

## Mean values of Dirichlet polynomials and applications to linear equations with prime variables

by

STEPHEN KWOK-KWONG CHOI (Burnaby) and  
ANGEL V. KUMCHEV (Towson, MD)

**1. Introduction.** In this paper we study Dirichlet polynomials of the form

$$(1.1) \quad D(s, \chi) = \sum_{n \leq N} a_n \chi(n) n^{-s}$$

where  $\chi(n)$  is a Dirichlet character,  $s = \sigma + it$  is a complex variable, and  $a_n$  are (complex) coefficients. Such Dirichlet polynomials are an important tool in multiplicative number theory and there is a vast literature on the subject. In particular, one often needs estimates for mean values of the form

$$\sum_{\chi \in \mathcal{H}} \int_{-T}^T \left| \sum_{N < n \leq 2N} \Lambda(n) \chi(n) n^{-it} \right| dt,$$

where  $\Lambda(n)$  is the von Mangoldt function and the outer summation is over some family of characters, possibly to various moduli. Our main result is Theorem 1.1 below, which deals with the most common types of such averages.

Let  $m \geq 1$ ,  $r \geq 1$ , and  $Q \geq r$ . We consider a set  $\mathcal{H}(m, r, Q)$  of characters  $\chi = \xi\psi$  modulo  $mq$ , where  $\xi$  is a character modulo  $m$  and  $\psi$  is a primitive character modulo  $q$ , with  $r \leq q \leq Q$ ,  $r | q$ , and  $(q, m) = 1$ . Our result is as follows.

**THEOREM 1.1.** *Let  $m \geq 1$ ,  $r \geq 1$ ,  $Q \geq r$ ,  $T \geq 2$ ,  $N \geq 2$ , and  $\mathcal{H}(m, r, Q)$  be a set of characters as described above. Then*

$$(1.2) \quad \sum_{\chi \in \mathcal{H}(m, r, Q)} \int_{-T}^T \left| \sum_{N < n \leq 2N} \Lambda(n) \chi(n) n^{-it} \right| dt \ll (N + HN^{11/20}) L^C,$$

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where  $C$  is an absolute constant,

$$H = mr^{-1}Q^2T \quad \text{and} \quad L = \log HN.$$

REMARK 1. A possible choice for  $C$  is  $C = 1100$ , and we have organized the proof as to make this obvious. On the other hand, we have spent no effort to optimize our estimates in that regard, because it is clear that our method will never yield a result with a “respectable” value of  $C$ , such as  $C = 10$ , or even  $C = 100$ .

REMARK 2. Under the Generalized Riemann Hypothesis (GRH), we have

$$\sum_{\chi \in \mathcal{H}(m,r,Q)} \int_{-T}^T \left| \sum_{N < n \leq 2N} \Lambda(n) \chi(n) n^{-it} \right| dt \ll NL + HN^{1/2}L^2,$$

where the term  $NL$  on the right side occurs only when the set  $\mathcal{H}(m,r,Q)$  contains a principal character. In contrast, because Theorem 1.1 is derived from a general result on bilinear forms (see Theorem 2.1 below), the first term on the right side of (1.2) occurs independent of the presence of a principal character in  $\mathcal{H}(m,r,Q)$ .

Using Theorem 1.1, we can make progress in an additive problem with prime variables. Consider the linear diophantine equation

$$(1.3) \quad a_1 p_1 + a_2 p_2 + a_3 p_3 = b$$

where  $a_1, a_2, a_3, b$  are integers with  $a_1 a_2 a_3 \neq 0$  and  $p_1, p_2, p_3$  are prime unknowns. Our goal is to prove the existence of solutions of (1.3) which do not grow too rapidly as  $B = \max\{|a_1|, |a_2|, |a_3|\} \rightarrow \infty$ . This problem was first raised and investigated by Baker [1] and was later settled, at least qualitatively, by M. C. Liu and Tsang [8]. A necessary condition for the solubility of (1.3) is

$$(1.4) \quad a_1 + a_2 + a_3 \equiv b \pmod{2}.$$

Without loss of generality, we may also assume that

$$(1.5) \quad (a_1, a_2, a_3) = (b, a_i, a_j) = 1, \quad 1 \leq i < j \leq 3.$$

M. C. Liu and Tsang [8] proved the following result.

THEOREM 1.2. *Suppose that  $a_1, a_2, a_3, b$  are integers with  $a_1 a_2 a_3 \neq 0$  and conditions (1.4) and (1.5) hold. Then there exists an absolute constant  $A > 0$  such that*

- (i) *if  $a_1, a_2, a_3$  are all positive, then (1.3) has solutions in primes whenever  $b \gg B^A$ ;*
- (ii) *if  $a_1, a_2, a_3$  are not all of the same sign, then (1.3) has solutions in primes satisfying*

$$(1.6) \quad |a_j| p_j \ll |b| + B^A.$$

It is not difficult to see that one cannot take the exponent  $A$  above arbitrarily small, so it remains to estimate the best possible value of  $A$ . The first numerical upper bound for  $A$  was obtained by Choi [2], who showed that  $A \leq 4190$ . This bound was subsequently reduced to  $A \leq 45$  by M. C. Liu and Wang [9] and to  $A \leq 38$  by Li [6]. Furthermore, Choi, M. C. Liu, and Tsang [3] showed that under GRH one has  $A \leq 5 + \varepsilon$  for any fixed  $\varepsilon > 0$ .

Recently, J. Y. Liu and Tsang [7] showed that when condition (1.5) is replaced by the somewhat more restrictive

$$(1.7) \quad (a_i, a_j) = (b, a_i) = (b, a_j) = 1, \quad 1 \leq i < j \leq 3,$$

then one can take (essentially)  $A = 17/2$ . In the last section of this paper, we obtain the following improvement on their result, thus reducing the value of  $A$  further to  $A = 23/3$ .

**THEOREM 1.3.** *Suppose that  $a_1, a_2, a_3, b$  are integers with  $a_1 a_2 a_3 \neq 0$  and conditions (1.4) and (1.7) hold.*

- (i) *If  $a_1, a_2, a_3$  are all positive, then (1.3) has solutions in primes whenever*

$$b \gg (a_1 a_2 a_3)^{20/9} B(\log B)^{26}.$$

- (ii) *If  $a_1, a_2, a_3$  are not all of the same sign, then (1.3) has solutions in primes satisfying*

$$|a_j| p_j \ll |b| + (a_1 a_2 a_3)^{20/9} B(\log B)^{26}.$$

**REMARK 3.** The proof of Theorem 1.2 uses the circle method and the Deuring–Heilbronn phenomenon to treat the major arcs, which need to be taken significantly larger than in classical applications. Under condition (1.7) in place of (1.5), one can show that the possible existence of Siegel zeros does not have special influence, and hence the Deuring–Heilbronn phenomenon can be avoided (see [7, Lemma 3.1]). As a result, better results can be obtained without recourse to the heavy numerical computations needed in [2, 6, 9].

**2. Mean values of products of Dirichlet polynomials.** We derive Theorem 1.1 from mean-value estimates for products of Dirichlet polynomials of the form

$$(2.1) \quad F(s, \chi) = \prod_{i=1}^3 \left\{ \sum_{N_i < n \leq N'_i} b_i(n) \chi(n) n^{-s} \right\}.$$

We assume that  $1 \leq N_i < N'_i \leq 2N_i$  and  $X = N_1 N_2 N_3 \geq 10$ . We also assume that the coefficients  $b_j(n)$  are subject to

$$(2.2) \quad |b_1(n)| \leq \tau_\kappa(n), \quad |b_2(n)| \leq \tau_\nu(n), \quad |b_3(n)| \leq 1$$

for some integers  $\kappa, \nu \geq 2$ . Here,  $\tau_\kappa(n)$  denotes the  $\kappa$ -fold divisor function. The main result of this section is the following theorem.

**THEOREM 2.1.** *Suppose that  $F(s, \chi)$  is a Dirichlet polynomial as above and  $\mathcal{H}(m, r, Q)$  is a set of characters as in Theorem 1.1 and such that either  $\mathcal{H}(m, r, Q)$  contains no principal character, or it contains only the trivial character  $\chi^0(n) = 1$  for all  $n \geq 1$ . Also, suppose that either*

- (i)  $\max(N_1, N_2) \ll X^{11/20}$  and  $b_3(n) = 1$  for all  $n \leq 2N_3$ , or
- (ii)  $\max(N_1, N_2) \ll X^{11/20}$  and  $N_3 \ll X^{8/35}$ .

Then

$$(2.3) \quad \sum_{\chi \in \mathcal{H}(m, r, Q)} \int_{-T}^T |F(it, \chi)| dt \ll (X + HX^{11/20})L^{c(\kappa, \nu)},$$

where  $c(\kappa, \nu) = 3 \max(\kappa^2, \nu^2) + \kappa + \nu + 20$ ,  $H = mr^{-1}Q^2T$ , and  $L = \log 2HX$ .

The main tool in the proof of Theorem 2.1 are bounds for the cardinality of a well-spaced set of points at which a Dirichlet polynomial of the form (1.1) is large. In this context, a ‘‘point’’ is an ordered pair  $(t, \chi)$ , where  $t$  is a real number such that  $|t| \leq T$  and  $\chi$  is a character from  $\mathcal{H}(m, r, Q)$ . We say that the points  $(t_1, \chi_1), \dots, (t_R, \chi_R)$  are *well-spaced* if  $|t_i - t_j| \geq 1$  whenever  $\chi_i = \chi_j$  and  $i \neq j$ .

**LEMMA 2.2.** *Suppose that  $(t_1, \chi_1), \dots, (t_R, \chi_R)$  are well-spaced and that for all  $j = 1, \dots, R$ ,*

$$\left| \sum_{n \leq N} a_n \chi_j(n) n^{-it_j} \right| \geq V.$$

Then

$$R \ll (NV^{-2} + H \min\{V^{-2}, NG^2V^{-6}\})GL^{18},$$

where  $L = \log 2HN$  and

$$G = \sum_{n \leq N} |a_n|^2.$$

*Proof.* When  $r = 1$ , the lemma is a direct consequence of [5, Theorems 9.16 and 9.18]. When  $r > 1$ , we need modifications of those results. The modifications, however, are straightforward because of the following observations:

- the trivial bound for the cardinality of  $\mathcal{H}(m, r, Q)$  is

$$|\mathcal{H}(m, r, Q)| \ll mr^{-1}Q^2;$$

- if  $1 \leq q_1, q_2 \leq Q$  and  $r \mid (q_1, q_2)$ , then  $[q_1, q_2] \leq r^{-1}Q^2$ . ■

**LEMMA 2.3.** *Let  $N < M \leq cN$  and define*

$$(2.4) \quad D(s, \chi) = \sum_{N < n \leq M} \chi(n)n^{-s}.$$

Suppose that  $(t_1, \chi_1), \dots, (t_R, \chi_R)$  are well-spaced and that  $|t_j| \geq N$  whenever  $\chi_j$  is principal. Then

$$(2.5) \quad \sum_{j=1}^R |D(it_j, \chi_j)|^4 \ll HN^2 L^{10}.$$

*Proof.* Without loss of generality we may assume that the distances from  $M$  and  $N$  to  $\mathbb{Z}$  equal  $1/2$ . For any character  $\chi \in \mathcal{H}(m, r, Q)$ , Perron's formula (see [5, Proposition 5.54]) yields

$$D(it, \chi) = \frac{1}{2\pi i} \int_{\alpha-iT_1}^{\alpha+iT_1} L(it+w, \chi) \frac{M^w - N^w}{w} dw + O(1),$$

where  $T_1 = 10HN$  and  $\alpha = 1 + (\log T_1)^{-1}$ . The integrand is holomorphic everywhere except possibly at  $w = 1 - it$ , where  $L(it + w, \chi)$  has a simple pole if  $\chi$  is principal. Thus, we can move the integration to the contour  $\mathfrak{C}$  consisting of the other three sides of the rectangle with vertices  $1/2 \pm iT_1$ ,  $\alpha \pm iT_1$ . By the convexity bound

$$L(\sigma + it, \chi) \ll (mq(|t| + 2))^{(1-\sigma)/2+\varepsilon} \quad (0 \leq \sigma \leq 1),$$

the integrals over the horizontal parts of  $\mathfrak{C}$  contribute at most

$$\sup_{1/2 \leq \sigma \leq \alpha} \{T_1^{-1} N^\sigma (mqT_1)^{(1-\sigma)/2+\varepsilon}\} \ll 1.$$

Also, the residue at  $w = 1 - it$  is  $\ll \delta_\chi N(1 + |t|)^{-1}$ , where  $\delta_\chi$  is 1 or 0 according as  $\chi$  is principal or not. Hence, for any point  $(t_j, \chi_j)$ ,  $j = 1, \dots, R$ , we have

$$\begin{aligned} D(it_j, \chi_j) &\ll N^{1/2} \int_{-T_1}^{T_1} |L(1/2 + i(t_j + u), \chi_j)| \frac{du}{1 + |u|} + \frac{\delta_{\chi_j} N}{1 + |t_j|} + 1 \\ &\ll N^{1/2} \int_{-T_1}^{T_1} |L(1/2 + i(t_j + u), \chi_j)| \frac{du}{1 + |u|} + 1, \end{aligned}$$

where the last inequality uses the hypothesis on points  $(t_j, \chi_j)$  with principal characters. Appealing to Hölder's inequality, we derive the estimate

$$\begin{aligned} |D(it_j, \chi_j)|^4 &\ll N^2 L^3 \int_{-T_1}^{T_1} |L(1/2 + i(t_j + u), \chi_j)|^4 \frac{du}{1 + |u|} + 1 \\ &\ll N^2 L^3 \int_{-2T_1}^{2T_1} |L(1/2 + iu, \chi_j)|^4 \frac{du}{1 + |u - t_j|} + 1, \end{aligned}$$

whence

$$\sum_{j=1}^R |D(it_j, \chi_j)|^4 \ll N^2 L^3 \sum_{j=1}^R \int_{-2T_1}^{2T_1} |L(1/2 + iu, \chi_j)|^4 \frac{du}{1 + |u - t_j|} + R.$$

This suffices, because

$$\begin{aligned} \sum_{j=1}^R \int_{-2T_1}^{2T_1} |L(1/2 + iu, \chi_j)|^4 \frac{du}{1 + |u - t_j|} \\ \ll \sum_{\chi \in \mathcal{H}(m,r,Q)} \int_{-2T_1}^{2T_1} |L(1/2 + iu, \chi)|^4 \left\{ \sum_{\substack{j=1 \\ \chi_j = \chi}}^R \frac{1}{1 + |u - t_j|} \right\} du \\ \ll TL \sum_{\chi \in \mathcal{H}(m,r,Q)} \int_{-2T_1}^{2T_1} |L(1/2 + iu, \chi)|^4 \frac{du}{T + |u|} \ll HL^7, \end{aligned}$$

where the final step uses the estimate for the fourth power moment of  $L(s, \chi)$  (see [10, Theorem 10.1]). ■

*Proof of Theorem 2.1.* Define the Dirichlet polynomials

$$F_i(s, \chi) = \sum_{N_i < n \leq N'_i} b_i(n) \chi(n) n^{-s} \quad (i = 1, 2, 3).$$

The proof is divided into four steps.

STEP 1. First, we treat the case where  $\max(N_1, N_2) \geq X^{9/20}$ . Suppose that  $N_1 \geq N_2$ . We recall the well known estimate (see [5, (1.80)])

$$\sum_{n \leq x} \tau_\kappa(n)^\nu \ll x(\log x)^{\kappa\nu - 1}.$$

By [5, Theorem 9.12] and (2.2),

$$(2.6) \quad \sum_{\chi \in \mathcal{H}(m,r,Q)} \int_{-T}^T |F_1(it, \chi)|^2 dt \ll (N_1 + H) N_1 L^{\kappa^2 + 2}.$$

Similarly,

$$(2.7) \quad \sum_{\chi \in \mathcal{H}(m,r,Q)} \int_{-T}^T |\tilde{F}_2(it, \chi)|^2 dt \ll (N_2 N_3 + H) N_2 N_3 L^{\nu^2 + 2\nu + 3},$$

where  $\tilde{F}_2(s, \chi) = F_2(s, \chi) F_3(s, \chi)$  is a Dirichlet polynomial with coefficients  $\tilde{b}_2(n)$  subject to

$$|\tilde{b}_2(n)| \leq \sum_{n=uv} \tau_\nu(u) \leq \tau_{\nu+1}(n).$$

Using (2.6), (2.7), and the Cauchy–Schwarz inequality, we find that the left side of (2.3) is bounded above by

$$\begin{aligned} & (X^{1/2} + (HN_1)^{1/2} + (HN_2N_3)^{1/2} + H)X^{1/2}L^{c_1} \\ & \ll (X + H^{1/2}X^{31/40} + HX^{1/2})L^{c_1} \ll (X + HX^{11/20})L^{c_1}, \end{aligned}$$

where  $c_1 = c_1(\kappa, \nu) = \kappa^2 + \nu^2 + 4$ . Since an obvious modification of this argument establishes (2.3) when  $N_2 \geq N_1$ , we may assume for the remainder of the proof that

$$(2.8) \quad \max(N_1, N_2) \leq X^{9/20}.$$

STEP 2. Suppose that hypothesis (ii) holds. By a standard argument,

$$(2.9) \quad \sum_{\chi \in \mathcal{H}(m,r,Q)} \int_{-T}^T |F(it, \chi)| dt \ll \sum_{j=1}^R |F(it_j, \chi_j)|,$$

where  $(t_1, \chi_1), \dots, (t_R, \chi_R)$  are well-spaced points. The points  $(t_j, \chi_j)$  such that

$$F_i(it_j, \chi_j) \ll X^{-1} \quad \text{for some } i = 1, 2, 3$$

contribute at most

$$RX^{-1}X^{1.01} \ll RX^{0.01} \ll HX^{0.01}$$

to the right side of (2.9). We divide the remaining points  $(t_j, \chi_j)$  into  $O(L^3)$  subsets so that for the points in a particular subset  $\mathcal{S}(V_1, V_2, V_3)$  we have

$$(2.10) \quad V_i \leq |F_i(it_j, \chi_j)| \leq 2V_i \quad (i = 1, 2, 3).$$

We obtain

$$(2.11) \quad \sum_{\chi \in \mathcal{H}(m,r,Q)} \int_{-T}^T |F(it, \chi)| dt \ll L^3V_1V_2V_3|\mathcal{S}(V_1, V_2, V_3)| + HX^{0.01}$$

for some  $V_1, V_2, V_3$  subject to

$$(2.12) \quad X^{-1} \leq V_i \leq N_iL^{\kappa+\nu}.$$

Thus, it suffices to show that

$$(2.13) \quad \Xi|\mathcal{S}(V_1, V_2, V_3)| \ll (X + HX^{11/20})L^{c_2+\kappa+\nu},$$

where  $\Xi = V_1V_2V_3$ ,  $c_2 = c_2(\kappa, \nu) = 3 \max(\kappa^2, \nu^2) + 15$ . To derive this bound, we apply Lemma 2.2 to  $F_1(s, \chi)$ ,  $F_2(s, \chi)$ , and  $F_3(s, \chi)^2$  and find that

$$(2.14) \quad |\mathcal{S}(V_1, V_2, V_3)| \ll \min\{N_1^2V_1^{-2} + HN_1 \min(V_1^{-2}, N_1^3V_1^{-6}), \\ N_2^2V_2^{-2} + HN_2 \min(V_2^{-2}, N_2^3V_2^{-6}), \\ N_3^4V_3^{-4} + HN_3^2 \min(V_3^{-4}, N_3^6V_3^{-12})\}L^{c_2}.$$

We now derive (2.13) from (2.8), (2.12), and (2.14) using a case-by-case analysis. We write

$$\Gamma_i = N_i^2 V_i^{-2}, \quad \Delta_i = \min(V_i^{-2}, N_i^3 V_i^{-6}), \quad \Delta_i(\alpha) = N_i^{3\alpha} V_i^{-2-4\alpha}$$

and remark that  $\Delta_i \leq \Delta_i(\alpha)$  for all  $0 \leq \alpha \leq 1$ .

CASE 1:  $\Gamma_1 \geq HN_1 \Delta_1$  and  $\Gamma_2 \geq HN_2 \Delta_2$ . Then, by (2.12) and (2.14),

$$\begin{aligned} \Xi |\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi \min\{\Gamma_1, \Gamma_2\} L^{c_2} \ll \Xi (\Gamma_1 \Gamma_2)^{1/2} L^{c_2} \\ &\ll N_1 N_2 V_3 L^{c_2} \ll X L^{c_2 + \kappa + \nu}. \end{aligned}$$

CASE 2:  $\Gamma_1 \leq HN_1 \Delta_1$ ,  $\Gamma_2 \leq HN_2 \Delta_2$ , and  $\Gamma_3^2 \geq HN_3^2 \Delta_3^2$ . By (2.14) and the hypothesis  $N_3 \leq X^{8/35}$ , we get

$$\begin{aligned} \Xi |\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi \min\{HN_1 \Delta_1, HN_2 \Delta_2, \Gamma_3^2\} L^{c_2} \\ &\ll \Xi (HN_1 \Delta_1 (1/6))^{3/8} (HN_2 \Delta_2 (1/6))^{3/8} \Gamma_3^{1/2} L^{c_2} \\ &\ll H^{3/4} (X^9 N_3^7)^{1/16} L^{c_2} \ll (X + HX^{11/20}) L^{c_2}, \end{aligned}$$

where the last step uses the fact that

$$H^{3/4} (X^9 N_3^7)^{1/16} \ll H^{3/4} X^{53/80} = X^{1/4} (HX^{11/20})^{3/4}.$$

CASE 3:  $\Gamma_1 \leq HN_1 \Delta_1$ ,  $\Gamma_2 \leq HN_2 \Delta_2$ , and  $\Gamma_3^2 \leq HN_3^2 \Delta_3^2$ . When  $N_3 \leq X^{1/5}$ , then (2.14) yields

$$\begin{aligned} \Xi |\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi \min\{HN_1 \Delta_1, HN_2 \Delta_2, HN_3^2 \Delta_3^2\} L^{c_2} \\ &\ll \Xi H (N_1 \Delta_1 (1/22) N_2 \Delta_2 (1/22))^{11/24} (N_3 \Delta_3 (1))^{1/6} L^{c_2} \\ &\ll H (X^{25} N_3^7)^{1/48} L^{c_2} \ll HX^{11/20} L^{c_2}. \end{aligned}$$

On the other hand, when  $N_3 \geq X^{1/5}$ , (2.14) yields

$$\begin{aligned} \Xi |\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi H (N_1 \Delta_1 (1/6) N_2 \Delta_2 (1/6))^{3/8} (N_3 \Delta_3 (0))^{1/2} L^{c_2} \\ &\ll H (X^9 N_3^{-1})^{1/16} L^{c_2} \ll HX^{11/20} L^{c_2}. \end{aligned}$$

CASE 4:  $\Gamma_1 \geq HN_1 \Delta_1$ ,  $\Gamma_2 \leq HN_2 \Delta_2$ , and  $\Gamma_3^2 \geq HN_3^2 \Delta_3^2$ . By (2.8), (2.14), and the hypothesis  $N_3 \leq X^{8/35}$ ,

$$\begin{aligned} \Xi |\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi \min\{\Gamma_1, HN_2 \Delta_2, \Gamma_3^2\} L^{c_2} \\ &\ll \Xi (\Gamma_1 \Gamma_3)^{1/2} (HN_2 \Delta_2 (1/2))^{1/4} L^{c_2} \\ &\ll H^{1/4} X^{5/8} (N_1 N_3)^{3/8} L^{c_2} \ll (X + HX^{11/20}) L^{c_2}, \end{aligned}$$

where the last step uses the fact that

$$H^{1/4} X^{5/8} (N_1 N_3)^{3/8} \ll H^{1/4} X^{197/224} \ll X^{3/4} (HX^{11/20})^{1/4}.$$

CASE 5:  $\Gamma_1 \geq HN_1\Delta_1$ ,  $\Gamma_2 \leq HN_2\Delta_2$ , and  $\Gamma_3^2 \leq HN_3^2\Delta_3^2$ . When  $N_3 \leq X^{1/5}$ , then (2.14) yields

$$\begin{aligned} \Xi|\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi \min\{\Gamma_1, HN_2\Delta_2, HN_3^2\Delta_3^2\}L^{c_2} \\ &\ll \Xi\Gamma_1^{1/2}(HN_2\Delta_2(1/10))^{5/12}(HN_3^2\Delta_3(1)^2)^{1/12}L^{c_2} \\ &\ll H^{1/2}(N_1^{11}N_3^3)^{1/24}X^{13/24}L^{c_2} \ll (X + HX^{11/20})L^{c_2}, \end{aligned}$$

where the last step uses the fact that

$$H^{1/2}(N_1^{11}N_3^3)^{1/24}X^{13/24} \ll H^{1/2}X^{371/480} \ll X^{1/2}(HX^{11/20})^{1/2}.$$

On the other hand, when  $N_3 \geq X^{1/5}$ , by (2.14),

$$\begin{aligned} \Xi|\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi\Gamma_1^{1/2}(HN_2\Delta_2(1/2))^{1/4}(HN_3^2\Delta_3(0)^2)^{1/4}L^{c_2} \\ &\ll H^{1/2}(X^5N_1^3N_3^{-1})^{1/8}L^{c_2} \ll (X + HX^{11/20})L^{c_2}, \end{aligned}$$

because

$$H^{1/2}(X^5N_1^3N_3^{-1})^{1/8} \ll H^{1/2}X^{123/160} \ll X^{1/2}(HX^{11/20})^{1/2}.$$

CASE 6:  $\Gamma_1 \leq HN_1\Delta_1$  and  $\Gamma_2 \geq HN_2\Delta_2$ . This case can be split into two subcases that can be handled similarly to Cases 4 and 5.

This completes the proof under hypothesis (ii).

STEP 3. Suppose that hypothesis (i) holds and  $\mathcal{H}(m, r, Q)$  contains no principal characters. If  $N_3 \leq X^{8/35}$ , hypothesis (i) implies hypothesis (ii), so we may assume that  $N_3 \geq X^{8/35}$ . Then we follow the argument from Step 2 until we reach (2.12). Under the present assumptions, Lemma 2.3 and (2.10) yield

$$|\mathcal{S}(V_1, V_2, V_3)| \ll HN_3^2V_3^{-4}L^{10}.$$

Thus, we may replace (2.14) by the bound

$$(2.15) \quad |\mathcal{S}(V_1, V_2, V_3)| \ll \min\{N_1^2V_1^{-2} + HN_1 \min(V_1^{-2}, N_1^3V_1^{-6}), N_2^2V_2^{-2} + HN_2 \min(V_2^{-2}, N_2^3V_2^{-6}), HN_3^2V_3^{-4}\}L^{c_2}.$$

We now embark on deriving (2.13) from (2.8), (2.12), (2.15), and the hypothesis  $N_3 \geq X^{8/35}$ . Let  $\Gamma_i$  and  $\Delta_i$  be as in Step 2. We consider four cases.

CASE 1:  $\Gamma_1 \geq HN_1\Delta_1$  and  $\Gamma_2 \geq HN_2\Delta_2$ . Then we argue as in Case 1 of Step 2.

CASE 2:  $\Gamma_1 \leq HN_1\Delta_1$  and  $\Gamma_2 \leq HN_2\Delta_2$ . Then (2.15) yields

$$\begin{aligned} \Xi|\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi H(N_1\Delta_1(1/6)N_2\Delta_2(1/6))^{3/8}(N_3V_3^{-2})^{1/2}L^{c_2} \\ &\ll H(X^9N_3^{-1})^{1/16}L^{c_2} \ll HX^{11/20}L^{c_2}. \end{aligned}$$

CASE 3:  $\Gamma_1 \geq HN_1\Delta_1$  and  $\Gamma_2 \leq HN_2\Delta_2$ . By (2.15),

$$\begin{aligned} \Xi|\mathcal{S}(V_1, V_2, V_3)| &\ll \Xi\Gamma_1^{1/2}(HN_2\Delta_2(1/2))^{1/4}(HN_3^2V_3^{-4})^{1/4}L^{c_2} \\ &\ll H^{1/2}(X^5N_1^3N_3^{-1})^{1/8}L^{c_2} \ll (X + HX^{11/20})L^{c_2}, \end{aligned}$$

because

$$H^{1/2}(X^5N_1^3N_3^{-1})^{1/8} \ll H^{1/2}X^{857/1120} \ll X^{1/2}(HX^{11/20})^{1/2}.$$

CASE 4:  $\Gamma_1 \leq HN_1\Delta_1$  and  $\Gamma_2 \geq HN_2\Delta_2$ . This is similar to Case 3.

STEP 4. Finally, suppose that hypothesis (i) holds and  $\mathcal{H}(m, r, Q)$  contains only the trivial character  $\chi^0(n)$ . Again, we may assume that  $N_3 \geq X^{8/35}$ . By [5, Corollary 8.11],

$$\sum_{N < n \leq 2N} n^{-it} \ll N(1 + |t|)^{-1} \quad (|t| < N).$$

Using this bound and the trivial estimates for the sums over  $n_1$  and  $n_2$ , we get

$$\int_{-N_3}^{N_3} |F(it, \chi^0)| dt \ll N_1N_2L^{\kappa+\nu-2} \int_{-N_3}^{N_3} \frac{N_3}{1 + |t|} dt \ll XL^{\kappa+\nu-1}.$$

On the other hand, when  $N_3 \leq T$ , we may repeat Steps 1 and 3 above to deduce that

$$(2.16) \quad \int_{N_3 \leq |t| \leq T} |F(it, \chi^0)| dt \ll (X + TX^{11/20})L^{c_3},$$

where  $c_3 = c_3(\kappa, \nu) = 3 \max(\kappa^2, \nu^2) + \kappa + \nu + 15$ . The only change occurs when we have to appeal to Lemma 2.3 to obtain a variant of (2.15). Because of the presence of a principal character, we now need to ensure that the points in  $\mathcal{S}(V_1, V_2, V_3)$  satisfy  $|t_j| \geq N_3$ . For the points appearing in the proof of (2.16), however, this condition obviously holds. ■

We conclude this section with a technical lemma, which will be needed in the next section.

LEMMA 2.4. *Suppose that  $2 \leq T \leq M < N$  and  $f : \mathbb{N}^2 \rightarrow \mathbb{C}$  is a function such that*

$$(2.17) \quad \sum_m \int_{-U}^U \left| \sum_{n \leq N} f(m, n)n^{it} \right| dt \leq A + BU$$

for all  $U \geq 2$ . Then

$$(2.18) \quad \sum_m \int_{-T}^T \left| \sum_{n \leq M} f(m, n)n^{it} \right| dt \ll (A + BT) \log^2 N.$$

*Proof.* Let  $g$  denote the indicator function of  $[-T, T]$  and let  $h$  be the function constructed in [5, Lemma 13.11] with  $z = N$ . Then

$$(2.19) \quad |h(u)| \ll \min\{\log N, |u|^{-1}, N|u|^{-2}\}$$

and

$$\int_{-\infty}^{\infty} h(u) \left(\frac{m}{n}\right)^{iu} du = \begin{cases} 1 & \text{if } m \leq n, \\ 0 & \text{if } m > n, \end{cases}$$

for any pair of integers  $m, n$  such that  $1 \leq m, n \leq N$ . Thus,

$$\sum_{n \leq M} f(m, n)n^{it} = \int_{-\infty}^{\infty} \left\{ \sum_{n \leq N} f(m, n)n^{i(t+u)} \right\} h(u)M^{-iu} du,$$

assuming (as we may) that  $M$  is an integer. It follows that the left side of (2.18) does not exceed

$$\begin{aligned} & \sum_m \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{\infty} |h(u)| \left| \sum_{n \leq N} f(m, n)n^{i(t+u)} \right| du dt \\ &= \sum_m \int_{-\infty}^{\infty} \left| \sum_{n \leq N} f(m, n)n^{i\tau} \right| \left\{ \int_{-\infty}^{\infty} g(\tau - u)|h(u)| du \right\} d\tau \\ &\ll T(\log N) \sum_m \int_{-\infty}^{\infty} \left| \sum_{n \leq N} f(m, n)n^{i\tau} \right| \min\{T^{-1}, |\tau|^{-1}, N|\tau|^{-2}\} d\tau, \end{aligned}$$

where the last step uses (2.19) and the definition of  $g$ . The desired conclusion now follows by a standard dyadic argument. ■

**3. Proof of Theorem 1.1.** In this section we deduce Theorem 1.1 from Theorem 2.1 and Heath-Brown’s identity for  $\Lambda(n)$ . We apply Heath-Brown’s identity in the following form (see [4, Lemma 1] or [5, Proposition 13.3] with  $k = 10$ ): if  $n \leq x$ , then

$$(3.1) \quad \Lambda(n) = \sum_{j=1}^{10} \binom{10}{j} (-1)^j \sum_{\substack{n=m_1 \cdots m_j \\ m_1, \dots, m_j \leq x^{1/10}}} \mu(m_1) \cdots \mu(m_j) \log m_{2j}.$$

First, we deal with the possibility that  $\mathcal{H}(m, r, Q)$  may contain the principal character  $\chi_0$  modulo  $m$ . We have

$$\sum_{N < n \leq 2N} \Lambda(n)\chi_0(n)n^{-s} = \sum_{N < n \leq 2N} \Lambda(n)n^{-s} + O(L^2).$$

Hence, if  $\chi_0 \in \mathcal{H}(m, r, Q)$ , the left side of (1.2) does not exceed

$$\sum_{\substack{\chi \in \mathcal{H}(m, r, Q) \\ \chi \neq \chi_0}} \int_{-T}^T \left| \sum_{N < n \leq 2N} \Lambda(n)\chi(n)n^{-it} \right| dt + \int_{-T}^T \left| \sum_{N < n \leq 2N} \Lambda(n)n^{-it} \right| dt + TL^2.$$

Thus, it suffices to establish (1.2) when  $\mathcal{H}(m, r, Q)$  is of one of the two types considered in Theorem 2.1.

By (3.1) with  $x = 2N$  and a standard splitting argument,

$$\sum_{N < n \leq 2N} \Lambda(n) \chi(n) n^{-s} \ll \sum_{\mathbf{M}} \left| \sum_{N < n \leq 2N} a(n; \mathbf{M}) \chi(n) n^{-s} \right|,$$

where  $\mathbf{M}$  runs over  $O(L^{19})$  vectors  $\mathbf{M} = (M_1, \dots, M_{2j})$ ,  $j \leq 10$ , subject to

$$M_1, \dots, M_j \ll N^{1/10}, \quad N \ll M_1 \cdots M_{2j} \ll N,$$

and

$$a(n; \mathbf{M}) = \sum_{\substack{n=m_1 \cdots m_{2j} \\ M_i < m_i \leq 2M_i}} \mu(m_1) \cdots \mu(m_j) (\log m_{2j}).$$

Hence, the left side of (1.2) is bounded above by

$$L^{19} \sum_{\chi \in \mathcal{H}(m, r, Q)} \int_{-T}^T \left| \sum_{N < n \leq 2N} a(n; \mathbf{M}) \chi(n) n^{-it} \right| dt$$

for some fixed choice of  $\mathbf{M}$  as above. Thus, if we show that

$$(3.2) \quad \sum_{\chi \in \mathcal{H}(m, r, Q)} \int_{-T}^T \left| \sum_n a(n; \mathbf{M}) \chi(n) n^{-it} \right| dt \ll (N + HN^{11/20}) L^{1020},$$

the desired result (with  $C = 1100$ ) will follow by Lemma 2.4.

The Dirichlet polynomial on the left side of (3.2) is the product of  $2j$ ,  $j \leq 10$ , Dirichlet polynomials of the form (1.1) with coefficients  $a_n = \mu(n)$ ,  $a_n = 1$ , or  $a_n = \log n$ . Furthermore, the single logarithmic weight can be removed by partial summation. Therefore, we may assume that

$$a(n; \mathbf{M}) = L \sum_{\substack{n=m_1 \cdots m_{2j} \\ M_i < m_i \leq M'_i}} \mu(m_1) \cdots \mu(m_j),$$

where  $M_i < M'_i \leq 2M_i$  (in reality,  $M'_i = 2M_i$  except for  $i = 2j$ ). We may now assume that  $M_{j+1} \leq \cdots \leq M_{2j}$ . We proceed to show that

$$a(n; \mathbf{M}) = L \sum_{n=n_1 n_2 n_3} b_1(n_1) b_2(n_2) b_3(n_3),$$

where the coefficients on the right yield a Dirichlet polynomial (2.1) that satisfies at least one of the hypotheses (i) or (ii) of Theorem 2.1. The analysis involves several cases depending on the sizes of  $M_1, \dots, M_{2j}$ .

CASE 1:  $M_{2j} \gg N^{9/20}$ . Assuming that  $j \geq 2$  (the case  $j = 1$  is similar and easier), we group the variables  $m_1, \dots, m_{2j}$  into  $n_1, n_2, n_3$  as follows:

$$n_1 = m_3 \cdots m_{2j-1}, \quad n_2 = m_1 m_2, \quad n_3 = m_{2j}.$$

Since  $M_1 \cdots M_{2j-1} \ll N^{11/20}$ , this yields a polynomial  $F(s, \chi)$  satisfying hypothesis (i) of Theorem 2.1.

CASE 2:  $M_{2j} \ll N^{9/20} \ll M_1 \cdots M_j M_{2j}$ . Let  $i$  be the least integer for which  $M_1 \cdots M_i M_{2j} \gg N^{9/20}$ . Since  $M_i \ll N^{1/10}$ , we have

$$N^{9/20} \ll M_1 \cdots M_i M_{2j} \ll N^{11/20}.$$

Hence, the choice

$$n_1 = m_1 \cdots m_i m_{2j}, \quad n_2 = m_{i+1} \cdots m_{2j-1}, \quad n_3 = 1$$

yields an  $F(s, \chi)$  that satisfies hypothesis (ii) of Theorem 2.1.

CASE 3:  $M_1 \cdots M_j M_{2j} \ll N^{9/20}$ . Let  $\ell$  be the least positive integer such that

$$M_1 \cdots M_j M_\ell \cdots M_{2j} \ll N^{9/20}.$$

We consider three subcases.

CASE 3.1:  $M_{\ell-1} \cdots M_{2j} \ll N^{11/20}$ . Then we can argue similarly to Case 2 to find an  $i$ ,  $0 \leq i \leq j$ , for which

$$N^{9/20} \ll M_1 \cdots M_i M_{\ell-1} \cdots M_{2j} \ll N^{11/20}.$$

Again, we will have  $F(s, \chi)$  that satisfies hypothesis (ii) of Theorem 2.1.

CASE 3.2:  $M_{\ell-1} \cdots M_{2j} \gg N^{11/20}$  and  $M_{\ell-1} \ll N^{8/35}$ . Then we define

$$n_1 = m_1 \cdots m_j m_\ell \cdots m_{2j}, \quad n_2 = m_{j+1} \cdots m_{\ell-2}, \quad n_3 = m_{\ell-1}.$$

Since  $M_{j+1} \cdots M_{\ell-2} \ll N^{9/20}$ , we again get an  $F(s, \chi)$  that satisfies hypothesis (ii) of Theorem 2.1.

CASE 3.3:  $M_{\ell-1} \cdots M_{2j} \gg N^{11/20}$  and  $M_{\ell-1} \gg N^{8/35}$ . This may occur only with  $\ell = 2j$ . Then

$$M_1 \cdots M_{2j-2} \ll N M_{2j-1}^{-2} \ll N^{19/35} \ll N^{11/20}, \quad M_{2j-1} \ll M_{2j} \ll N^{9/20}.$$

We write

$$b_1(n) = \sum_{n=m_1 \cdots m_{2j-2}} \mu(m_1) \cdots \mu(m_j), \quad n_2 = m_{2j-1}, \quad n_3 = m_{2j},$$

and we obtain an  $F(s, \chi)$  that satisfies hypothesis (i) of Theorem 2.1.

The desired bound (3.2) follows on noting that the arising coefficients satisfy (2.2) with  $\kappa, \nu$  for which  $c(\kappa, \nu) \leq c(18, 2) = 1012$ .

**4. Exponential sums twisted by characters.** In this section we estimate the exponential sum

$$(4.1) \quad W(\beta, \chi) = \sum_{N < p \leq 2N} (\log p) \chi(p) e(\beta p^k),$$

where  $k$  is a positive integer,  $\beta$  is “small”, and  $\chi$  is Dirichlet character. Such exponential sums arise in dealings with the major arcs in the Waring–Goldbach problem and related questions. In particular, in the proof of Theorem 1.3 we need the case  $k = 1$  of our estimates.

LEMMA 4.1. *Suppose that  $N \geq 2$  and  $0 \leq \Delta \leq N^{1-k}$ . Suppose also that  $\mathcal{H}(m, r, Q)$  is a set of characters as in Theorem 1.1 and  $W(\beta, \chi)$  is defined by (4.1). Then*

$$(4.2) \quad \sum_{\chi \in \mathcal{H}(m, r, Q)} \max_{\Delta \leq |\beta| \leq 2\Delta} |W(\beta, \chi)| \ll T_0^{-1/2} L^{C+1} (N + HN^{11/20}),$$

where  $T_0 = 1 + \Delta N^k$ ,  $H = mr^{-1}Q^2T_0$ ,  $L = \log N$ , and  $C$  is the constant appearing in Theorem 1.1.

*Proof.* We first replace  $W(\beta, \chi)$  by the exponential sum

$$\widetilde{W}(\beta, \chi) = \sum_{N < n \leq 2N} \Lambda(n) \chi(n) e(\beta n^k)$$

using the fact that

$$(4.3) \quad W(\beta, \chi) = \widetilde{W}(\beta, \chi) + O(N^{1/2}).$$

By Perron’s formula [5, Proposition 5.54], for  $N < M \leq 2N$ ,

$$(4.4) \quad \sum_{N < n \leq M} \Lambda(n) \chi(n) = \frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} F(s, \chi) \frac{M^s - N^s}{s} ds + O\left(\frac{NL^2}{1 + T_1 \|M\|}\right),$$

where  $0 < b < (\log N)^{-1}$ ,  $T_1 = (HN)^{10}$ ,  $\|M\|$  is the distance from  $M$  to the nearest integer, and

$$F(s, \chi) = \sum_{N < n \leq 2N} \Lambda(n) \chi(n) n^{-s}.$$

Hence, by partial summation,

$$(4.5) \quad \widetilde{W}(\beta, \chi) = \frac{1}{2\pi i} \int_{b-iT_1}^{b+iT_1} F(s, \chi) V(s, \beta) ds + O(1),$$

where

$$V(s, \beta) = \int_N^{2N} y^{s-1} e(\beta y^k) dy.$$

By [5, Lemma 8.10], for  $\Delta \leq |\beta| \leq 2\Delta$ ,

$$(4.6) \quad V(\sigma + it, \beta) \ll N^\sigma \min\{T_0^{-1/2}, \sup_{N \leq y \leq 2N} |t + 2k\pi\beta y^k|^{-1}\}.$$

Combining (4.5) and (4.6) and letting  $b \downarrow 0$ , we obtain

$$\widetilde{W}(\beta, \chi) \ll T_0^{1/2} \int_{-T_1}^{T_1} |F(it, \chi)| \frac{dt}{T_0 + |t|} + 1.$$

Recalling (4.3), we deduce that the left side of (4.2) is bounded above by

$$(4.7) \quad LT_0^{1/2} T^{-1} \sum_{\chi \in \mathcal{H}(m,r,Q) - T} \int_{-T}^T |F(it, \chi)| dt + |\mathcal{H}| N^{1/2},$$

for some  $T$  in the range  $T_0 \leq T \leq T_1$ . The result now follows from (1.2). ■

We now define the exponential integral

$$(4.8) \quad v(\beta; X) = \int_X^{2X} e(\beta y^k) dy.$$

LEMMA 4.2. *Suppose that  $N \geq 2$ ,  $1 \leq Q \leq N$ , and  $0 \leq \Delta \leq N^{1-k}$ . Let  $W(\beta, \chi)$  be defined by (4.1). Then, for any fixed  $A > 0$  and  $\delta > 0$ ,*

$$(4.9) \quad \sum_{Q < q \leq 2Q} \sum_{\chi \bmod q}^* \max_{\Delta \leq |\beta| \leq 2\Delta} |W(\beta, \chi)| \ll N Q^\delta L^{-A} + Q^2 T_0^{1/2} N^{11/20} L^{C+1},$$

where  $T_0 = 1 + \Delta N^k$ ,  $L = \log N$ , and  $C$  is the constant appearing in Theorem 1.1. Furthermore, for any fixed  $A > 0$  we have

$$(4.10) \quad W(\beta, \chi^0) - v(\beta; N) \ll N L^{-A} + T_0^{1/2} N^{11/20} L^{C+1},$$

where  $v(\beta; N)$  is defined by (4.8) and  $\chi^0$  is the trivial character. In both (4.9) and (4.10) the implied constants may depend on  $A$ , and the implied constant in (4.9) may also depend on  $\delta$ .

*Proof.* The first claim follows from Lemma 4.1 and the Siegel–Walfisz theorem in the form of [5, (5.79)]. Put  $B = (2 + \delta^{-1})(A + C + 1)$ . If  $Q \geq L^B$  or  $\Delta \geq L^B N^{-k}$ , we have

$$N T_0^{-1/2} L^{C+1} \ll N Q^\delta L^{-A}$$

and (4.9) follows from (4.2) with  $m = r = 1$ . On the other hand, if  $Q \leq L^B$  and  $\Delta \leq L^B N^{-k}$ , we find by partial summation that the left side of (4.9) is bounded above by

$$L^{3B+1} \max_{N < N_1 \leq 2N} \left| \sum_{N < p \leq N_1} \chi(p) \right| \ll N L^{-A},$$

by the aforementioned version of the Siegel–Walfisz theorem.

The proof of the second claim is similar, except that it appeals to the case  $m = r = Q = 1$  of Lemma 4.1 and to the prime number theorem (which is why we need to include the term  $v(\beta; N)$  on the left side of (4.10)). ■

LEMMA 4.3. *Suppose that  $N \geq 2$  and  $N^{-k} \leq \Delta \leq N^{1-k}$ . Suppose also that  $\mathcal{H}(m, r, Q)$  is a set of characters as in Theorem 1.1 and  $W(\beta, \chi)$  is defined by (4.1). Then*

$$(4.11) \quad \sum_{\chi \in \mathcal{H}(m, r, Q)} \left\{ \int_{-\Delta}^{\Delta} |W(\beta, \chi)|^2 d\beta \right\}^{1/2} \ll N^{-k/2} L^{C+1} (N + HN^{11/20}),$$

where  $H = mr^{-1}Q^2\Delta N^k$ ,  $L = \log N$ , and  $C$  is the constant from Theorem 1.1.

*Proof.* By [10, Lemma 1.9], we have

$$(4.12) \quad \int_{-\Delta}^{\Delta} |W(\beta, \chi)|^2 d\beta \ll \Delta^2 \int_{-\infty}^{\infty} \left| \sum_{u(y) < p \leq v(y)} (\log p)\chi(p) \right|^2 dy \\ \ll \Delta^2 N^k \left| \sum_{M < n \leq M+Y} \Lambda(n)\chi(n) \right|^2 + \Delta^2 N^{k+1},$$

where  $u(y) = \max(N, y^{1/k})$ ,  $v(y) = \min(2N, (y + (2\Delta)^{-1})^{1/k})$ , and

$$(4.13) \quad N < M \leq 2N, \quad Y \ll \Delta^{-1}N^{1-k}.$$

Without loss of generality, we may assume that the distance from  $M$  to the nearest integer is  $1/2$  and that  $Y$  is an integer. We then appeal to Perron’s formula to derive

$$\sum_{M < n \leq M+Y} \Lambda(n)\chi(n) \ll \left| \int_{b-iT_1}^{b+iT_1} F(s, \chi) \frac{(M+Y)^s - M^s}{s} ds \right| + 1,$$

where  $0 < b < L^{-1}$ ,  $T_1 = (HN)^{10}$ , and  $F(s, \chi)$  is the Dirichlet polynomial appearing in the proof of Lemma 4.1. Hence, as in that proof,

$$(4.14) \quad \sum_{M < n \leq M+Y} \Lambda(n)\chi(n) \ll \int_{-T_1}^{T_1} |F(it, \chi)| \frac{dt}{T_0 + |t|} + 1,$$

where  $T_0 = \Delta N^k$ . By (4.12) and (4.14), the left side of (4.11) is bounded above by

$$\Delta N^{k/2} L T^{-1} \sum_{\chi \in \mathcal{H}(m, r, Q)} \int_{-T}^T |F(it, \chi)| dt + HN^{(1-k)/2},$$

where  $T$  is subject to  $T_0 \leq T \leq T_1$ . The result now follows from (1.2). ■

**5. Proof of Theorem 1.3.** Since the proof follows closely the proof of the main result in [7], we only describe the necessary changes. Let  $N$  be a large parameter chosen as in [7, Lemma 2.3] and set

$$(5.1) \quad P = (N/B)^{9/20}, \quad L = \log N, \quad Q = N/(PL^2).$$

We note that the improvement on the result of Liu and Tsang arises from the choice of  $P$  in (5.1): the respective choice in [7] is  $P = (N/B)^{2/5}$  (see [7, (2.1)]). In order to justify the analysis in [7] for this larger value of  $P$ , we need appropriate variants of [7, Lemmas 3.2 and 3.3]. If we weaken the bound in part (i) of [7, Lemma 3.3] to

$$J_j(R) \ll \tau(\varpi)R^{-1/9}N|a_j|^{-1}L^c,$$

then the modified lemma (which suffices for the proof of Theorem 1.3) holds with  $P = (N/B)^{9/20}$ .

Now, let  $N_j = N/|a_j|$ ,  $N^{1/10} \leq R \leq P$ , and  $g, D$  be positive integers. Define

$$K_j(g; R) = \sum_{R < r \leq 2R} \frac{\sqrt{([g, r], D)}}{[g, r]} \sum_{\chi \bmod r}^* \left( \int_{-1/(RQ)}^{1/(RQ)} |W_j(a_j\beta, \chi)|^2 d\beta \right)^{1/2},$$

where  $W_j(\beta, \chi)$  is the sum (4.1) with  $N = N_j$  and  $k = 1$ . In order to prove [7, Lemma 3.2] with  $P$  as in (5.1), we need to show that

$$(5.2) \quad K_j(g; R) \ll g^{-1} \sqrt{(g, D)} \tau(gD)^2 N_j N^{-1/2} L^c$$

for some absolute constant  $c$ . By [7, (5.20)],

$$(5.3) \quad K_j(g; R) \ll \frac{\sqrt{(g, D)}}{gR} \sum_{\substack{d|gD \\ d \leq 2R}} d\tau(d) \tilde{K}_j(d; R),$$

where

$$\tilde{K}_j(d; R) = \sum_{\chi \in \mathcal{H}(1, d, 2R)} \left( \int_{-1/(RQ)}^{1/(RQ)} |W_j(a_j\beta, \chi)|^2 d\beta \right)^{1/2}.$$

By Lemma 4.3 with  $k = 1$ ,

$$\begin{aligned} \tilde{K}_j(d; R) &\ll |a_j|^{-1/2} \sum_{\chi \in \mathcal{H}(1, d, 2R)} \left( \int_{-|a_j|/(RQ)}^{|a_j|/(RQ)} |W_j(\beta, \chi)|^2 d\beta \right)^{1/2} \\ &\ll N^{-1/2} L^{C+1} (N_j + H_j N_j^{11/20}), \end{aligned}$$

where  $C$  is the constant appearing in Theorem 1.1 and

$$H_j = d^{-1} R^2 (|a_j|/(RQ)) N_j \ll d^{-1} P R L^2 \ll d^{-1} R N_j^{9/20} L^2.$$

Thus,

$$\tilde{K}_j(d; R) \ll N_j N^{-1/2} L^{C+3} (R/d + 1).$$

Clearly, this inequality and (5.3) imply (5.2). This completes the proof of the theorem.

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Department of Mathematics and Statistics  
Simon Fraser University  
Burnaby, Canada V5A 1S6  
E-mail: kkchoi@math.sfu.ca

Department of Mathematics  
Towson University  
8000 York Road  
Towson, MD 21252, U.S.A.  
E-mail: akumchev@towson.edu

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