\,dy \biggr) \\
&= \sideset{}{^*}\sum_{\chi(-1)=1} \biggl(\sum_{1\le n_1\le N} \frac{\overline{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1} \biggr)\biggl(\sum_{1\le n_2\le N} \frac{\chi\chi_4(n_2)\tau_k(n_2)}{n_2} \biggr) \\
&\quad + \sideset{}{^*}\sum_{\chi(-1)=1} \biggl(\sum_{1\le n_1\le N} \frac{\overline{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1} \biggr)\biggl(\int_N^\infty \frac{A(y,\chi\chi_4)}{y^2}\,dy \biggr) \\
&\quad + \sideset{}{^*}\sum_{\chi(-1)=1} \biggl(\sum_{1\le n_2\le N} \frac{\chi\chi_4(n_2)\tau_k(n_2)}{n_2} \biggr)\biggl(\int_N^\infty \frac{A(y,\overline{\chi}\chi_4)}{y^2}\,dy \biggr) \\
&\quad + \sideset{}{^*}\sum_{\chi(-1)=1} \biggl(\int_N^\infty \frac{A(y,\overline{\chi}\chi_4)}{y^2}\,dy \biggr)\biggl(\int_N^\infty \frac{A(y,\chi\chi_4)}{y^2}\,dy \biggr) \\
&= M_1 + M_2 + M_3 + M_4.
\end{aligned}
$$

Now we shall calculate each term in the expression (9).

(i) From Lemma 3 we have

$$
\begin{aligned}
(10)\qquad \sideset{}{^*}\sum_{\chi(-1)=1} \chi\chi_4(a) &= \sideset{}{^*}\sum_{\chi\chi_4(-1)=-1} \chi\chi_4(a) \\
&= \frac{1}{2} \sideset{}{^*}\sum_{\chi\chi_4 \bmod 4q} (1-\chi\chi_4(-1))\chi\chi_4(a) \\
&= \frac{1}{2} \sideset{}{^*}\sum_{\chi\chi_4 \bmod 4q} \chi\chi_4(a) - \frac{1}{2} \sideset{}{^*}\sum_{\chi\chi_4 \bmod 4q} \chi\chi_4(-a) \\
&= \frac{1}{2} \sum_{d\mid(4q,a-1)} \mu\Bigl(\frac{4q}{d}\Bigr)\phi(d) - \frac{1}{2} \sum_{d\mid(4q,a+1)} \mu\Bigl(\frac{4q}{d}\Bigr)\phi(d).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(11)\qquad M_1 &= \sideset{}{^*}\sum_{\chi(-1)=1} \biggl(\sum_{1\le n_1\le N} \frac{\overline{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1} \biggr)\biggl(\sum_{1\le n_2\le N} \frac{\chi\chi_4(n_2)\tau_k(n_2)}{n_2} \biggr) \\
&= \frac{1}{2} \sideset{}{'}\sum_{1\le n_1\le N} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \sum_{d\mid(4q,\overline{n}_1 n_2-1)} \mu\Bigl(\frac{4q}{d}\Bigr)\phi(d)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} \sideset{}{'}\sum_{1\le n_1\le N} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2} \sum_{d\mid(4q,\overline{n}_1n_2+1)} \mu\left(\frac{4q}{d}\right)\phi(d)\\
&= \frac{1}{2}\sum_{d\mid 4q}\mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N\\ n_2\equiv n_1 \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}\\
&\quad - \frac{1}{2}\sum_{d\mid 4q}\mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N\\ n_2\equiv -n_1 \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2},
\end{aligned}
$$

where $\sum'_{1\le n\le N}$ denotes summation over n from 1 to N such that $(n, 2q) = 1$.

For convenience, we split the sum over n_1 or n_2 into four cases: (i) $d \le n_1, n_2 \le N$; (ii) $d \le n_1 \le N$ and $1 \le n_2 \le d-1$; (iii) $1 \le n_1 \le d-1$ and $d \le n_2 \le N$; (iv) $1 \le n_1, n_2 \le d-1$. So we have

$$
\begin{aligned}
&\sum_{d\mid 4q}\mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{d\le n_1\le N\\ n_2\equiv n_1 \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{d\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}\\
&\quad \ll \sum_{d\mid 4q}\phi(d) \sum_{1\le r_1\le N/d}\ \sum_{1\le r_2\le N/d}\ \sideset{}{'}\sum_{\substack{l_1=1\\ l_2\equiv l_1 \,(\mathrm{mod}\, d)}}^{d-1} \sideset{}{'}\sum_{l_2=1}^{d-1} \frac{\tau_k(r_1d+l_1)\tau_k(r_2d+l_2)}{(r_1d+l_1)(r_2d+l_2)}\\
&\quad \ll \sum_{d\mid 4q}\phi(d) \sum_{1\le r_1\le N/d}\ \sum_{1\le r_2\le N/d}\ \sideset{}{'}\sum_{l_1=1}^{d-1} \frac{[(r_1d+l_1)(r_2d+l_1)]^{\varepsilon}}{(r_1d+l_1)(r_2d+l_1)}\\
&\quad \ll \sum_{d\mid 4q}\frac{\phi(d)}{d} \sum_{1\le r_1\le N/d}\ \sum_{1\le r_2\le N/d} \frac{[(r_1d+1)(r_2d+1)]^{\varepsilon}}{r_1r_2} \ll q^{\varepsilon},\\
&\sum_{d\mid 4q}\mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{d\le n_1\le N\\ n_2\equiv n_1 \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1\le n_2\le d-1} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}\\
&\qquad\qquad \ll \sum_{d\mid 4q}\phi(d) \sum_{1\le r_1\le N/d}\ \sum_{1\le n_2\le d-1} (r_1n_2d)^{\varepsilon-1} \ll q^{\varepsilon},\\
&\sum_{d\mid 4q}\mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le d-1\\ n_2\equiv n_1 \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{d\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}\\
&\qquad\qquad \ll \sum_{d\mid 4q}\phi(d) \sum_{1\le n_1\le d-1}\ \sum_{1\le r_2\le N/d} (n_1r_2d)^{\varepsilon-1} \ll q^{\varepsilon},
\end{aligned}
$$

where we have used the estimate $\tau_k(n) \ll n^{\varepsilon}$.

For the case $1 \le n_1, n_2 \le d-1$, the solution of the congruence $n_2 \equiv n_1 \pmod d$ is $n_2 = n_1$. Hence,

$$\begin{aligned}\sum_{d\mid 4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le d-1 \\ n_2\equiv n_1 \pmod d}} \sideset{}{'}\sum_{1\le n_2\le d-1} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \\ = \sum_{d\mid 4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{1\le n_2\le d-1} \frac{\tau_k^2(n_2)}{n_2^2} \\ = \sum_{d\mid 4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sum_{\substack{n_2=1 \\ (n_2,2q)=1}}^{\infty} \frac{\tau_k^2(n_2)}{n_2^2} + O(q^{\varepsilon}).\end{aligned}$$

Now from Lemma 4, we immediately get

$$\begin{aligned}&\frac{1}{2}\sum_{d\mid 4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ n_2\equiv n_1 \pmod d}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \\ &= \frac{J(4q)}{2}\,\zeta^{2k-1}(2)\prod_{p\mid 2q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)+O(q^{\varepsilon}).\end{aligned}$$

Since $(q,2)=1$,

$$\begin{aligned}J(4q) &= \sum_{d\mid 4q}\mu\left(\frac{4q}{d}\right)\phi(d) = \left(\sum_{d\mid q}\mu\left(\frac{q}{d}\right)\phi(d)\right)\left(\sum_{d\mid 4}\mu\left(\frac{4}{d}\right)\phi(d)\right) \\ &= \sum_{d\mid q}\mu\left(\frac{q}{d}\right)\phi(d) = J(q).\end{aligned}$$

So we have

$$\begin{aligned}&\frac{1}{2}\sum_{d\mid 4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ n_2\equiv n_1 \pmod d}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \\ &= \frac{J(q)}{2}\left(\frac{\pi^2}{8}\right)^{2k-1}\prod_{p\mid q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)+O(q^{\varepsilon}),\end{aligned} \tag{12}$$

where we used the identity $\zeta(2)=\pi^2/6$.

Similarly, we can also get the estimate

$$\begin{aligned}&\frac{1}{2}\sum_{d\mid 4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ n_2\equiv -n_1 \pmod d}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2} \\ &= \frac{1}{2}\sum_{d\mid 4q} \mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ n_2+n_1=d}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1 n_2}\end{aligned} \tag{13}$$

$$
\begin{aligned}
&+ \frac{1}{2}\sum_{d|4q}\mu\left(\frac{4q}{d}\right)\phi(d) \sideset{}{'}\sum_{1\le n_1\le N}\ \sideset{}{'}\sum_{\substack{1\le n_2\le N\\ n_2+n_1=ld,\, l\ge 2}} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}\\
\ll &\sum_{d|4q}\phi(d)\sum_{1\le n\le d-1}\frac{\tau_k(n)\tau_k(d-n)}{n(d-n)}\\
&+ \sum_{d|4q}\phi(d)\sideset{}{'}\sum_{1\le n_1\le N}\ \sum_{l=[n_1/d]+2}^{[(N+n_1)/d]}\frac{\tau_k(n_1)\tau_k(ld-n_1)}{ldn_1-n_1^2}\\
\ll &\sum_{d|4q}\frac{\phi(d)}{d}\sum_{1\le n\le d-1}\frac{\tau_k(n)\tau_k(d-n)}{n}\\
&+ \sum_{d|4q}\frac{\phi(d)}{d}\sideset{}{'}\sum_{1\le n_1\le N}\ \sum_{l=[n_1/d]+2}^{[(N+n_1)/d]}\frac{n_1^{\varepsilon}(ld-n_1)^{\varepsilon_1}}{ln_1-n_1^2/d}\\
\ll &\, q^{\varepsilon}+\sum_{d|4q}\frac{\phi(d)d^{\varepsilon_1}}{d}\sum_{n_1=1}^{N}\sum_{l=1}^{N}\frac{n_1^{\varepsilon}l^{\varepsilon_1}}{ln_1}\ll q^{\varepsilon}.
\end{aligned}
$$

Then from (11)–(13), we have

$$
M_1=\frac{J(q)}{2}\left(\frac{\pi^2}{8}\right)^{2k-1}\prod_{p|q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right)+O(q^{\varepsilon}). \tag{14}
$$

(ii) From Lemma 4 of [10], we have the estimate

$$
\sum_{\chi\ne\chi_0}|A(y,\chi)|^2\ll y^{2-4/2^k+\varepsilon}\phi^2(q),
$$

where χ_0 denotes the principal character modulo q. Then from the Cauchy inequality we can easily get

$$
\sum_{\chi(-1)=-1}|A(y,\chi)|\ll\sum_{\chi\ne\chi_0}|A(y,\chi)|\ll y^{1-2/2^k+\varepsilon}q^{3/2}.
$$

Using this estimate we have

$$
\begin{aligned}
M_2 &= \sideset{}{^*}\sum_{\chi(-1)=-1}\left(\sum_{1\le n_1\le N}\frac{\overline{\chi}\chi_4(n_1)\tau_k(n_1)}{n_1}\right)\left(\int_N^{\infty}\frac{A(y,\chi\chi_4)}{y^2}\,dy\right) \qquad (15)\\
&\ll \sum_{1\le n_1\le N} n_1^{\varepsilon-1}\int_N^{\infty}\frac{1}{y^2}\Big(\sum_{\chi\chi_4(-1)=-1}|A(y,\chi\chi_4)|\Big)\,dy\\
&\ll N^{\varepsilon}\int_N^{\infty}\frac{q^{3/2}y^{1-2/2^k+\varepsilon_1}}{y^2}\,dy\ll\frac{q^{3/2}}{N^{2/2^k-\varepsilon}}.
\end{aligned}
$$

(iii) Similar to (ii), we can also get

$$M_3 \ll \frac{q^{3/2}}{N^{2/2^k-\varepsilon}}. \tag{16}$$

(iv) By the same argument as in (ii), and noting the absolute convergence of the integrals, we can write

$$\begin{aligned} M_4 &= \sideset{}{^*}\sum_{\chi(-1)=-1} \Big(\int_N^\infty \frac{A(y,\overline{\chi}\chi_4)}{y^2}\,dy\Big)\Big(\int_N^\infty \frac{A(y,\chi\chi_4)}{y^2}\,dy\Big) \\ &\le \int_N^\infty\int_N^\infty \frac{1}{y^2z^2} \sideset{}{^*}\sum_{\chi(-1)=-1} |A(y,\overline{\chi}\chi_4)|\,|A(z,\chi\chi_4)|\,dy\,dz \\ &\ll \int_N^\infty \frac{1}{y^2}\int_N^\infty \frac{1}{z^2}\Big(\sum_{\chi\chi_4\neq\chi_0} |A(y,\overline{\chi}\chi_4)|^2\Big)^{1/2} \\ &\quad\times\Big(\sum_{\chi\chi_4\neq\chi_0} |A(z,\chi\chi_4)|^2\Big)^{1/2}\,dy\,dz \\ &\ll \Big(\int_N^\infty \frac{1}{y^2}\Big(\sum_{\chi\chi_4\neq\chi_0} |A(y,\chi\chi_4)|^2\Big)^{1/2}\,dy\Big)^2 \\ &\ll \Big(\int_N^\infty \frac{\phi(q)}{y^{1+2/2^k-\varepsilon}}\,dy\Big)^2 \ll \frac{\phi^2(q)}{N^{4/2^k-\varepsilon}}. \end{aligned} \tag{17}$$

Now taking $N=q^{2^k}$ and $\varepsilon<2/2^k$, combining (9) and (14)–(17) we obtain the asymptotic formula

$$\begin{aligned} &\sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\overline{\chi}\chi_4)|^{2k} \\ &\quad= \frac{J(q)}{2}\Big(\frac{\pi^2}{8}\Big)^{2k-1}\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)^{2k-1}\prod_{p\nmid 2q}\Big(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\Big)+O(q^\varepsilon). \end{aligned}$$

This proves Lemma 5.

3. Proof of the Theorem. In this section, we will complete the proof of the Theorem. Noting that $\chi\chi_4$ is a primitive character modulo $4q$ if χ is a primitive character modulo q, we get $|\tau(\chi\chi_4)|=2\sqrt{q}$. So from Lemma 2 and $\overline{\chi}(4)\neq 0$, we can write

$$\sideset{}{^*}\sum_{\chi(-1)=1}\Big|\sum_{a<q/4}\chi(a)\Big|^{2k} = \frac{q^k}{\pi^{2k}} \sideset{}{^*}\sum_{\chi(-1)=1} |L(1,\overline{\chi}\chi_4)|^{2k}. \tag{18}$$

From Lemma 5 and (18), we can easily get

$$\sum_{\chi(-1)=1}^{*} \Big| \sum_{a<q/4} \chi(a) \Big|^{2k} = \frac{J(q)q^k}{16} \left(\frac{\pi}{8}\right)^{2k-2} \prod_{p\mid q} \left(1-\frac{1}{p^2}\right)^{2k-1} \prod_{p\nmid 2q} \left(1-\frac{1-C_{2k-2}^{k-1}}{p^2}\right) + O(q^{k+\varepsilon}).$$

This completes the proof of the Theorem.

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