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# Bounds for the solutions of $S$-unit equations and decomposable form equations

by

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The main purpose of this paper is to considerably improve (in completely explicit form) the best known effective upper bounds for the solutions of $S$-unit equations and decomposable form equations.

**1. Introduction.** Several effective bounds have been established for the heights of the solutions of unit equations and, more generally, of $S$-unit equations in two unknowns; see e.g. [12], [13], [1], [5], [2], [17], [4], [21] and the references given there. Except in [1] and [2], their proofs rely on Baker's method and its $p$-adic analogue as well as certain quantitative results concerning fundamental/independent systems of units. In our Theorems 1 and 2 we improve upon the best known estimates for $S$-unit equations in terms of the parameters of $S$ and the ground field $K$. As a consequence of Theorem 2 we deduce a completely explicit result (cf. Corollary 2) in the direction of the $abc$ conjecture over number fields.

To prove our results we use, among other things, some recent improvements due to Matveev [25] and Yu [33] concerning linear forms in logarithms of algebraic numbers, a recent theorem of Loher and Masser [22] on multiplicatively independent algebraic numbers, and our improved estimates for fundamental/independent systems of $S$-units. In proving our Theorem 1 we follow the arguments of [5] with some refinements and utilize the improvements mentioned above.

In the bound in Theorem 1 there is a factor of the form $s^{2s}$, where $s$ denotes the cardinality of $S$. This factor arises from the use of estimates concerning fundamental $S$-units. To avoid such a factor in Theorem 2, we

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do not employ fundamental $S$-units and $S$-regulator in the proof. Instead we combine some arguments of [12] with the aforementioned new ingredients, and reduce the proof to the special case of our Theorem 1 with $S = S_\infty$. The removal of $s^{2s}$ is crucial in some applications, e.g. in our Corollaries 4 and 5.

The first author reduced a large class of decomposable form equations to ($S$-)unit equations and then, using his effective results concerning such equations (cf. [10], [12], [13], [17] and the joint work [5] with Bugeaud), gave upper bounds for the solutions of the decomposable form equations in question; see e.g. [10], [19], [11], [14]–[16], [6], [17]. Our Theorems 1 and 2 together with thorough refinements upon the arguments of [17] enable us to improve the earlier bounds for the solutions of decomposable form equations (cf. Theorem 3) and, in particular, of Thue equations in $S$-integers (cf. Corollary 3). As an application, we obtain lower bounds for the greatest prime factors of decomposable forms at integral points (cf. Corollary 4), and get some new information about the arithmetical properties of integers represented by decomposable forms (cf. Corollary 5). Further applications of Theorem 2 are given in [18] and [20].

**2. Bounds for the solutions of $S$-unit equations.** The following standard notation will be used throughout this paper. Let $K$ be an algebraic number field of degree $d$ with regulator $R$, class number $h$ and unit rank $r$. Let $S$ denote a finite set of places on $K$ containing the set $S_\infty$ of infinite places. Denote by $s$ the cardinality of $S$, by $t$ the number of finite places in $S$, and by $P$ the largest norm of the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ corresponding to the finite places in $S$ with the convention that $P = 1$ if $S = S_\infty$ (i.e. $t = 0$). Further, denote by $O_S$ the ring of $S$-integers, and by $O_S^*$ the group of $S$-units in $K$, which has rank $s - 1 = r + t$. The case $s = 1$ being trivial, we assume throughout the paper that $s \geq 2$. We denote by $R_S$ the $S$-regulator of $K$ (for its definition see e.g. [5]). For $S = S_\infty$ (i.e. $t = 0$) we have $R_S = R$, and $O_S$ is just the ring of integers $O_K$ of $K$.

For any algebraic number $\alpha$, we denote by $h(\alpha)$ the absolute logarithmic height of $\alpha$ (cf. Section 4). By height we shall always mean the absolute logarithmic height. We use the notation $\log^* a$ for $\max\{\log a, 1\}$.

Let $\alpha$ and $\beta$ be non-zero elements of $K$ with

$$\max\{h(\alpha), h(\beta)\} \leq H,$$

where, for technical reasons, we assume that $H \geq \max\{1, \pi/d\}$. Consider the $S$-unit equation

$$\alpha x + \beta y = 1 \quad \text{in } x, y \in O_S^*. \tag{1.a}$$

For $S = S_\infty$, this is an ordinary unit equation.

Theorem 1. *All solutions $x$, $y$ of* (1.a) *satisfy*

$$\max\{h(x), h(y)\} < c_1 P R_S (1 + (\log^* R_S)/\log^* P) H \tag{2}$$

*where*

$$c_1 = c_1(d, s) = s^{2s+3.5} 2^{7s+27} \log(2s) d^{2(s+1)} (\log^*(2d))^3.$$

*Further, if in particular $S = S_\infty$ (i.e. $t = 0$), then the bound in* (2) *can be replaced by*

$$c_2 R(\log^* R) H \tag{3}$$

*where*

$$c_2 = c_2(d, r) = (r+1)^{2r+9} 2^{3.2(r+12)} \log(2r+2)(d \log^*(2d))^3.$$

Remark 1. It is clear that the factor $(1 + (\log^* R_S)/\log^* P)$ in (2) does not exceed $2 \log^* R_S$, and if $\log^* R_S \le \log^* P$, then it is at most 2.

Remark 2. Theorem 1 is an improvement of the Theorem of Bugeaud and Győry [5]. Our constants $c_1$ and $c_2$ are smaller than the corresponding ones in [5] (and do not contain any parameter related to the Lehmer problem). Further, from the upper bound in [5] concerning $\max\{h(x), h(y)\}$ an extra factor $\log^* R_S$ has been eliminated. We recall that in [5], [6] and [17] the absolute height is used.

Consider now equation (1.a) in homogeneous form

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \quad \text{in } x_1, x_2, x_3 \in O_S^*, \tag{1.b}$$

where $\alpha_1, \alpha_2, \alpha_3$ are non-zero numbers in $K$ with $\max_k h(\alpha_k) \le H$ $(H \ge 2)$. For $t > 0$, set

$$\mathcal{T} = \left\{ \begin{array}{ll} 1 & \text{if } r = 0 \\ 2^t & \text{if } r \ge 1 \end{array} \right\} \cdot \prod_{i=1}^{t} \max\{h_i \log N(\mathfrak{p}_i), c_3 d R\},$$

where $h_i$ denotes the smallest positive integer for which the ideal $\mathfrak{p}_i^{h_i}$ is principal (and thus $h_i \mid h$). The constant $c_3$ (coming from Lemma 3) is defined by

$$c_3 = \begin{cases} 0 & \text{if } r = 0, \\ 1/d & \text{if } r = 1, \\ 29 e r! r \sqrt{r-1} \log d & \text{if } r \ge 2. \end{cases}$$

Further, let

$$\mathcal{R} = \max\{h, c_3 d R\},$$

and for brevity, write $\mathscr{S} = O_K \cap O_S^*$.

Theorem 2. *Let $t > 0$. For every solution $x_1, x_2, x_3$ of* (1.b) *there are $\sigma \in O_S^*$ and $\varrho_1, \varrho_2, \varrho_3 \in \mathscr{S}$ such that*

$$x_k = \sigma \varrho_k, \qquad k = 1, 2, 3, \tag{4}$$

*and*

$$\max_{1\le k\le 3} h(\varrho_k) < c_4 hR^2(\log^* R)\mathcal{R}(1+(\log^*\mathcal{R})/\log^* P)(P/\log^* P)\mathcal{T}H, \tag{5}$$

*where*

$$c_4 = c_4(d,r,t) = (r+1)^{4r+10}2^{10(r+t)+63}(r+t+1)^{3.5}d^{r+t+5}(\log^*(2d))^6.$$

*If in particular* $r=0$, *then the bound in* (5) *can be replaced by*

$$c_5 h^2(1+(\log^* h)/\log^* P)(P/\log^* P)\Big\{\prod_{i=1}^{t} h_i \log N(\mathfrak{p}_i)\Big\}H, \tag{6}$$

*with*

$$c_5 = c_5(d,t) = 2^{10t+21}\, t^{3.5} d^{t+2}(\log^*(2d))^3.$$

*Finally, if* $x_k \in \mathscr{S}$ *for* $k=1,2,3$, *then* $\sigma$ *can be chosen from* $\mathscr{S}$.

Remark 3. Equations (1.a) and (1.b) can be transformed into each other. For $t>0$, the inequalities

$$R\prod_{i=1}^{t}\log N(\mathfrak{p}_i) \le R_S \le hR\prod_{i=1}^{t}\log N(\mathfrak{p}_i) \tag{7}$$

(see e.g. [5]) and

$$\prod_{i=1}^{t}\log N(\mathfrak{p}_i) \le \mathcal{T} \le (2\mathcal{R})^t\prod_{i=1}^{t}\log^* N(\mathfrak{p}_i)$$

make it easier to compare the upper bounds in Theorems 1 and 2. In the important special case $K=\mathbb{Q}$, the bound in (2) takes the form

$$c_1(t)P(\log p_1)\cdots(\log p_t)H,$$

where $c_1(t) = (t+1)^{2t+6.5}2^{7t+34}\log(2t+2)$. The same bound can be deduced from Theorem 2 for the solutions of (1.a) but with $c_1(t)$ replaced by $2^{10t+23}t^{3.5}/\log^* P$, which is smaller than $c_1(t)$ for all $t\ge 1$. Here $p_1,\dots,p_t$ denote the rational primes corresponding to the finite places in $S$, and $P$ is the maximum of these primes.

In terms of $S$, $s^{2s}$ is the dominating factor in the bound in (2) whenever $t>\log P$. In the bounds of Theorem 2 there is no factor of the form $s^s$ or $t^t$. This improvement plays an important role in some applications; see [18], [20] and Section 3 of the present paper.

Remark 4. The factor $s^{2s}$ occurring in the bound of Theorem 1 is a consequence of the use of Lemma 2 concerning $S$-units. To obtain a bound in Theorem 2 without this factor $s^{2s}$, we shall combine the proof of Lemma 6 of [12] with our Theorem 1 with $t=0$.

Let $\alpha_1, \alpha_2, \alpha_3$ be non-zero elements in $K$ with heights at most $H$ ($H \geq 2$). In some applications, it is more convenient to consider the following equation instead of (1.b):

$$\text{(1.c)} \qquad \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \quad \text{in } x_k \in O_S \setminus \{0\} \quad \text{with } N_S(x_k) \leq N \text{ for } k = 1, 2, 3,$$

where $N_S$ denotes the $S$-norm (see Section 4). Then setting

$$Q = N(\mathfrak{p}_1 \cdots \mathfrak{p}_t) \quad \text{if } t > 0, \qquad Q = 1 \quad \text{if } t = 0$$

and

$$\mathcal{N} = c_3 R + \frac{h}{d} \log Q + H + \frac{1}{d} \log N,$$

it is easy to deduce from Theorems 1 and 2 (for $t > 0$) and Theorem 1 (for $t = 0$) the following.

Corollary 1. *For every solution $x_1, x_2, x_3$ of* (1.c) *there is an $\varepsilon \in O_S^*$ such that* $\max_{1 \leq k \leq 3} h(\varepsilon x_k)$ *is bounded above by*

$$\text{(8.a)} \qquad 2.001 c_1 P R_S (1 + (\log^* R_S)/\log^* P) \mathcal{N}$$

*and, for $t > 0$, by*

$$\text{(8.b)} \qquad c_4 h R^2 (\log^* R) \mathcal{R} (1 + (\log^* \mathcal{R})/\log^* P)(P/\log^* P) \mathcal{T} \mathcal{N}$$

*with $c_1$ and $c_4$ occurring in Theorems* 1 *and* 2*. Further, if in particular $t = 0$, the bound in* (8.a) *can be replaced by*

$$\text{(8.c)} \qquad 2.001 c_2 R (\log^* R) \mathcal{N},$$

*where $c_2$ denotes the constant specified in Theorem* 1*.*

We note that $\log Q \leq t \log P$. Our Corollary 1 improves upon Lemma 6 of [12] and the Corollary of [5].

Denote by $D$ the discriminant of $K$, and by $\log_i$ the $i$th iterate of the logarithmic function with $\log_1 = \log$. Further, let $Q_0$ denote the product of the distinct prime factors of $Q = N(\mathfrak{p}_1 \cdots \mathfrak{p}_t)$. Then we have

$$Q_0 \leq Q \leq Q_0^d.$$

The next corollary is a consequence of Theorem 2. We recall that $\mathscr{S} = O_K \cap O_S^*$. Put

$$Q_0^* = \max(Q_0, 16).$$

Corollary 2. *Let $t > 0$. If $x_1$, $x_2$, $x_3 \in O_S^*$ satisfy* (1.b)*, then there exist $\sigma \in O_S^*$ and $\varrho_1, \varrho_2, \varrho_3 \in \mathscr{S}$ such that $x_k = \sigma \varrho_k$ $(1 \leq k \leq 3)$ and*

$$\text{(9)} \qquad \max_{1 \leq k \leq 3} h(\varrho_k) \leq c_6 |D|^{3/2} (\log^* |D|)^{3d-1} \frac{P}{\log^* P} \times Q_0^{d(c_7 \log^* |D| + 19.2 \log_3 Q_0^*)/\log_2 Q_0^*} H,$$

*where*

$$c_6 = \begin{cases} 2^{22} & \text{if } d = 1, \\ 2^{26} & \text{if } d = 2 \text{ with } r = 0, \\ 2^{96} d^8 (\log(2d))^6 & \text{if } r = 1, \\ (r+1)^{5r+15} 2^{9r+71} d^{r+9} (\log(2d))^8 & \text{if } r \geq 2, \end{cases}$$

*and*

$$c_7 = \begin{cases} 12.4 & \text{if } d = 1, \\ 14.7 & \text{if } d = 2 \quad \text{with } r = 0, \\ 9.7d & \text{if } r = 1, \\ 8.9 d^2 \log d & \text{if } r \geq 2. \end{cases}$$

*Further if $d = 2$ with $r = 0$, the expression $|D|^{3/2}(\log^* |D|)^{3d-1}$ can be replaced by $|D|(\log^* |D|)^{2d-1}$. Finally, if $x_k \in \mathscr{S}$ $(1 \leq k \leq 3)$, then $\sigma$ may be chosen from $\mathscr{S}$.*

We note that $P \leq Q_0^d$. Corollary 2 can be readily compared with [21, Theorem 3.1] and [29, Theorem 1.5], and may be considered as an explicit result related to the *abc* conjecture over number fields; see e.g. [3] and [24]. In the special case $K = \mathbb{Q}$, in order to apply Corollary 2 to the equation

$$x + y = z \quad \text{ with } (x, y, z) = 1 \text{ and } z > 2,$$

in positive rational integers $x, y, z$, which is the equation in Stewart and Yu [28, Theorem 2], we take $K = \mathbb{Q}$, and $S \setminus S_\infty$ to be the set of all distinct prime factors of $xyz$. Then we have $D = 1$, $H = 2$, $\sigma = \pm 1$. Let $p_x, p_y, p_z$ be the greatest prime factors of $x, y, z$, respectively, with the convention that if $x = 1$ $(y = 1)$, then $p_x = 1$ $(p_y = 1)$. Put $P = \max\{p_x, p_y, p_z\}$ and $p' = \min\{p_x, p_y, p_z\}$. In the notation of [28], we have $Q_0 = G$, $Q_0^* = G^*$. Now our Corollary 2 implies, on noting $12.4 + 19.2 \log_3 G^* \leq 653 \log_3 G^*$, that

$$z < \exp\left(2^{23} \frac{P}{\log P} G^{653(\log_3 G^*)/\log_2 G^*}\right).$$

Although this is completely explicit, it is still weaker than [28, Theorem 2] in general, since there $p'$ occurs in place of the expression $2^{23} P/\log P$. Furthermore, Chim Kwok Chi [7], following the proof of [28], has proved that

$$z < \exp(p' G^{710(\log_3 G^*)/\log_2 G}).$$

**3. Bounds for the solutions of decomposable form equations.** Keeping the notation of Section 2, consider the equation

$$F(\mathbf{x}) = \beta \quad \text{ in } \mathbf{x} = (x_1, \ldots, x_m) \in O_S^m, \tag{10}$$

where $\beta \in K \setminus \{0\}$, and $F(\mathbf{X}) = F(X_1, \ldots, X_m)$ is a decomposable form of degree $n \geq 3$ in $m \geq 2$ variables which factorizes into linear forms over $K$.

These linear factors of $F$ are uniquely determined over $K$ up to proportional factors from $K$. Fix a factorization of $F$ into linear forms, and denote by $\mathcal{L}_F$ the system of these linear forms.

The first author established several effective bounds for the solutions of equation (10), subject to certain assumptions on $\mathcal{L}_F$ (see e.g. [10], [19], [14]–[17] and the references given there). The most general effective results were obtained in [17]. Here we slightly refine the assumptions on $\mathcal{L}_F$ in [17], in order to make them more transparent.

For a system $\mathcal{L}$ of non-zero linear forms in $X_1, \dots, X_m$ over $K$, let $\mathcal{L}^*$ denote a maximal subset of pairwise linearly independent linear forms of $\mathcal{L}$. We denote by $\mathcal{G}(\mathcal{L}^*)$ the graph with vertex set $\mathcal{L}^*$ in which distinct $l, l'$ in $\mathcal{L}^*$ are connected by an edge if $\lambda l + \lambda' l' + \lambda'' l'' = 0$ for some $l'' \in \mathcal{L}^*$ and some non-zero $\lambda$, $\lambda'$, $\lambda''$ in $K$. Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be the vertex sets of the connected components of $\mathcal{G}(\mathcal{L}^*)$. When $k = 1$ and $\mathcal{L}^*$ has at least three elements, $\mathcal{L}$ is said to be *triangularly connected* (cf. [19]). If $k > 1$, we introduce the graph $\mathcal{H}(\mathcal{L}_1, \dots, \mathcal{L}_k)$ with vertex set $\{\mathcal{L}_1, \dots, \mathcal{L}_k\}$, in which the pair $[\mathcal{L}_i, \mathcal{L}_j]$ is an edge if there exists a non-zero linear form which can be expressed simultaneously as a linear combination over $K$ of the forms in $\mathcal{L}_i$ and of the forms in $\mathcal{L}_j$.

Now we apply the above terminology to $\mathcal{L}_F$. We suppose that the decomposable form $F$ in (10) satisfies the following conditions:

(i) $\mathcal{L}_F$ has rank $m$;

(ii) denoting by $\mathcal{L}_1, \dots, \mathcal{L}_k$ the vertex sets of the connected components of $\mathcal{G}(\mathcal{L}_F^*)$, either $k = 1$ or $k > 1$ with the graph $\mathcal{H}(\mathcal{L}_1, \dots, \mathcal{L}_k)$ being connected.

It is obvious that (ii) depends only on $\mathcal{L}_F$, but not on the choice of $\mathcal{L}_F^*$. For $k = 1$, assumptions (i) and (ii) imply that $\mathcal{L}_F$ is triangularly connected.

In (ii) with $k > 1$, for each edge $[\mathcal{L}_i, \mathcal{L}_j]$ of the graph $\mathcal{H}(\mathcal{L}_1, \dots, \mathcal{L}_k)$ there is one (and apart from proportional factors at most finitely many) non-zero linear form $l_{i,j}$ which can be expressed as $\sum_{l \in \mathcal{L}_i} \lambda_l l = \sum_{l \in \mathcal{L}_j} \lambda_l l$ such that the total number of non-zero terms on both sides of the equality is minimal. We pick up for each edge $[\mathcal{L}_i, \mathcal{L}_j]$ such an $l_{i,j}$, and we denote by $\mathcal{L}'_F$ the set of the $l_{i,j}$'s so chosen ([1]).

We recall that, throughout the paper, by height we mean the absolute logarithmic height.

THEOREM 3. *Let $F$ be a decomposable form as above with properties* (i) *and* (ii). *Further, let $\beta \in K \setminus \{0\}$ with $h(\beta) \le B$, and suppose that the heights of the coefficients of the linear forms in $\mathcal{L}_F$ do not exceed $A$ $(\ge 1)$.*

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([1]) As will be seen in the proof, it is enough to consider an $\mathcal{L}'_F$ which consists of $l_{i,j}$ for a minimal number of edges $[\mathcal{L}_i, \mathcal{L}_j]$ ensuring the connectedness of $\mathcal{H}(\mathcal{L}_1, \dots, \mathcal{L}_k)$.

*With the above notation, all solutions* $\mathbf{x} = (x_1, \dots, x_m) \in O_S^m$ *of* (10) (*with* $l(\mathbf{x}) \neq 0$ *for all* $l \in \mathcal{L}'_F$ *if* $k > 1$) *satisfy*

$$\max_{1\le i\le m} h(x_i) < c'_1 P R_S (1 + (\log^* R_S)/\log^* P) \times \left(c_3 R + \frac{h}{d}\log Q + mndA + B\right) \tag{11}$$

*and*, *for* $t > 0$,

$$\max_{1\le i\le m} h(x_i) < c'_4 h R^2 (\log^* R)\mathcal{R}(1 + (\log^* \mathcal{R})/\log^* P) \times (P/\log^* P)\mathcal{T}\left(c_3 R + \frac{h}{d}\log Q + mndA + B\right). \tag{12}$$

*Further*, *if* $t = 0$ (*i.e.* $O_S = O_K$), *then the bound in* (11) *can be replaced by*

$$c'_2 R(\log^* R)(c_3 R + mndA + B). \tag{13}$$

*Here if* $k = 1$, *then* $c'_i = 25m(n-1)c_i$ $(i = 1, 2)$ *and* $c'_4 = 12.5m(n-1)c_4$, *and if* $k > 1$, *then* $c'_i = 50m(m + 1/2)(n-1)c_i$ $(i = 1, 2)$ *and* $c'_4 = 25m(m + 1/2) \times (n-1)c_4$, *where* $c_1, c_2$ *are the constants specified in Theorem* 1, *and* $c_4$ *is specified in Theorem* 2.

Our bounds improve upon the corresponding estimates of Theorem 1 of [16] and Theorem 1 of [17]. Further, (12) implies an improved and explicit version of Theorem 3.4 of [21]. It should be observed that there is no factor of the form $s^s$ or $t^t$ in the bound in (12). This will be important for Corollaries 4 and 5.

It is clear that binary forms having at least three pairwise non-proportional linear factors are triangularly connected. Further, as is known (see e.g. [19], [16] and [17]), discriminant forms and index forms are also triangularly connected, and a large class of norm forms in $m$ variables satisfies conditions (i), (ii), with $k > 1$ and $\mathcal{L}'_F = \{X_m\}$. Therefore our Theorem 3 improves upon the bounds in Corollaries 2, 3, 4.1 and 5 of [16] on the $S$-integer solutions of norm form, discriminant form and index form equations.

We present a consequence of Theorem 3 for the Thue equation

$$F(x_1, x_2) = \beta \quad \text{in } x_1, x_2 \in O_S, \tag{14}$$

where $F(X_1, X_2)$ denotes a binary form of degree $n$ with splitting field $K$ and with at least three pairwise non-proportional linear factors. Suppose that the heights of the coefficients of $F$ do not exceed $A$ $(\geq 1)$.

The next corollary is a significant improvement of Corollary 1 of [17].

Corollary 3. *All solutions* $(x_1, x_2) \in O_S^2$ *of* (14) *satisfy* (11) *and* (12) *for* $t > 0$ *and* (13) *for* $t = 0$ (*when* $O_S = O_K$), *with* $c'_i$ *for* $k = 1$ *replaced by* $5d^2n^5c'_i$ *for* $i = 1, 4, 2$, *respectively.*

As is known (see e.g. [16]), equation (10) is in fact equivalent to the equation of Mahler type

$$F(\mathbf{x}) \in \beta\mathscr{S} \quad \text{in } \mathbf{x} = (x_1, \ldots, x_m) \in O_K^m, \tag{15}$$

where, as above, $\mathscr{S} = O_K \cap O_S^*$. If $F$ satisfies the assumptions of Theorem 3 and $x_1, \ldots, x_m$ is a solution of (15) for which the norm $N((x_1, \ldots, x_m))$ of the ideal $(x_1, \ldots, x_m)$ is bounded, then Theorem 3 implies an explicit upper bound for $\max_{1 \le i \le m} h(\varepsilon x_i)$ with an appropriate $\varepsilon \in O_K^*$.

We formulate a further consequence of Theorem 3. We denote by $\omega(\alpha)$ the number of distinct prime ideal divisors of $\alpha \in O_K \setminus \{0\}$, and by $P(\alpha)$ the greatest of the norms of these prime ideals (with the convention that $P(\alpha) = 1$ if $\alpha \in O_K^*$).

Corollary 4. *Let $F$ be a decomposable form as in Theorem 3 with coefficients in $O_K$, and let $N_0$ be a positive integer. If $\mathbf{x} = (x_1, \ldots, x_m) \in O_K^m$ and $N((x_1, \ldots, x_m)) \le N_0$ with $F(\mathbf{x}) \neq 0$ (and with $l(\mathbf{x}) \neq 0$ for $l \in \mathcal{L}'_F$ if $k > 1$) then*

$$P(\log P)^{\omega} > c_8 (\log N)^{c_9} \tag{16}$$

*and*

$$P > \begin{cases} c_{10}(\log N)^{c_{11}} & \text{if } \omega \le \log P/\log_2 P, \\ c_{12}(\log_2 N)(\log_3 N)/(\log_4 N) & \text{otherwise}, \end{cases} \tag{17}$$

*provided that $N = \max_{1 \le i \le m} |N_{K/\mathbb{Q}}(x_i)| \ge N_1$, where $P = P(F(\mathbf{x}))$ and $\omega = \omega(F(\mathbf{x}))$. Here $c_8, \ldots, c_{12}$ and $N_1$ are effectively computable positive numbers which depend at most on $F$, $K$, and $N_0$.*

An important special case is when $k = 1$, $m = 2$, i.e. when $F$ is a binary form with splitting field $K$ and with at least three pairwise non-proportional linear factors. Our Corollary 4 can be compared with the estimate (10) in [11], Theorem 7 in [15], and with Theorems 3.3 and 3.5 in [21] where, for $k = 1$, the second of our lower estimates in (17) is proved for all $\omega$.

We note that if $F(X) \in O_K[X]$ is a polynomial of degree $n$ with splitting field $K$ and with at least two distinct zeros, then, applying Corollary 4 to the binary form $Y^{n+1}F(X/Y)$, we obtain (16) and (17) for $P = P(F(x))$, $\omega = \omega(F(x))$, $N = |N_{K/\mathbb{Q}}(x)|$ with $x \in O_K$, provided that $N$ is sufficiently large.

Corollary 4 motivates the following.

Conjecture. *With the assumptions and notation of Corollary 4, we have*

$$P > c_{13}\,(\log N)^{c_{14}} \qquad \text{if } N \ge N_1,$$

*where $c_{13}$, $c_{14}$ and $N_1$ are effectively computable positive constants depending at most on $F$, $K$ and $N_0$.*

The following corollary enables us to obtain some new information about the arithmetical structure of those algebraic integers of $K$ which can be represented by a decomposable form of the above type.

Corollary 5. *Suppose $F$ and $N_0$ are as in Corollary 4. Let $F_0$ be any non-zero integer in $K$ represented by $F(x_1, \ldots, x_m)$, where $x_1, \ldots, x_m \in O_K$ with $N((x_1, \ldots, x_m)) \leq N_0$ (and with $l(x_1, \ldots, x_m) \neq 0$ for $l \in \mathcal{L}'_F$ if $k > 1$). Then*

$$P > \begin{cases} c_{15}(\log N)^{c_{16}} & \text{if } \omega \leq \log P/\log_2 P, \\ c_{17}(\log_2 N)(\log_3 N)/(\log_4 N) & \text{otherwise}, \end{cases}$$

*provided that $N = |N_{K/\mathbb{Q}}(F_0)| \geq N_2$, where $P = P(F_0)$ and $\omega = \omega(F_0)$. Here $c_{15}$, $c_{16}$, $c_{17}$ and $N_2$ are effectively computable positive numbers which depend at most on $F$, $K$, and $N_0$.*

This is a generalization and a considerable improvement of Corollary 1 of [11]. As was mentioned above, binary forms, discriminant forms and index forms (with $k = 1$) and a large class of norm forms satisfy the conditions of our Corollaries 4 and 5.

**4. Auxiliary results.** Keeping the notation of the preceding sections, let again $K$ denote an algebraic number field with the parameters $d$, $R$, $h$ and $r$ specified above. Denote by $M_K$ the set of places on $K$. For every place $v$ we choose a valuation $|\cdot|_v$ in the usual way: if $v$ is infinite and corresponds to $\sigma : K \to \mathbb{C}$, then we put, for $\alpha \in K$, $|\alpha|_v = |\sigma(\alpha)|^{d_v}$, where $d_v = 1$ or $2$ according as $\sigma(K)$ is contained in $\mathbb{R}$ or not; if $v$ is a finite place corresponding to the prime ideal $\mathfrak{p}$ in $K$, then we put $|\alpha|_v = N(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}} \alpha}$ for $\alpha \in K \setminus \{0\}$, and $|0|_v = 0$. Here, for $\alpha \neq 0$, $\operatorname{ord}_{\mathfrak{p}} \alpha$ denotes the exponent to which $\mathfrak{p}$ divides the principal fractional ideal $(\alpha)$.

The absolute logarithmic height $h(\alpha)$ of $\alpha \in K$ is defined by

$$h(\alpha) = \frac{1}{d} \sum_{v \in M_K} \log \max\{1, |\alpha|_v\}.$$

It depends only on $\alpha$, and not on the choice of the number field $K$ containing $\alpha$. For properties of this height, we refer to [31].

As in Section 2, $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ will denote the prime ideals of $K$ corresponding to the finite places of $S$. For $\alpha \in K \setminus \{0\}$, the fractional ideal $(\alpha)$ can be written uniquely as a product of two fractional ideals $\mathfrak{a}_1, \mathfrak{a}_2$, where $\mathfrak{a}_1$ is composed of $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ and $\mathfrak{a}_2$ is composed solely of prime ideals different from $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Then the $S$-norm of $\alpha$ is defined as $N_S(\alpha) = N(\mathfrak{a}_2)$.

Finally, $\omega_K$ will denote the number of roots of unity in $K$.

PROPOSITION 1. *For $n \geq 1$, let $\alpha_1, \ldots, \alpha_n$ be multiplicatively independent non-zero elements of $K$. If $K$ is of degree $d \geq 2$, then*

$$58(n!e^n/n^n)d^{n+1}(\log d)h(\alpha_1)\cdots h(\alpha_n) \geq \omega_K,$$

*while if $d = 1$, the expression $58d^{n+1}(\log d)$ can be replaced by* 17.

*Proof.* This is a consequence of Theorem 3 of Loher and Masser [22]. ∎

As is known, $n!e^n/n^n$ is asymptotic to $\sqrt{2\pi n}$ and

$$n!e^n/n^n \leq e\sqrt{n}. \tag{18}$$

For simplicity, we shall apply Proposition 1 together with (18).

For $s \geq 2$, let

$$c_{18} = ((s-1)!)^2/(2^{s-2}d^{s-1}), \quad c'_{18} = (s-1)!/d^{s-1}.$$

Further, for $s \geq 3$, let

$$c_{19} \text{ (resp. } c'_{19}) = \begin{cases} 8.5e\sqrt{s-2}\ c_{18} \text{ (resp. } c'_{18}) & \text{if } d = 1, \\ 29e\sqrt{s-2}\ d^{s-1}(\log d)\, c_{18} \text{ (resp. } c'_{18}) & \text{if } d \geq 2, \end{cases}$$

and

$$c_{20} = \begin{cases} ((s-1)!)^2/(2^{s-2}\log 2) & \text{if } d = 1, \\ (((s-1)!)^2/2^{s-1})(\log(3d))^3 & \text{if } d \geq 2. \end{cases}$$

LEMMA 2. *Let $s \geq 2$. There exists in $K$ a fundamental* (*resp. independent*) *system $\{\varepsilon_1, \ldots, \varepsilon_{s-1}\}$ of S-units with the following properties*:

(i) $$\prod_{i=1}^{s-1} h(\varepsilon_i) \leq c_{18}R_S \quad (\textit{resp. } c'_{18}R_S);$$

(ii) $$\max_{1\leq i\leq s-1} h(\varepsilon_i) \leq c_{19}R_S \quad (\textit{resp. } c'_{19}R_S) \quad \textit{if } s \geq 3;$$

(iii) *the absolute values of the entries of the inverse matrix of $(\log|\varepsilon_i|_{v_j})_{i,j=1,\ldots,s-1}$ of the fundamental system $\{\varepsilon_1, \ldots, \varepsilon_{s-1}\}$ do not exceed $c_{20}$.*

We note that (i) and (iii) were proved in [5] and [6], respectively, in the "fundamental" case, and (i) was obtained in [4] in the "independent" case. The inequality (ii) is an improvement, at least in terms of $s$, of the corresponding statements of [5], [6] and [4].

*Proof of Lemma 2.* For the proof of (i), see Lemma 1 in [5] and its proof. (ii) is an immediate consequence of (i), Proposition 1 and (18). To prove (iii), it is enough to combine the proof of (iii) in Lemma 1 of [5] with the inequality

$$dh(\varepsilon_i) \geq \begin{cases} \log 2 & \text{if } d = 1, \\ 2/(\log 3d)^3 & \text{if } d \geq 2, \end{cases} \tag{19}$$

which, for $d \geq 2$, is due to Voutier [30]. ∎

The next lemma has various variants in the literature.

LEMMA 3. *For every* $\alpha \in O_S \setminus \{0\}$ *and for every integer* $n \geq 1$ *there exists an* $\varepsilon \in O_S^*$ *such that*

$$h(\varepsilon^n \alpha) \leq \frac{1}{d} \log N_S(\alpha) + n\left(c_3 R + \frac{h}{d} \log Q\right)$$

*with* $c_3$ *defined in Section* 2.

Lemma 3 was proved in [5] and [17] with a larger $c_3$. We remark that in the special case $t = 0$, the unit $\varepsilon \in O_K^*$ occurring in Lemma 3 can be chosen from the group generated by independent units having properties specified in (i) and (ii) of Lemma 2.

*Proof of Lemma 3.* We combine the proof of Lemma 2 of [5] with our Lemma 2. First consider the case $t = 0$, when $\alpha \in O_K \setminus \{0\}$. If $r = 0$, the assertion immediately follows with $\varepsilon = 1$. Suppose that $r \geq 1$, and choose a system of independent units $\varepsilon_1, \ldots, \varepsilon_r$ in $K$ with the properties specified in Lemma 2. As in [5], consider the system of linear equations

$$\sum_{j=1}^{r} (\log |\varepsilon_j|_{v_i}) X_j = -\log(M^{-d_{v_i}/d} |\alpha|_{v_i}), \qquad i = 1, \ldots, r+1,$$

where $M = |N_{K/\mathbb{Q}}(\alpha)|$ and $v_1, \ldots, v_{r+1}$ denote the infinite places on $K$. This system has a unique solution $(x_1, \ldots, x_r) \in \mathbb{R}^r$. Let $(b_1, \ldots, b_r)$ be the unique point in $\mathbb{Z}^r$ such that

$$x_j = nb_j + \varrho_j \quad \text{ with } -\frac{1}{2}n < \varrho_j \leq \frac{1}{2}n,\ j = 1, \ldots, r.$$

Putting $\varepsilon = \varepsilon_1^{b_1} \cdots \varepsilon_r^{b_r}$, we infer that

$$|\log(M^{-d_{v_i}/d} |\varepsilon^n \alpha|_{v_i})| \leq \frac{n}{2} \sum_{j=1}^{r} |\log |\varepsilon_j|_{v_i}|$$

for $i = 1, \ldots, r+1$. Then using the product formula for $\varepsilon_j$, we deduce that

$$h(\varepsilon^n \alpha) \leq \frac{1}{d} \sum_{i=1}^{r+1} |\log |\varepsilon^n \alpha|_{v_i}| \leq \frac{1}{d} \log M + \frac{n}{d} \sum_{j=1}^{r} \sum_{i=1}^{r} |\log |\varepsilon_j|_{v_i}|.$$

We assert that if $r > 1$, then the inner sum in the extreme right-hand side of the above inequality is at most $(d/r)c_3 R$. This can be seen by using [5, (9), (10)], the second inequality of [5, (12)] and by applying Proposition 1 to any $r - 1$ of the $\varepsilon_i$ $(1 \leq i \leq r)$. Thus Lemma 3 is proved for $r > 1$. If $r = 1$, we can use (i) of Lemma 2 to prove the assertion.

The case $t > 0$ of our lemma follows from the case $t = 0$ in the same way as in the proof of Lemma 10 of [8], observing that $Q$ can be taken everywhere in place of $P^{td}$ with $P = \max\{p_1, \dots, p_t\}$ considered in [8]. ∎

Let

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1, \tag{20}$$

where $\alpha_1, \dots, \alpha_n$ are $n$ $(\geq 2)$ non-zero elements of $K$, and $b_1, \dots, b_n$ are rational integers, not all zero, with

$$B^* = \max\{|b_1|, \dots, |b_n|\}.$$

Set

$$A_i \geq \max\{dh(\alpha_i), \pi\}, \qquad i = 1, \dots, n.$$

The following result is a consequence of a deep theorem of Matveev [25].

Proposition 4. *Suppose* $\Lambda \neq 0$, $b_n = \pm 1$ *and* $B$ *satisfies*

$$B \geq \max\{B^*, 2e \max(n\pi/\sqrt{2}, A_1, \dots, A_{n-1})A_n\}. \tag{21}$$

*Then*

$$\log|\Lambda| > -c_{21}(n, d)A_1 \cdots A_n \log(B/(\sqrt{2}\,A_n)), \tag{22}$$

*where*

$$c_{21}(n, d) = \min\{1.451(30\sqrt{2})^{n+4}(n + 1)^{5.5}, \pi 2^{6.5n+27}\}d^2 \log(ed).$$

*Proof.* Let log denote the principal value of the logarithm. There exists an even rational integer $b_0$ such that $|b_0| \leq |b_1| + \dots + |b_n| \leq nB^*$ and that $|\mathrm{Im}(\Sigma)| \leq \pi$, where

$$\Sigma := b_0 \log \alpha_0 + b_1 \log \alpha_1 + \dots + b_n \log \alpha_n$$

and $\alpha_0 = -1$. The assumption $\Lambda \neq 0$ implies that $\Sigma \neq 0$. We may assume that $|e^{\Sigma} - 1| = |\Lambda| \leq 1/3$. Then $|\Sigma| \leq 0.6$, whence

$$|\Lambda| \geq \tfrac{1}{2}\,|\Sigma|. \tag{23}$$

Using $|\log|\alpha_i|| \leq dh(\alpha_i)$, it is easy to show that

$$|\log \alpha_i| \leq \sqrt{2} \max(dh(\alpha_i), \pi), \qquad i = 1, \dots, n.$$

Thus, setting $A_0 = \pi/\sqrt{2}$, we have

$$\sqrt{2}\,A_i \geq \max\{dh(\alpha_i), |\log \alpha_i|, 0.16\}, \qquad i = 0, 1, \dots, n.$$

Further, (21) implies

$$\left(\frac{B}{\sqrt{2}\,A_n}\right)^2 \geq e \max\left\{1, \max_{0 \leq i \leq n}\left(\frac{|b_i|A_i}{A_n}\right)\right\}.$$

By applying now Corollary 2.3 of [25] to $|\Sigma|$ and using (23), we obtain (22). ∎

For $s \geq 3$, let

$$c_{22} = e\sqrt{s-2}\,(((s-1)!)^2/2^{s-2})\pi^{s-2} \cdot \begin{cases} 8.5 & \text{if } d = 1, \\ 29d\log d & \text{if } d \geq 2. \end{cases}$$

When we apply Proposition 4, we shall get better bounds by using the following technical lemma.

Lemma 5. *Let $\{\varepsilon_1, \ldots, \varepsilon_{s-1}\}$ be a fundamental system of $S$-units in $K$ with the properties specified in Lemma 2. Then*

$$\prod_{i=1}^{s-1} \max(dh(\varepsilon_i), \pi) \leq \begin{cases} \max(R_S, \pi) & \text{if } s = 2, \\ c_{22}R_S & \text{if } s \geq 3. \end{cases} \tag{24}$$

*Proof.* The case $s = 2$ is trivially true by Lemma 2. Suppose $s \geq 3$. Let $k$ denote the number of indices $i$ with $1 \leq i \leq s-1$ such that $dh(\varepsilon_i) < \pi$.

Suppose first $1 \leq k \leq s-2$ and, without loss of generality, $dh(\varepsilon_i) < \pi$ for $i = 1, \ldots, k$ and $dh(\varepsilon_j) \geq \pi$ for $j = k+1, \ldots, s-1$. Thus, using Proposition 1 and Lemma 2, we infer that

$$\begin{aligned} \prod_{i=1}^{s-1} \max(dh(\varepsilon_i), \pi) &= \frac{\pi^k}{d^k h(\varepsilon_1)\cdots h(\varepsilon_k)}\, d^{s-1}h(\varepsilon_1)\cdots h(\varepsilon_{s-1}) \\ &\leq \frac{\pi^k}{d^k} \left\{ \begin{matrix} 8.5 & \text{if } d = 1 \\ 29d^{k+1}\log d & \text{if } d \geq 2 \end{matrix} \right\} e\sqrt{k}\, \frac{((s-1)!)^2}{2^{s-2}}\, R_S \leq c_{22}R_S, \end{aligned}$$

which proves (24).

If $k = 0$, then (24) immediately follows from (i) of Lemma 2. Consider now the case $k = s-1$. Then (7) and $R_K \geq 0.2052$ (cf. [9]) imply that if $s \geq 3$ then

$$R_S \geq \begin{cases} (\log 3)(\log 2) & \text{if } d = 1, \\ 0.2052(\log 2)^{s-2} & \text{if } d \geq 2. \end{cases} \tag{25}$$

By $k = s-1$ we have $dh(\varepsilon_i) < \pi$ for all $i$. Hence (24) follows from (25). ∎

Consider again $\Lambda$ defined by (20). Let $B$ and $B_n$ be real numbers satisfying

$$B \geq \max\{|b_1|, \ldots, |b_n|\}, \qquad B \geq B_n \geq |b_n|. \tag{26}$$

Denote by $\mathfrak{p}$ a prime ideal of $O_K$ lying above the prime number $p$, and by $e_\mathfrak{p}$ and $f_\mathfrak{p}$ the ramification index and the residue class degree of $\mathfrak{p}$, respectively. Thus $N(\mathfrak{p}) = p^{f_\mathfrak{p}}$.

The following result is due to Yu [33].

Proposition 6. *Assume that* $\mathrm{ord}_p\, b_n \leq \mathrm{ord}_p\, b_j$ *for* $j = 1, \ldots, n$, *and set*

$$h'_j = \max\{h(\alpha_j), 1/(16e^2d^2)\} \qquad (j = 1, \ldots, n).$$

*If $\Lambda \neq 0$, then for any real $\delta$ with $0 < \delta \leq 1/2$ we have*

$$\operatorname{ord}_{\mathfrak{p}} \Lambda < c_{23}(n,d) e_{\mathfrak{p}}^{n} \frac{N(\mathfrak{p})}{(\log N(\mathfrak{p}))^2} \max\left\{ h_1' \cdots h_n' \log M, \frac{\delta B}{B_n c_{24}(n,d)} \right\},$$

*where*

$$c_{23}(n,d) = (16ed)^{2(n+1)} n^{3/2} \log(2nd) \log(2d),$$
$$c_{24}(n,d) = (2d)^{2n+1} \log(2d) \log^3(3d),$$

*and*

$$M = (B_n/\delta) c_{25}(n,d) N(\mathfrak{p})^{n+1} h_1' \cdots h_{n-1}'$$

*with*

$$c_{25}(n,d) = 2e^{(n+1)(6n+5)} d^{3n} \log(2d).$$

*Proof.* This is the Corollary of Theorem 4 in [33]. As is remarked in [33], for $p > 2$, the expression $(16ed)^{2(n+1)}$ can be replaced by $(10ed)^{2(n+1)}$. ∎

Proposition 6 will be used in the proof of Theorem 1. In the proof of Theorem 2 we shall apply the next proposition, which is sharper than Proposition 6 in the dependence on $d$ and $n$ when all $\alpha_j$ ($j = 1, \ldots, n$) are $\mathfrak{p}$-adic units.

PROPOSITION 7. *Suppose that $\operatorname{ord}_p b_n \leq \operatorname{ord}_p b_j$ and $\operatorname{ord}_{\mathfrak{p}} \alpha_j = 0$ for $j = 1, \ldots, n$, and that $\alpha_1, \ldots, \alpha_{n-1}$ are multiplicatively independent. Set*

$$h_n'' = \max\{h(\alpha_n), 1/(8e^2 d)\}.$$

*If $\Lambda \neq 0$, then for any real $\delta$ with $0 < \delta \leq 1/2$ we have*

$$\operatorname{ord}_{\mathfrak{p}} \Lambda < c_{23}'(n,d) e_{\mathfrak{p}}^{n} \frac{N(\mathfrak{p})}{(\log N(\mathfrak{p}))^2} \times \max\left\{ h(\alpha_1) \cdots h(\alpha_{n-1}) h_n'' \log M', \frac{\delta B}{B_n c_{24}'(n,d)} \right\},$$

*where*

$$c_{23}'(n,d) = c a^n n^{3/2} d^{n+2} \log(2nd) \log(2d)$$

*with*

$$c = \begin{cases} 1692, & p > 2, \\ 292, & p = 2, \end{cases} \qquad a = \begin{cases} 48e^2, & p > 2, \\ 128e^2, & p = 2, \end{cases}$$
$$c_{24}'(n,d) = (2d)^{n+1} \log(2d) \log^3(3d),$$

*and*

$$M' = (B_n/\delta) c_{25}'(n,d) N(\mathfrak{p})^{n+1} h(\alpha_1) \cdots h(\alpha_{n-1})$$

*with*

$$c_{25}'(n,d) = 2e^{(n+1)(6n+5)} d^{2n+1} \log(2d).$$

*Proof.* This is again a consequence of Theorem 4 in [33]. ∎

## 5. Proofs of the theorems

*Proof of Theorem 1.* We follow the proof of the Theorem of [5], and only those steps will be detailed which differ from those in [5].

Let $x, y$ be a solution of (1.a). We may assume that $h(x) \geq h(y)$. Let $\varepsilon_1, \dots, \varepsilon_{s-1}$ be a fundamental system of $S$-units in $K$ with the properties specified in Lemma 2. Then $y$ can be written in the form

$$y = \zeta \varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}}, \tag{27}$$

where $\zeta$ is a root of unity in $K$ and $b_1, \dots, b_{s-1}$ are rational integers. We derive as in [5] that

$$\max\{|b_1|, \dots, |b_{s-1}|\} \leq 2c_{20} d h(x). \tag{28}$$

Set $\alpha_s = \zeta\beta$ and $b_s = 1$. Let $v \in S$ for which $|x|_v$ is minimal. Then, using (1.a), we deduce that

$$\log |\varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}} \alpha_s^{b_s} - 1|_v = \log |\alpha x|_v \leq -\frac{d}{s}\, h(x) + dH. \tag{29}$$

First assume that $v$ is infinite. We shall prove that

$$h(x) < c_{26}(s,d) R_S (\log^* R_S) H, \tag{30}$$

where

$$c_{26}(s,d) = \min\{s^{2s+7} 2^{3.2s+35.2}, s^{2s+1.5} 2^{4.3s+\lambda}\} \log(2s)(d \log^*(2d))^3$$

with

$$\lambda = \begin{cases} 40 & \text{if } s \geq 3,\ d \geq 2, \\ 37.3 & \text{if } s = 2 \text{ with } d \geq 2, \text{ or } s \geq 3 \text{ with } d = 1, \\ 35.4 & \text{if } s = 2,\ d = 1. \end{cases}$$

Set

$$\begin{aligned} A_i &= \max(dh(\varepsilon_i), \pi), \qquad i = 1, \dots, s-1, \\ A_s &= dH \geq \max(dh(\alpha_s), \pi). \end{aligned} \tag{31}$$

We may assume that

$$2c_{20} d h(x) > 2e \max(s\pi/\sqrt{2}, A_1, \dots, A_{s-1}) A_s,$$

since otherwise (30) follows easily from Lemma 2 and Proposition 1. By applying Proposition 4 and Lemma 5, and using (29) and (4), we infer that

$$\log |\alpha x|_v > -d_v c_{21}(s,d) \left\{ \begin{matrix} \max\{R_S, \pi\} & \text{if } s = 2 \\ c_{22} R_S & \text{if } s \geq 3 \end{matrix} \right\} \cdot dH \log\left(\frac{2c_{20} h(x)}{\sqrt{2}\, H}\right)$$

with the $c_{21}(s,d)$, $c_{22}$ occurring in Proposition 4 and Lemma 5, respectively. Together with (29) this implies (30).

We note that for $t = 0$, (30) implies the second part of Theorem 1.

Next assume that $v$ is finite, corresponding to the prime ideal $\mathfrak{p}$. So the equality in (29) implies that

$$\log|\alpha x|_v = -\operatorname{ord}_{\mathfrak{p}}(\varepsilon_1^{b_1}\cdots\varepsilon_{s-1}^{b_{s-1}}\alpha_s^{b_s}-1)\cdot\log N(\mathfrak{p}). \tag{32}$$

We set $B = 2c_{20}dh(x)$. Further, we assume that

$$B \geq 2c_{24}(s,d)h(\varepsilon_1)\cdots h(\varepsilon_{s-1})H, \tag{33}$$

since otherwise, using Lemma 2, we obtain (2). In view of (19) and $H \geq 1$, for $i = 1,\ldots,s-1$ we have

$$h_i' = \max\left\{h(\varepsilon_i), \frac{1}{16e^2d^2}\right\} = h(\varepsilon_i), \qquad h_s' = \max\left\{h(\alpha_s), \frac{1}{16e^2d^2}\right\} \leq H.$$

We choose

$$\delta = c_{24}(s,d)h(\varepsilon_1)\cdots h(\varepsilon_{s-1})H/B.$$

Then, by (33), we have $\delta \leq 1/2$. Applying Proposition 6 we get the following lower bound for the right side of (32):

$$-c_{23}(s,d)d^s\,\frac{N(\mathfrak{p})}{\log N(\mathfrak{p})}\max\left\{h(\varepsilon_1)\cdots h(\varepsilon_{s-1})H\log M, \frac{\delta B}{c_{24}(s,d)}\right\}, \tag{34}$$

where

$$M = \delta^{-1}c_{25}(s,d)N(\mathfrak{p})^{s+1}h(\varepsilon_1)\cdots h(\varepsilon_{s-1})$$

and $c_{23}$, $c_{24}$, $c_{25}$ denote the expressions occurring in Proposition 6, with $n$ replaced by $s$. Using (29), (34), our choice of $\delta$, and Lemma 2(i), we infer that

$$\frac{d}{s}\,h(x) < (1+10^{-11})c_{18}(s,d)c_{23}(s,d)d^s\,\frac{N(\mathfrak{p})}{\log N(\mathfrak{p})}\,R_S H\log Y_1,$$

where

$$Y_1 = \frac{c_{25}(s,d)}{c_{24}(s,d)}\,\frac{2c_{20}(s,d)dh(x)}{H}\,N(\mathfrak{p})^{s+1},$$

whence

$$\frac{Y_1}{\log Y_1} < 2(1+10^{-11})c_{18}(s,d)c_{20}(s,d)c_{23}(s,d)\,\frac{c_{25}(s,d)}{c_{24}(s,d)}\,sd^s\,\frac{N(\mathfrak{p})^{s+2}}{\log N(\mathfrak{p})}\,R_S$$
$$=: M_1.$$

This gives

$$Y_1 < 1.059M_1\log M_1,$$

since $M_1 > 2.24\cdot 10^{32}$. Observe that $N(\mathfrak{p})/\log N(\mathfrak{p}) \leq (1/\log 2)P/\log^* P$ and

$$\log M_1 < 10.2\,s^2\log^*(2d)(\log^* P + \log^* R_S),$$

where 10.2 can be replaced by 7.9 when $d \geq 2$. Now

$$h(x) < s^{2s+3.5}2^{7s+19.4}\log(2s)d^{2s+2}(\log^*(2d))^3P\mathcal{R}s(1+(\log^* R_S)/\log^* P)H$$

by a careful computation. On combining this with (30), we arrive at (2). ∎

*Proof of Theorem 2.* To obtain better bounds, we combine the proof of Lemma 6 of [12] with the case $t = 0$ of our Theorem 1. Further, we use Lemmas 2 and 3 in the present improved forms, and replace the estimate used in [12] for linear forms in logarithms in the $p$-adic case by a recent improved bound of Yu's (cf. Proposition 7).

We may assume without loss of generality that, in (1.b), $x_k \in O_K \cap O_S^*$ for $k = 1, 2, 3$. This can be achieved by multiplying (1.b) by an appropriate $S$-unit. We write

$$(x_k) = \mathfrak{p}_1^{u_{1k}} \cdots \mathfrak{p}_t^{u_{tk}} \quad \text{and} \quad u_{ik} = h_i v_{ik} + r_{ik}$$

with rational integers $u_{ik}, v_{ik} \geq 0$ and $0 \leq r_{ik} < h_i$ for $k = 1, 2, 3$ and $i = 1, \dots, t$. There are integers $\pi_i$ and $\gamma_k$ in $O_K$ such that $\mathfrak{p}_i^{h_i} = (\pi_i)$ and $(\gamma_k) = \mathfrak{p}_1^{r_{1k}} \cdots \mathfrak{p}_t^{r_{tk}}$. Further, by Lemma 3 with $t = 0$, $\pi_i$ and $\gamma_k$ can be chosen so that

$$h(\pi_i) \leq 2 \max\left\{\frac{h_i}{d} \log N(\mathfrak{p}_i), c_3 R\right\} \leq \frac{2}{d}\, \mathcal{R} \log^* P, \qquad i = 1, \dots, t, \tag{35}$$

and

$$\begin{aligned} h(\gamma_k) &\leq \frac{1}{d} \sum_{i=1}^{t} h_i \log N(\mathfrak{p}_i) + c_3 R \\ &\leq 2 \max\left\{\frac{1}{d} \sum_{i=1}^{t} h_i \log N(\mathfrak{p}_i), c_3 R\right\} =: \mathcal{Z}, \qquad k = 1, 2, 3. \end{aligned} \tag{36}$$

Then we have

$$x_k = \varepsilon_k \gamma_k \pi_1^{v_{1k}} \cdots \pi_t^{v_{tk}}, \qquad k = 1, 2, 3, \tag{37}$$

with some units $\varepsilon_k$ from $K$. We note that if $r = 0$, then $c_3 = 0$ and the factor 2 in (35) and (36) can be replaced by 1.

Let $a_i = \min_k v_{ik}$ and $v'_{ik} = v_{ik} - a_i$ for $k = 1, 2, 3$ and $i = 1, \dots, t$. We may assume that $V := \max_{i,k} v'_{ik} = v'_{11} > 0$ and $v'_{13} = 0$. If $r \geq 1$, let $\eta_1, \dots, \eta_r$ be a fundamental system of units in $K$ with the properties specified in Lemma 2. Then

$$\varepsilon_k / \varepsilon_3 = \zeta_k \eta_1^{w_{1k}} \cdots \eta_r^{w_{rk}}, \qquad k = 1, 2, 3, \tag{38}$$

where $\zeta_k$ is a root of unity in $K$ and $w_{1k}, \dots, w_{rk}$ are rational integers such that $\zeta_3 = 1$ and $w_{13} = \cdots = w_{r3} = 0$. Obviously, (38) holds for $r = 0$ as well. Putting $\sigma = \varepsilon_3 \pi_1^{a_1} \cdots \pi_t^{a_t}$, we infer that

$$x_k = \sigma \varrho_k, \qquad k = 1, 2, 3,$$

where

$$\varrho_k = \zeta_k \gamma_k \eta_1^{w_{1k}} \cdots \eta_r^{w_{rk}} \pi_1^{v'_{1k}} \cdots \pi_t^{v'_{tk}} \in O_K \cap O_S^*, \qquad k = 1, 2, 3. \tag{39}$$

We are going to derive an upper bound for $V$. In order to be able to apply Proposition 7 and avoid the use of $\gamma_k$ (which yields a slight improvement in our bound on $V$), we introduce some further notation. Put

$$W = \max_{j,k} |w_{jk}| \quad \text{and} \quad B = \max\{V, W\}.$$

For $r \geq 1$, there are rational integers $t_{1k}, \ldots, t_{rk}$ and a root of unity $\zeta'_k$ in $K$ such that

$$\gamma_k^h = \zeta'_k \eta_1^{t_{1k}} \cdots \eta_r^{t_{rk}} \pi_1^{r'_{1k}} \cdots \pi_t^{r'_{tk}}, \qquad k = 1, 2, 3, \tag{40}$$

where $r'_{ik} = r_{ik} h / h_i$ for $i = 1, \ldots, t$. This implies that

$$\sum_{j=1}^{r} t_{jk} \log |\eta_j|_v = h \log |\gamma_k|_v - \sum_{i=1}^{t} r'_{ik} \log |\pi_i|_v$$

for each infinite place $v$ of $K$ (which are normalized as in Lemma 2). Using the fact that $|\log |\alpha|_v| \leq d h(\alpha)$ for $\alpha \in O_K \setminus \{0\}$ and applying (35), (36) and Lemma 2, we deduce that

$$\max_{j,k} |t_{jk}| \leq c_{27} \mathcal{Z} \tag{41}$$

with $c_{27} = (t+3) h d c_{20}^\star$, where $c_{20}^\star$ denotes the constant $c_{20}$ with the choice $s = r+1$, i.e. $c_{20}^\star = (r!)^2 (\log(3d))^3 / 2^r$ for $r \geq 1$. For $r = 0$, let $c_{27} = 0$. We may assume that

$$V > 17 d H. \tag{42}$$

We show that $\alpha_2 \varrho_2 / (\alpha_3 \varrho_3)$ is not a root of unity. Indeed, if $\alpha_2 \varrho_2 = \zeta \alpha_3 \varrho_3$ with some root of unity $\zeta$, then (1.b) gives

$$\alpha_1 \varrho_1 = -(1 + \zeta) \alpha_3 \varrho_3.$$

But we have

$$|\mathrm{ord}_{\mathfrak{p}_1} \alpha| \leq \frac{d}{\log N(\mathfrak{p}_1)} h(\alpha) \tag{43}$$

for each $\alpha \in K$, $\alpha \neq 0$ (see e.g. [32, p. 124]). Hence we deduce that

$$h_1 V \leq \mathrm{ord}_{\mathfrak{p}_1} \varrho_1 \leq \mathrm{ord}_{\mathfrak{p}_1}((1+\zeta)\alpha_3/\alpha_1) + \mathrm{ord}_{\mathfrak{p}_1} \varrho_3 \leq \frac{3d}{\log 2} H + h_1,$$

which contradicts (42).

In view of (42) it follows that $\mathrm{ord}_{\mathfrak{p}_1}\left(\frac{\alpha_1 \varrho_1}{\alpha_3 \varrho_3}\right) > 0$. Thus we infer from (1.b), (39) and (42) that

$$0.8 h_1 V < \mathrm{ord}_{\mathfrak{p}_1}\left(\frac{\alpha_1 \varrho_1}{\alpha_3 \varrho_3}\right) \leq \mathrm{ord}_{\mathfrak{p}_1}\left(\left(\frac{-\alpha_2 \varrho_2}{\alpha_3 \varrho_3}\right)^h - 1\right). \tag{44}$$

To apply Proposition 7 to the right side of (44) we have to make some preparation. In view of (39) and (40) we can write

$$\left(-\frac{\alpha_2\varrho_2}{\alpha_3\varrho_3}\right)^h = \eta_1^{b_1}\cdots\eta_r^{b_r}\pi_2^{b_{r+2}}\cdots\pi_t^{b_{r+t}}(\pi_1^{b_{r+1}}\beta_{r+t}), \tag{45}$$

where $b_1,\dots,b_{r+t}$ are rational integers and $\beta_{r+t} = \zeta'(-\alpha_2/\alpha_3)^h$ with an appropriate root of unity $\zeta'$. Further, by virtue of (41) and $h < 2c_{27}\mathcal{Z}$ if $r \geq 1$, we obtain

$$\max_{1\leq j\leq r+t,\, j\neq r+1} |b_j| \leq hB + 2c_{27}\mathcal{Z}. \tag{46}$$

We infer from (44) that $(-\alpha_2\varrho_2/(\alpha_3\varrho_3))^h$ is a $\mathfrak{p}_1$-adic unit. Then, by (45), $\pi_1^{b_{r+1}}\beta_{r+t}$ is also a $\mathfrak{p}_1$-adic unit, that is, we have

$$\operatorname{ord}_{\mathfrak{p}_1}\beta_{r+t} + h_1 b_{r+1} = 0. \tag{47}$$

Putting $\beta'_{r+t} = \beta_{r+t}\pi_1^{b_{r+1}}$ and using (43) and (47), we obtain

$$h(\beta'_{r+t}) \leq \frac{6}{h_1\log N(\mathfrak{p}_1)}\max\{h_1\log N(\mathfrak{p}_1), c_3 dR\}hH. \tag{48}$$

Here 6 may be replaced by 4 if $r = 0$. We note that

$$h''_{r+t} := \max\left\{h(\beta'_{r+t}), \frac{1}{8e^2 d}\right\}$$

has the same upper bound. Further, we recall that $\eta_1,\dots,\eta_r,\pi_2,\dots,\pi_t$ are $\mathfrak{p}_1$-adic units and are multiplicatively independent.

We are now in a position to apply Proposition 7. We may assume that in (46),

$$hB + 2c_{27}\mathcal{Z} \leq \frac{5}{4}hB =: B', \tag{49}$$

since otherwise we get at once a better upper bound for $B$, and hence also in (5), than required. For brevity, we write

$$\Pi = h(\eta_1)\cdots h(\eta_r)h(\pi_2)\cdots h(\pi_t).$$

We may assume that

$$B' \geq c'_{24}(r+t,d)e^{4(r+t+1)}P^{2/3}h''_{r+t}\max\{\Pi,1\}, \tag{50}$$

since otherwise we again obtain a better upper bound for $B$ than required. We choose

$$\delta = \frac{c'_{24}(r+t,d)h''_{r+t}\Pi}{B'}.$$

Then, by (50), we have $0 < \delta < 1/2$. Proposition 7 gives the upper bound

$$c'_{23}(r+t,d)d^{r+t}\,\frac{N(\mathfrak{p}_1)}{(\log N(\mathfrak{p}_1))^2}\,\Pi h''_{r+t}\log M'$$

for the right side of (44), where

$$M' := \frac{c'_{25}(r+t,d)}{c'_{24}(r+t,d)}\, N(\mathfrak{p}_1)^{r+t+1}\, \frac{B'}{h''_{r+t}}.$$

In view of (50) we have

$$\log M' \le 2(r+t+1)\log\left(\frac{B'}{h''_{r+t}}\right).$$

Set

$$\mathcal{T}' = 2^{t-1}\prod_{i=2}^{t}\max\{h_i \log N(\mathfrak{p}_i), c_3 dR\},$$

where $2^{t-1}$ may be replaced by 1 if $r=0$. Using now (35), (45), (46), (48), (49), (50) and Lemma 2 we deduce that

$$c_{28}\,\frac{N(\mathfrak{p}_1)}{(\log N(\mathfrak{p}_1))^2}\, R\mathcal{T}' h''_{r+t}\log\left(\frac{B'}{h''_{r+t}}\right)$$

is an upper bound for the right side of (44), where

$$c_{28} = 2(r+t+1)d^{r+1}c^\star_{18}c'_{23}(r+t,d).$$

Here $c^\star_{18}$ denotes the constant $c_{18}$ with the choice $s=r+1$, i.e. $c^\star_{18} = (r!)^2/(2^{r-1}d^r)$ if $r\ge 1$, and $c^\star_{18}=1$ if $r=0$. In view of (44) we infer that

$$V < c_{29}\,\frac{N(\mathfrak{p}_1)}{(\log N(\mathfrak{p}_1))^2}\, R\mathcal{T}' h''_{r+t}\log\left(\frac{B'}{h''_{r+t}}\right), \tag{51}$$

where $c_{29} = 1.25c_{28}$.

If $V=B$ then (51) and (49) yield

$$\frac{Y_2}{\log Y_2} < 1.25c_{29}\,\frac{N(\mathfrak{p}_1)}{(\log N(\mathfrak{p}_1))^2}\, hR\mathcal{T}' =: M_2$$

for $Y_2 = B'/h''_{r+t}$. Now $M_2 > 1.53\cdot 10^6$ if $r=0$ and $M_2 > 2.41\cdot 10^{11}$ if $r\ge 1$. Thus

$$Y_2 < 1.2\, M_2 \log M_2,$$

where 1.2 can be replaced by 1.13 if $r\ge 1$. By the definition of $\mathcal{T}$ in Section 2, the definition of $\mathcal{T}'$ and (48), we have

$$\mathcal{T}' h''_{r+t} \le 4\mathcal{T} hH/\log N(\mathfrak{p}_1),$$

where 4 can be replaced by 3 if $r\ge 1$. Observe further that

$$\log M_2 < 0.646(\log c_{29})(\log^* \mathcal{P} + \log^* \mathcal{R}),$$

and that

$$\frac{N(\mathfrak{p}_1)}{(\log N(\mathfrak{p}_1))^3} < \frac{2}{19}\left(\frac{\log 19}{\log 2}\right)^3 \frac{P}{(\log^* P)^3}.$$

It follows from the above estimates that

$$B < c_{30}\,\frac{P}{(\log^* P)^3}\,Rh(\log^* P+\log^*\mathcal{R})\mathcal{T}H, \tag{52}$$

where $c_{30}=25.02c_{29}(\log c_{29})$, and 25.02 may be replaced by 17.68 if $r\ge 1$.

If in particular $r=0$, then obviously $V=B$. In this case $c_3=0$, $R=1$, $\mathcal{R}=h$, and the right-hand side of (52) is greater than $4.59\cdot 10^8$. Thus using (39), (35), (36), we obtain

$$\begin{aligned}\max_k h(\varrho_k) &< (B+1)\,\frac{t}{d}\,h\log^* P\\ &< (1+2.18\cdot 10^{-9})c_{30}\,\frac{t}{d}\,\frac{P}{(\log^* P)^2}\,h^2(\log^* P+\log^* h)\mathcal{T}H,\end{aligned}$$

which yields the bound (6) for $\max_k h(\varrho_k)$, since

$$(1+2.18\cdot 10^{-9})c_{30}t < c_5(d,t)d$$

(in fact the left-hand side of the above inequality reaches its maximum $0.789\ldots$ at $r=0$, $d=2$, $t=13$). If $r\ge 1$ and $B=V$, then (39), (35), (36), Lemma 2(ii) and (52) imply

$$\begin{aligned}\max_k h(\varrho_k) &< (B+1)(2t/d+r!/(d\,2^{r-1}))\mathcal{R}\log^* P\\ &< c_{31}\,\frac{P}{(\log^* P)^2}\,Rh\mathcal{R}(\log^* P+\log^*\mathcal{R})\mathcal{T}H,\end{aligned} \tag{53}$$

where $c_{31}=(4t/d)(r!/2^{r-1})c_{30}$.

There remains the case $r\ge 1$ with $B=W$. In this case we shall use (51). We reduce equation (1.b) to the case $t=0$ of equation (1.a). Let

$$\alpha=-\zeta_1\left(\frac{\alpha_1\gamma_1}{\alpha_3\gamma_3}\right)\prod_{i=1}^{t}\pi_i^{v'_{i1}-v'_{i3}},\qquad \beta=-\zeta_2\left(\frac{\alpha_2\gamma_2}{\alpha_3\gamma_3}\right)\prod_{i=1}^{t}\pi_i^{v'_{i2}-v'_{i3}}.$$

Then

$$x=\eta_1^{w_{11}}\cdots\eta_r^{w_{r1}},\qquad y=\eta_1^{w_{12}}\cdots\eta_r^{w_{r2}}$$

is a solution of equation (1.a) in $x$, $y\in O_K^*$. We may assume that $h(x)\ge h(y)$. Then we deduce as in (28) that $B=W\le 2dc_{20}^\star h(x)$, where $c_{20}^\star = c_{20}(d,r+1)=((r!)^2/2^r)(\log(3d))^3$. We may assume further that $h(x)\ge 2.5hdc_{20}^\star h''_{r+t}$. Thus we have, by (49), $B'/h''_{r+t}\le (h(x)/h''_{r+t})^2$. In view of (35), (36), (42) and (51) we obtain

$$\begin{aligned}\max\{h(\alpha),h(\beta)\} &\le 2H+(t+1+Vt)(2/d)\mathcal{R}\log^* P\\ &\le V\,\frac{2t}{d}\,\mathcal{R}\log^* P\left(\frac{1}{17t}+\frac{t+1}{68t}+1\right)\le \frac{37}{17}\,\frac{t}{d}\,V\mathcal{R}\log^* P\\ &< 4.353c_{29}\,\frac{t}{d}\,\frac{N(\mathfrak{p}_1)\log^* P}{(\log N(\mathfrak{p}_1))^2}\,R\mathcal{R}\mathcal{T}'h''_{r+t}\log\left(\frac{h(x)}{h''_{r+t}}\right)=:H'.\end{aligned} \tag{54}$$

With the above choice of $\alpha, \beta$ we can now apply the case $t = 0$ of our Theorem 1 to equation (1.a) and we get

$$\begin{aligned} h(x) &< c_2(d,r)R(\log^* R)H' \\ &\le c_{32}\,\frac{N(\mathfrak{p}_1)\log^* P}{(\log N(\mathfrak{p}_1))^2}\,R^2(\log^* R)\mathcal{R}\mathcal{T}'h''_{r+t}\log\left(\frac{h(x)}{h''_{r+t}}\right) \end{aligned}$$

with $c_{32} = 4.353(t/d)c_{29}c_2(d,r)$, where $c_2(d,r)$ denotes the constant occurring in (3). This implies that, with $Y_3 := h(x)/h''_{r+t}$,

$$\frac{Y_3}{\log Y_3} < c_{32}\,\frac{N(\mathfrak{p}_1)\log^* P}{(\log N(\mathfrak{p}_1))^2}\,R^2(\log^* R)\mathcal{R}\mathcal{T}' =: M_3.$$

On noting $M_3 > 1.74\cdot 10^{28}$, we get $Y_3 < 1.066 M_3\log M_3$. Observing further that

$$\log M_3 < 0.646(\log c_{31})(\log^* P + \log^* \mathcal{R}), \qquad \mathcal{T}'h''_{r+t} \le 3\mathcal{T}hH/\log N(\mathfrak{p}_1),$$
$$N(\mathfrak{p}_1)/(\log N(\mathfrak{p}_1))^3 < (2/19)(\log 19/\log 2)^3 P/(\log^* P)^3,$$

we obtain

$$h(x) < c_{33}\,\frac{P}{(\log^* P)^2}\,hR^2(\log^* R)\mathcal{R}(\log^* P + \log^* \mathcal{R})\mathcal{T}H =: X_0, \tag{55}$$

where $c_{33} = 16.67c_{32}\log c_{32}$. Putting

$$\tau_k = \zeta_k\gamma_k\pi_1^{v'_{1k}}\cdots\pi_t^{v'_{tk}}, \qquad k = 1,2,3,$$

we have

$$\varrho_1 = x\tau_1, \qquad \varrho_2 = y\tau_2, \qquad \varrho_3 = \tau_3. \tag{56}$$

Fix $k \in \{1,2,3\}$. If $h(\tau_k) \le h(x)$, then (56) and (55) give

$$h(\varrho_k) \le 2X_0 \quad \text{for this } k. \tag{57}$$

Now suppose that $h(\tau_k) > h(x)$. We deduce as in (54) that

$$\begin{aligned} h(\tau_k) &\le (t+1+2Vt)(1/d)\mathcal{R}\log^* P \\ &\le V\,\frac{t}{d}\,\mathcal{R}\log^* P\left(\frac{t+1}{68t} + 2\right) \le \frac{69}{34}\,\frac{t}{d}\,V\mathcal{R}\log^* P \\ &< 4.06c_{29}\,\frac{t}{d}\,\frac{N(\mathfrak{p}_1)\log^* P}{(\log N(\mathfrak{p}_1))^2}\,R\mathcal{R}\mathcal{T}'h''_{r+t}\log\left(\frac{h(\tau_k)}{h''_{r+t}}\right) \\ &< c_{32}\,\frac{N(\mathfrak{p}_1)\log^* P}{(\log N(\mathfrak{p}_1))^2}\,R^2(\log^* R)\mathcal{R}\mathcal{T}'h''_{r+t}\log\left(\frac{h(\tau_k)}{h''_{r+t}}\right) \end{aligned} \tag{58}$$

since $c_2(d,r)R\log^* R > 1$ by the fact that $R > 0.2052$ (cf. [7]). As before, this gives $h(\tau_k) < X_0$. Hence we obtain (57) again. On combining (57) with (53) and noting that

$$c_{31}/R \le c_{31}/0.2052 \le 2c_{33} < c_4(d,r,t)$$

(in fact $2c_{33}/c_4(d,r,t)$ reaches its maximum $0.59\ldots$ at $r=1$, $d=2$, $t=13$), we see that (5) holds when $r \geq 1$. It is readily seen that the right-hand side of (5) with $r=0$ is greater than the quantity in (6). Thus Theorem 2 follows. ∎

*Proof of Corollary 1.* The assertion with the bounds (8.a) and (8.c) follows from Theorem 1 in the same way as the Corollary was deduced from the Theorem in [5], but using the fact that, by (7),

$$R_S \geq 0.2052(\log 2)^t \geq 0.2052(\log 2)^s.$$

Next suppose that $t>0$. If the bound in (8.b) is greater than that in (8.a) then we are done. Consider now the case when the bound (8.b) does not exceed (8.a). Let $x_1, x_2, x_3$ be a solution of (1.c). Then, by Lemma 3, there are $\varepsilon_k \in O_S^*$ such that

$$h(x_k/\varepsilon_k) \leq \mathcal{N} - H, \qquad k = 1,2,3.$$

Now $\varepsilon_1, \varepsilon_2, \varepsilon_3$ satisfy

$$\beta_1\varepsilon_1 + \beta_2\varepsilon_2 + \beta_3\varepsilon_3 = 0,$$

where $\beta_k = \alpha_k x_k/\varepsilon_k$, $k=1,2,3$. Note that $h(\beta_k) \leq h(\alpha_k) + h(x_k/\varepsilon_k) \leq \mathcal{N}$. By Theorem 2, there exists $\varepsilon \in O_S^*$ such that

$$h(\varepsilon\varepsilon_k) < 0.6c_4hR^2(\log^* R)\mathcal{R}(1+(\log^*\mathcal{R})/\log^* P)(P/\log^* P)\mathcal{T}\mathcal{N} \qquad (k=1,2,3).$$

Here for the factor 0.6 see the end of the proof of Theorem 2. Thus for $k=1,2,3$ we have

$$\begin{aligned} h(\varepsilon x_k) &\leq h(\varepsilon\varepsilon_k) + h(x_k/\varepsilon_k) \\ &< c_4hR^2(\log^* R)\mathcal{R}(1+(\log^*\mathcal{R})/\log^* P)(P/\log^* P)\mathcal{T}\mathcal{N}, \end{aligned}$$

which completes the proof of Corollary 1. ∎

*Proof of Corollary 2.* In view of Theorem 2 it suffices to deduce (9) from (5) for $r \geq 1$ and from (6) for $r=0$. We have

$$hR \leq |D|^{1/2}(\log^* |D|)^{d-1}. \tag{59}$$

This can be seen as follows. If $K=\mathbb{Q}$ or $K=\mathbb{Q}(\sqrt{-3})$, we have $h=R=1$, and $D=1$ or $D=-3$, respectively, hence (59) holds trivially. The remaining cases of (59) follow from (2) in [23] and

$$\omega_K \leq 20d\log_2 d \qquad \text{if } d \geq 3, \tag{60}$$

where $\omega_K$ denotes the number of roots of unity in $K$. Since Euler's function $\phi(\omega_K)$ divides $d$, (60) is an immediate consequence of [27, Theorem 15].

We treat first the case $r \geq 1$. Using the notation of Theorem 2, we infer from (59) and $R \geq 0.2052$ that

$$\mathcal{R} \leq c_{34}|D|^{1/2}(\log^* |D|)^{d-1} \tag{61}$$

with $c_{34} = \max\{c_3 d, 4.88\}$, and

$$\mathcal{T} \le (c_{35}|D|^{1/2}(\log^*|D|)^{d-1})^t \prod_{i=1}^{t} \log N(\mathfrak{p}_i), \tag{62}$$

where $c_{35} = (2/\log 2)c_{34}$. Further it follows from (59) and (61) that

$$hR^2(\log^* R)\mathcal{R}\left(1 + \frac{\log^* \mathcal{R}}{\log^* P}\right) \le 4d^2 c_{34}(\log c_{34})|D|^{3/2}(\log^*|D|)^{3d-1}; \tag{63}$$

here we have used the facts that $1.5 + (d-1)/e \le 1.5d$ and $\log c_{34} + 0.5 + (d-1)/e \le 1.32 d \log c_{34}$.

Denote by $t_0$ the number of distinct prime factors of $Q = N(\mathfrak{p}_1 \cdots \mathfrak{p}_t)$. Then $t \le dt_0$. It follows from explicit estimates in [27] or [26] that

$$t_0 < 1.5\,\frac{\log Q_0}{\log_2 Q_0^*}. \tag{64}$$

Further, from (64), $t \le dt_0$, and

$$\prod_{i=1}^{t} \log N(\mathfrak{p}_i) \le \left(\frac{\log Q}{t}\right)^t \le \left(\frac{d\log Q_0}{t}\right)^t,$$

it follows that

$$\prod_{i=1}^{t} \log N(\mathfrak{p}_i) \le Q_0^{19.16d(\log_3 Q_0^*)/\log_2 Q_0^*}. \tag{65}$$

Indeed, let

$$\begin{aligned}\eta &= t\log\left(\frac{d\log Q_0}{t}\right)\left(\frac{d(\log Q_0)\log_3 Q_0^*}{\log_2 Q_0^*}\right)^{-1}\\ &= \left(\frac{d\log Q_0}{t}\right)^{-1}\log\left(\frac{d\log Q_0}{t}\right)\frac{\log_2 Q_0^*}{\log_3 Q_0^*}.\end{aligned}$$

If $\log_2 Q_0^* < 1.5e$, then

$$\eta \le \frac{1}{e}\max\left(\frac{\log_2 16}{\log_3 16}, \frac{1.5e}{\log(1.5e)}\right) < 19.16.$$

In the opposite case we have

$$\frac{d\log Q_0}{t} \ge \frac{\log Q_0}{t_0} \ge \frac{\log_2 Q_0^*}{1.5} \ge e.$$

Hence

$$\left(\frac{d\log Q_0}{t}\right)^{-1}\log\left(\frac{d\log Q_0}{t}\right) \le \left(\frac{\log_2 Q_0^*}{1.5}\right)^{-1}\log\left(\frac{\log_2 Q_0^*}{1.5}\right) \le 1.5\,\frac{\log_3 Q_0^*}{\log_2 Q_0^*}$$

and $\eta \le 1.5$. Thus we get (65).

Now using the facts that $r \geq 1$, $t \geq 1$, whence $r+t+1 \leq (r+1)(t+0.5)$ and $(t+0.5)^{3.5} \leq 2^{2.32t}$, we see that

$$c_4 \leq (2^{12.32}d)^t c_{36} \quad \text{with } c_{36} = (r+1)^{4r+13.5} 2^{10r+63} d^{r+5} (\log^*(2d))^6$$

and

$$\log(2^{12.32} d c_{35} |D|^{1/2} (\log^* |D|)^{d-1}) \leq 2.42 d (\log c_{35}) \log^* |D|. \tag{66}$$

By $t \leq dt_0$, (64) and (66), we obtain

$$(2^{12.32} d c_{35} |D|^{1/2} (\log^* |D|)^{d-1})^t \leq Q_0^{3.63 d^2 (\log c_{35})(\log^* |D|)/\log_2 Q_0^*}. \tag{67}$$

Let $c_7' = 3.63 d \log c_{35}$. Then the product of the left-hand sides of (67) and (65) does not exceed

$$Q_0^{d(c_7' \log^* |D| + 19.16 \log_3 Q_0^*)/\log_2 Q_0^*}.$$

If $r = 1$, then $c_{35} = 9.76/\log 2$, while if $r \geq 2$, then $\log c_{35} \leq 2.43 d \log d$ (here we used the fact that $r+1 \leq d$). Thus $c_7' \leq c_7$ if $r \geq 1$.

Let

$$c_6' = 4d^2 c_{34} (\log c_{34}) c_{36}.$$

We recall that $c_{34} = 4.88$ if $r = 1$, and $c_{34} = c_3 d$ if $r \geq 2$. Hence we have $\log c_{34} \leq 1.3 d (\log 2d)$ if $r \geq 2$. It is readily verified that $c_6' \leq c_6$ if $r \geq 1$.

Summing up, we obtain (9) for the case $r \geq 1$ from inequality (5) in Theorem 2. The results for the cases $d = 1$ and $d = 2$ with $r = 0$ can be deduced from inequality (6) in Theorem 2 and we omit the details here. ∎

Denote by $\overline{|\alpha|}$ the maximum absolute value of the conjugates of an algebraic number $\alpha$, and by $\mathrm{den}(\alpha)$ the denominator of $\alpha$. The fact will be used in the next proofs that $\overline{|\alpha+\beta|} \leq \overline{|\alpha|} + \overline{|\beta|}$, $\overline{|\alpha\beta|} \leq \overline{|\alpha|}\,\overline{|\beta|}$ for $\alpha, \beta \in K$, $h(\alpha) \leq \log \overline{|\alpha|} \leq d h(\alpha)$ for $\alpha \in O_K \setminus \{0\}$ and $\mathrm{den}(\alpha) \leq \exp\{d h(\alpha)\}$ for $\alpha \in K$.

*Proof of Theorem 3.* In fact we follow the proof of Theorem 1 of [17] with some modifications, corresponding to the refined assumptions on $\mathcal{L}_F$ introduced in Section 3. Moreover, to obtain as good upper bounds as possible, we shall need more detailed deduction. Hence we give here a self-contained proof for our theorem.

Multiplying (10) by the product of the denominators of the coefficients of the linear factors of $F$, we can write (10) in the form

$$\prod_{i=1}^{n} l_i(\mathbf{x}) = \beta,$$

where the linear forms $l_i(\mathbf{X})$ already have integral coefficients in $K$ with heights $A_1 = (md+1)A$ and $\beta$ is of height at most $B_1 = mndA + B$. We may assume that $\beta \in O_S$, since otherwise our equation is not solvable.

Let $\mathbf{x} = (x_1, \ldots, x_m) \in O_S^m$ be a solution of (10) (with $l(\mathbf{x}) \neq 0$ for all $l \in \mathcal{L}'_F$ if $k > 1$). Put $l_i(\mathbf{x}) = \beta_i$ for $i = 1, \ldots, n$. Let $\mathcal{L}_F^*$ be a maximal subset of pairwise linearly independent linear forms from $\mathcal{L}_F$, and consider the vertex sets $\mathcal{L}_1, \ldots, \mathcal{L}_k$ of the connected components of $\mathcal{G}(\mathcal{L}_F^*)$. Then $\mathcal{L}_1, \ldots, \mathcal{L}_k$ is a partition of $\mathcal{L}_F^*$.

For $j$ with $1 \leq j \leq k$, denote by $\mathcal{I}_j$ the set of $i$ with $l_i \in \mathcal{L}_j$ and by $n_j$ the cardinality of $\mathcal{I}_j$. Then either $n_j \geq 3$ or $n_j = 1$. If $n_j \geq 3$ and $l_{i_1}, l_{i_2} \in \mathcal{L}_j$ are connected by an edge in $\mathcal{G}(\mathcal{L}_j)$, then there are $l_{i_{1,2}} \in \mathcal{L}_j$ and non-zero $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_{1,2}}$ in $O_K$ with heights at most $H := 4A_1 + \log 2$ such that $\lambda_{i_1} l_{i_1} + \lambda_{i_2} l_{i_2} + \lambda_{i_{1,2}} l_{i_{1,2}} = 0$, whence

$$\lambda_{i_1}\beta_{i_1} + \lambda_{i_2}\beta_{i_2} + \lambda_{i_{1,2}}\beta_{i_{1,2}} = 0. \tag{68}$$

For each $i$ with $1 \leq i \leq n$, $\beta_i$ is an $S$-integer which divides $\beta$ in $O_S$, hence $N_S(\beta_i) \leq N_S(\beta) \leq \exp\{dB_1\} = N$. By applying Corollary 1 to equation (68) we infer that there is an $\eta_j \in O_S^*$ such that

$$\max_{q=1,2} h(\eta_j \beta_{i_q}) \leq E'_1,$$

where $E'_1$ denotes the bound from (8.a) or (8.b) for $t > 0$ and the bound from (8.c) for $t = 0$ with

$$\mathcal{N} = c_3 R + \frac{h}{d}\log Q + H + \frac{1}{d}\log N,$$

where $H$ and $N$ are given above. It is easy to see that for $t \geq 0$,

$$\mathcal{N} \leq 3.12\left(c_3 R + \frac{h}{d}\log Q + mndA + B\right). \tag{69}$$

Write $E_1$ for $E'_1$ with $\mathcal{N}$ replaced by its upper bound in (69).

If now $l_{i_2}, l_{i_3}$ are also connected by an edge in $\mathcal{G}(\mathcal{L}_j)$, then we deduce in the same way that $\max_{q=2,3} h(\varepsilon\beta_{i_q}) \leq E_1$ with some $\varepsilon \in O_S^*$, whence it follows that

$$\max_{1 \leq q \leq 3} h(\eta_j \beta_{i_q}) \leq 3E_1.$$

Using the connectedness of $\mathcal{G}(\mathcal{L}_j)$, we proceed as follows. Given any $l_i$ in $\mathcal{L}_j$, we choose the shortest path $\mathcal{P}$ between $l_{i_1}$ and $l_i$. If $\mathcal{P}$ goes through $l_{i_2}$, then by repeating the above procedure we obtain $h(\eta_j\beta_i) \leq (2n_j - 3)E_1$. If $\mathcal{P}$ does not go through $l_{i_2}$, then we use the path between $l_{i_2}$ and $l_i$, which is $[l_{i_2}, l_{i_1}]$ combined with $\mathcal{P}$. In this way we get $h(\eta_j\beta_i) \leq (2n_j - 3)E_1$ again. Hence we have

$$h(\eta_j\beta_i) \leq \max(2n_j - 3, 1)E_1 \quad \text{for } i \in \mathcal{I}_j. \tag{70}$$

If $n_j = 1$, then, by Lemma 3, there is also an $\eta_j \in O_S^*$ with a bound for $h(\eta_j\beta_i)$ smaller than $E_1$. Hence (70) holds for each $j$ with $1 \leq j \leq k$ and for each $i \in \mathcal{I}_j$.

We now consider the case $k = 1$. Thus $n \geq n_1 \geq 3$. If $l_{i'} \in \mathcal{L}_F \setminus \mathcal{L}_F^*$ is proportional to a linear form $l_i \in \mathcal{L}_F^*$, then $l_{i'} = \delta l_i$ with some non-zero $\delta \in K$ of height at most $2A_1$. Then $\beta_{i'} = \delta\beta_i$, and so (70) implies

$$h(\eta_1\beta_i) \leq (2n-3)E_1 + 2A_1 \quad \text{for } i = 1, \ldots, n. \tag{71}$$

Then it follows that

$$h(\eta_1) \leq \frac{1}{n}\, h\biggl(\frac{(\eta_1\beta_1)\cdots(\eta_1\beta_n)}{\beta_1\cdots\beta_n}\biggr) \leq (2n-3)E_1 + 2A_1 + \frac{1}{n}\, B_1.$$

Together with (71) this gives

$$h(\beta_i) \leq (4n-6)E_1 + 7mdA + \frac{1}{n}\, B = E_2 \quad \text{for } i = 1, \ldots, n.$$

We may assume, without loss of generality, that $l_1, \ldots, l_m$ are linearly independent. Denote by $\mathcal{A}$ the $m \times m$ matrix whose $i$th row consists of the coefficients, say $a_{i1}, \ldots, a_{im}$, of $l_i$. Then

$$\mathcal{A}(x_1, \ldots, x_m)^\tau = (\beta_1, \ldots, \beta_m)^\tau, \tag{72}$$

where $\tau$ signifies matrix transposition. Since $\det \mathcal{A} \in O_K$, we infer that

$$\begin{aligned} h(\det \mathcal{A}) &\leq \log \overline{\left|\sum a_{1i_1}\cdots a_{mi_m}\right|} \leq \log(\max \overline{|a_{1i_1}\cdots a_{mi_m}|}) + \log(m!) \\ &\leq mdA_1 + \log(m!) = A_2. \end{aligned}$$

Let $\mathcal{A}_i$ be the $m \times m$ matrix obtained by replacing the $i$th column of $\mathcal{A}$ by $(\beta_1, \ldots, \beta_m)^\tau$. Expanding $\det \mathcal{A}_i$ by its $i$th column, we have

$$\det \mathcal{A}_i = \beta_1 C_{1i} + \beta_2 C_{2i} + \cdots + \beta_m C_{mi},$$

where $C_{ji}$ is the $(j,i)$-cofactor of $\mathcal{A}_i$ and hence

$$h(C_{ji}) \leq (m-1)dA_1 + \log((m-1)!) = A_3$$

for $1 \leq j \leq m$. We deduce that

$$h(\det \mathcal{A}_i) \leq m(E_2 + A_3) + \log m.$$

Hence we get, for each $i$,

$$h(x_i) = h(\det \mathcal{A}_i / \det \mathcal{A}) \leq m(E_2 + A_3) + A_2 + \log m. \tag{73}$$

If $E_1$ denotes the bound from (8.a), one can show by careful computation that each of $mdA$, $mdA_1$, $n^{-1}B$, $\log(m!)$ is smaller than $10^{-14}E_1$. Thus it follows from (73) that

$$h(x_i) < 4m(n-1)E_1 \quad \text{for } i = 1, \ldots, m.$$

This gives (11) for $k = 1$. Using the bounds $E_1$ from (8.b) and (8.c), one can deduce in a similar manner (12) and (13) for $k = 1$.

Now we treat the case $k > 1$. By assumption (ii) of Theorem 3, the graph $\mathcal{H}(\mathcal{L}_1, \ldots, \mathcal{L}_k)$ is connected. Assume, for convenience, that $[\mathcal{L}_1, \mathcal{L}_2]$ is an edge in this graph. Then there is a non-zero $l_{1,2} \in \mathcal{L}_F'$ which can be represented in the form

$$\sum_{i\in\mathcal{I}_1}\lambda_i l_i=\sum_{i\in\mathcal{I}_2}\lambda_i l_i, \tag{74}$$

such that the total number of non-zero $\lambda_i\in K$ in both sides of (74) is minimal. Denote by $n'_1$ and $n'_2$, respectively, the number of non-zero $\lambda_i$ in these sums. Putting $m'=n'_1+n'_2$, it is easy to see that among the linear forms $l_i$ in (74) with non-zero coefficients exactly $m'-1$ are linearly independent, whence $m'\le m+1$. Note that $m'\ge 4$, since $\mathcal{L}_1$ and $\mathcal{L}_2$ are the vertex sets of distinct connected components of $\mathcal{G}(\mathcal{L}_F^*)$. Comparing the coefficients of $x_1,\dots,x_m$ in (74), we obtain a homogeneous linear system of $m'-1$ linearly independent equations in $m'$ unknowns $\lambda_i$, among which exactly one, say $\lambda_{i_0}$, is a free variable. Moving $\lambda_{i_0}$ to the right-hand side of each equation, we obtain a system of $m'-1$ linearly independent equations in $m'-1$ unknowns, with the coefficient matrix denoted by $\mathcal{A}'$. Setting $\lambda_{i_0}=-\det\mathcal{A}'$, this system of linear equations determines uniquely the values $\lambda_i\in O_K\setminus\{0\}$ for which $h(\lambda_i)\le A_2$. With this set of $\lambda_i$'s, the two linear combinations in (74) are equal to $\lambda_{1,2}l_{1,2}$ for some $\lambda_{1,2}\in K\setminus\{0\}$.

For the solution $\mathbf{x}$ considered above we deduce from (74) and (70) that

$$h(\eta_q\lambda_{1,2}l_{1,2}(\mathbf{x}))\le n'_q(A_2+(2n_q-1)E_1)+\log n'_q\quad\text{for } q=1,2.$$

But $l_{1,2}(\mathbf{x})\ne 0$, hence it follows that

$$h(\eta_1/\eta_2)\le(m+1)A_2+m((2n_1-1)+(2n_2-1))E_1+2\log m=E_3.$$

In view of (70) this implies

$$h(\eta_1\beta_i)\le E_3+(2n_2-1)E_1\quad\text{for each } i\in\mathcal{I}_2.$$

Using the fact that $\mathcal{H}(\mathcal{L}_1,\dots,\mathcal{L}_k)$ is connected and repeating this procedure with the shortest path connecting two vertices, we infer that

$$\begin{aligned}h(\eta_1\beta_i)\le{}&(m(4n-2k-2)+2n-2k+1)E_1\\&+(k-1)(m+1)A_2+2(k-1)\log m=E_4\end{aligned}$$

for each $i$ in $\mathcal{I}_1\cup\dots\cup\mathcal{I}_k$. It follows as above in the case $k=1$ that $h(\eta_1\beta_i)\le E_4+2A_1=E_5$ for $i=1,\dots,n$, and so

$$h(\beta_i)\le 2E_5+\frac{1}{n}B_1=E_6\quad\text{for } i=1,\dots,n.$$

We now infer in the same way as in the case $k=1$ that for $i=1,\dots,m$,

$$h(x_i)\le m(E_6+A_3)+A_2+\log m. \tag{75}$$

We deduce from (75) with careful computation that

$$\max_{1\le i\le m}h(x_i)\le 8m(m+1/2)(n-1)E_1.$$

Finally, this implies (11), (12) or (13) according as $E_1$ is from (8.a), (8.b) or (8.c). ∎

*Proof of Corollary 3.* We follow the proof of Corollary 1.1 in [16], but we use here our Theorem 3 in place of Theorem 1 of [16] and we work with logarithmic height instead of the usual height $H(\cdot)$.

There is an $a \in \mathbb{Z}$ with $1 \le a \le n$ such that $F(1,a) \neq 0$. Consider the binary form $G(X,Y) = F(X, aX+Y)$ in which the coefficient of $X^n$ is $F(1,a) \neq 0$ and the heights of the coefficients of $G$ do not exceed $(n+1) \times (A + n\log n) + \log(n+1) = A_1$. Denoting by $d_0$ the product of the denominators of the coefficients of $G$, we can write

$$d_0 G(X,Y) = a_0 X^n + a_1 X^{n-1} Y + \cdots + a_n Y^n = a_0(X - \alpha_1 Y)\cdots(X - \alpha_n Y) \tag{76}$$

where $a_0, \ldots, a_n$ are already integers in $K$ with heights not exceeding $d(n+1)A_1 + A_1 = A_2$. Further, at least three from among $\alpha_1, \ldots, \alpha_n$ are pairwise distinct.

We infer that for each solution $x_1, x_2$ of (14),

$$x = a_0 x_1, \qquad y = -a x_1 + x_2 \tag{77}$$

is a solution of the equation

$$(x - a_0\alpha_1 y)\cdots(x - a_0\alpha_n y) = \beta', \tag{78}$$

where $\beta' = d_0 a_0^{n-1}\beta$. It follows from (76) that $a_0\alpha_i \in O_K$ and

$$(a_0\alpha_i)^n + a_1 a_0 (a_0\alpha_i)^{n-1} + \cdots + a_n a_0^{n-1} = 0 \tag{79}$$

for each $i$. Put $\max_{0\le i\le n} \overline{|a_i|} = A_0$. Then (79) implies that $\overline{|a_0\alpha_i|} \le nA_0^n$, whence

$$h(a_0\alpha_i) \le ndA_2 + \log n = A_3 \quad \text{for each } i.$$

Further,

$$h(\beta') \le d(n+1)A_1 + (n-1)A_2 + B = B_1.$$

Applying now our Theorem 3 to (78), we obtain

$$\max\{h(x), h(y)\} \le E_1, \tag{80}$$

where $E_1$ denotes the bound in (11) or (12) for $t > 0$ and (13) for $t = 0$, with the choice $k = 1$ and with $A$ and $B$ replaced by $A_3$ and $B_1$, respectively. It follows from (80) and (77) that

$$\max\{h(x_1), h(x_2)\} \le 2E_1 + A_2 + \log 2n.$$

But it is easy to see that

$$2ndA_3 + B_1 \le 2.45 d^2 n^5 (2ndA + B),$$

hence $x_1, x_2$ satisfy (11), (12) for $t > 0$ and (13) for $t = 0$, with $c_i'$ for $k = 1$ replaced by $5d^2n^5c_i'$ for $i = 1, 4, 2$. ∎

*Proof of Corollary 4.* Below, $c_{37}, \ldots, c_{43}$ will denote effectively computable positive constants which depend at most on $F$, $K$ and $N_0$. Let

$\mathbf{x} = (x_1, \ldots, x_m) \in O_K^m$ with $N((x_1, \ldots, x_m)) \le N_0$. Suppose that $F(\mathbf{x}) \neq 0$ and that $l(\mathbf{x}) \neq 0$ for $l \in \mathcal{L}'_F$ if $k > 1$. Denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ the distinct prime ideal divisors of $F(\mathbf{x})$, and by $S$ the set consisting of $S_\infty$ and of the finite places corresponding to $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Keeping the above notation, by Lemma 3 there is an $\varepsilon \in O_S^*$ such that

$$h(F(\varepsilon x_1, \ldots, \varepsilon x_m)) \le c_{37} \log Q. \tag{81}$$

Then (12) and (13) in Theorem 3 and, for $t > 0$, the inequality $\log Q \le t \log^* P$ imply that

$$\max_{1 \le i \le m} h(\varepsilon x_i) < c_{38} c_{39}^t P \prod_{i=1}^{t} \log^* N(\mathfrak{p}_i) := C_1. \tag{82}$$

For $t = 0$ this gives $\log N \le dc_{38}$, where $N = \max_{1 \le i \le m} |N_{K/\mathbb{Q}}(x_i)|$. Hence, if $\log N > dc_{38}$, then $t > 0$ must hold.

Inequality (82) implies that $-\operatorname{ord}_{\mathfrak{p}_j} \varepsilon x_i (\log 2) \le dC_1$ for each $i$ and $j$. Further, in view of $N((x_1, \ldots, x_m)) \le N_0$ we infer that for each $j$ there is an $i$ such that $\operatorname{ord}_{\mathfrak{p}_j} x_i \le \log N_0 / \log 2$. Thus $-\operatorname{ord}_{\mathfrak{p}_j} \varepsilon \le (dC_1 + \log N_0)/\log 2$ $:= C_2$ for all $j$. By Lemma 3 we can choose a $\varrho \in O_K \setminus \{0\}$ such that.

$$(\varrho) = (\mathfrak{p}_1 \cdots \mathfrak{p}_t)^{[C_2+1]h} \quad \text{and} \quad h(\varrho) \le c_{40} C_2 \log Q. \tag{83}$$

Then $\varrho\varepsilon \in O_K$ and, for each $i$, we have

$$\log |N_{K/\mathbb{Q}}(x_i)| \le \log |N_{K/\mathbb{Q}}(\varrho\varepsilon x_i)| \le dh(\varrho\varepsilon x_i) \le c_{41} C_2 \log Q. \tag{84}$$

If $N > N_1$ for a sufficiently large and effectively computable $N_1$ depending only on $F$, $K$ and $N_0$ then (16) follows from (84) and (82). For $t \le \log P / \log_2 P$, the first inequality of (17) is an immediate consequence of (16). It follows from Theorem 1 of [27] that $t \le c_{42} P / \log P$. Now the second inequality of (17) follows from (16), provided that $N_1$ is large enough. ∎

*Proof of Corollary 5.* Let $F(x_1, \ldots, x_m) = F_0$ with some $x_1, \ldots, x_m \in O_K$ such that $N((x_1, \ldots, x_m)) \le N_0$ (and with $l(x_1, \ldots, x_m) \neq 0$ for $l \in \mathcal{L}'_F$ if $k > 1$). Following the proof of Corollary 4 and using its notation, we deduce from (81) and (83) that

$$h(F(\varrho\varepsilon x_1, \ldots, \varrho\varepsilon x_m)) \le c_{44} C_1 (t+1) \log^* P, \tag{85}$$

where $P = P(F_0)$, $t = \omega(F_0)$ and $c_{44}$ is an effectively computable positive number which depends only on $F$, $K$ and $N_0$. But $\varrho\varepsilon \in O_K$, hence

$$\log N \le dh(F(\varrho\varepsilon x_1, \ldots, \varrho\varepsilon x_m)) \tag{86}$$

where $N = |N_{K/\mathbb{Q}}(F_0)|$. If now $N \ge N_2$ with a sufficiently large and effectively computable $N_2$, then (85) and (86) imply the required lower estimates for $P$. ∎

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