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Cyclic sum of multiple zeta values

by

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1. Introduction. In recent years, the multiple zeta values and multiple zeta-star values ("MZVs" and "MZSVs", respectively, for short) have appeared in various fields of mathematics and physics and attracted much interest (cf. [1], [3], [4], [12], [15]). At present, one of the central problems in this area is to clarify the structure of $\mathbb{Q}$-algebras generated by MZVs and MZSVs. (These algebras coincide with each other.) MZVs and MZSVs satisfy many relations, but their global structure is not yet fully understood. Here we will review their definitions and the well known basic identities, called the *sum formulas* for MZVs and MZSVs. The main purpose of this paper is to give a clean-cut decomposition of the sum formula for MZSVs and get a new family of relations between MZSVs and Riemann zeta values.

For any multi-index $\mathbf{k} = (k_1, \ldots, k_n)$ ($k_i \in \mathbb{Z}$, $k_i > 0$), the *weight* $\mathrm{wt}(\mathbf{k})$, *depth* $\mathrm{dep}(\mathbf{k})$, and *height* $\mathrm{ht}(\mathbf{k})$ of $\mathbf{k}$ are by definition the integers $k = k_1 + \cdots + k_n$, n, and $s = \#\{i \mid k_i > 1\}$, respectively. We denote by $I(k,n,s)$ the set of multi-indices $\mathbf{k}$ of weight k, depth n, and height s, and by $I_0(k,n,s)$ the subset of *admissible* indices, i.e., indices with the extra requirement that $k_1 \geq 2$. We also use the set $I(k,n) = \bigcup_{s=1}^{\min(n,k-n)} I(k,n,s)$. For any admissible index $\mathbf{k} = (k_1, \ldots, k_n) \in I_0(k,n,s)$, the multiple harmonic series MZV $\zeta(\mathbf{k})$ and MZSV $\zeta^*(\mathbf{k})$ are defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$
$$\zeta^*(\mathbf{k}) = \zeta^*(k_1, \ldots, k_n) = \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

Note that there exist natural linear relations between MZVs and MZSVs, for example,

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$$\zeta^*(k_1,k_2)=\zeta(k_1,k_2)+\zeta(k_1+k_2),\quad \zeta(k_1,k_2)=\zeta^*(k_1,k_2)-\zeta^*(k_1+k_2),$$
$$\begin{aligned}\zeta^*(k_1,k_2,k_3)&=\zeta(k_1,k_2,k_3)+\zeta(k_1+k_2,k_3)+\zeta(k_1,k_2+k_3)\\&\quad+\zeta(k_1+k_2+k_3),\\ \zeta(k_1,k_2,k_3)&=\zeta^*(k_1,k_2,k_3)-\zeta^*(k_1+k_2,k_3)-\zeta^*(k_1,k_2+k_3)\\&\quad+\zeta^*(k_1+k_2+k_3),\end{aligned}$$

and so the $\mathbb{Q}$-algebras of MZVs and MZSVs coincide with each other. Multiple zeta-star values $\zeta^*(\mathbf{k})$ were studied by Euler ([5]), and his study is the origin of various investigations of multiple zeta values $\zeta(\mathbf{k})$.

The sum formulas were conjectured by C. Moen and M. E. Hoffman ([7]) and M. Schmidt and C. Markett ([10]) independently, and proved by A. Granville ([6]) and D. Zagier ([16]) independently. The statement of the formulas is as follows: For any integers $k>n>0$, we have

$$\sum_{(k_1,k_2,\dots,k_{n+1})\in I(k,n+1)}\zeta(k_1+1,k_2,\dots,k_{n+1})=\zeta(k+1)$$

and

$$\sum_{(k_1,k_2,\dots,k_{n+1})\in I(k,n+1)}\zeta^*(k_1+1,k_2,\dots,k_{n+1})=\binom{k}{n}\zeta(k+1)$$

(see also [8], [9], [11], [13], [14]).

In this article, we introduce a new classification of MZSVs to give a clean-cut decomposition of the sum formula. The definition is an analogy of Hoffman's cyclic equivalence classes for MZVs ([8]). However, the totals of the values in each class for the present case are much simpler than the ones for MZVs. They are integral multiples of a single Riemann zeta value. We shall state the result more precisely in the next section.

2. Cyclic equivalence classes of MZSVs and the main result. In this section we define the cyclic classes of MZSVs and state the main result. The proof is given in the next section, together with a new proof of the sum formula for MZSVs.

The cyclic sum formula for MZVs was conjectured by M. E. Hoffman and proved by the first author ([8]). Similarly we define cyclic equivalence classes of multiple indices in the set

$$I(k,n)=\{(k_1,\dots,k_n)\mid k_1+\dots+k_n=k,\ k_1,\dots,k_n\geq 1\}.$$

We say two elements of $I(k,n)$ are *cyclically equivalent* if they are cyclic permutations of each other, i.e., for $\sigma=(k_1,\dots,k_n)$ and $j=1,\dots,n$, we define $(k_1,\dots,k_n)\equiv(\sigma^j(k_1),\dots,\sigma^j(k_n))$. Let $\mathit{\Pi}(k,n)$ be the set of cyclic equivalence classes of $I(k,n)$. We denote by $|\alpha|$ the cardinality of the set α. Then our main result is as follows:

THEOREM 1. *For any index set $\alpha \in \Pi(k,n)$ with $0 < n < k$, we have*

$$\sum_{(k_1,k_2,\dots,k_n)\in\alpha} \sum_{i=0}^{k_1-2} \zeta^*(k_1-i,k_2,\dots,k_n,i+1) = \frac{k|\alpha|}{n}\zeta(k+1)$$

where the inner sum on the left-hand side is treated as 0 whenever $k_1 = 1$.

In general, the cyclic sum formula for MZVs, given in [8], has linear combinations on both sides of the identity. The right-hand side of the above identity is an integral multiple of a single Riemann zeta value.

Here we give a table of classification of MZSVs of weights less than 8.

wt.	dep.	rep. of α	MZSVs	total	$n/\lvert\alpha\rvert$
3	2	(2)	$\zeta^*(2,1)$	$2\zeta(3)$	1
4	2	(3)	$\zeta^*(3,1), \zeta^*(2,2)$	$3\zeta(4)$	1
	3	$(2,1)$	$\zeta^*(2,1,1)$	$3\zeta(4)$	1
5	2	(4)	$\zeta^*(4,1), \zeta^*(3,2), \zeta^*(2,3)$	$4\zeta(5)$	1
	3	$(3,1)$	$\zeta^*(3,1,1), \zeta^*(2,1,2)$	$4\zeta(5)$	1
	3	$(2,2)$	$\zeta^*(2,2,1)$	$2\zeta(5)$	2
	4	$(2,1,1)$	$\zeta^*(2,1,1,1)$	$4\zeta(5)$	1
6	2	(5)	$\zeta^*(5,1), \zeta^*(4,2), \zeta^*(3,3), \zeta^*(2,4)$	$5\zeta(6)$	1
	3	$(4,1)$	$\zeta^*(4,1,1), \zeta^*(3,1,2), \zeta^*(2,1,3)$	$5\zeta(6)$	1
	3	$(3,2)$	$\zeta^*(3,2,1), \zeta^*(2,2,2), \zeta^*(2,3,1)$	$5\zeta(6)$	1
	4	$(3,1,1)$	$\zeta^*(3,1,1,1), \zeta^*(2,1,1,2)$	$5\zeta(6)$	1
	4	$(2,2,1)$	$\zeta^*(2,2,1,1), \zeta^*(2,1,2,1)$	$5\zeta(6)$	1
	5	$(2,1,1,1)$	$\zeta^*(2,1,1,1,1)$	$5\zeta(6)$	1
7	2	(6)	$\zeta^*(6,1), \zeta^*(5,2), \zeta^*(4,3), \zeta^*(3,4), \zeta^*(2,5)$	$6\zeta(7)$	1
	3	$(5,1)$	$\zeta^*(5,1,1), \zeta^*(4,1,2), \zeta^*(3,1,3), \zeta^*(2,1,4)$	$6\zeta(7)$	1
	3	$(4,2)$	$\zeta^*(4,2,1), \zeta^*(3,2,2), \zeta^*(2,2,3), \zeta^*(2,4,1)$	$6\zeta(7)$	1
	3	$(3,3)$	$\zeta^*(3,3,1), \zeta^*(2,3,2)$	$3\zeta(7)$	2
	4	$(4,1,1)$	$\zeta^*(4,1,1,1), \zeta^*(3,1,1,2), \zeta^*(2,1,1,3)$	$6\zeta(7)$	1
	4	$(3,2,1)$	$\zeta^*(3,2,1,1), \zeta^*(2,2,1,2), \zeta^*(2,1,3,1)$	$6\zeta(7)$	1
	4	$(3,1,2)$	$\zeta^*(3,1,2,1), \zeta^*(2,1,2,2), \zeta^*(2,3,1,2)$	$6\zeta(7)$	1
	4	$(2,2,2)$	$\zeta^*(2,2,2,1)$	$2\zeta(7)$	3
	5	$(3,1,1,1)$	$\zeta^*(3,1,1,1,1), \zeta^*(2,1,1,1,2)$	$6\zeta(7)$	1
	5	$(2,2,1,1)$	$\zeta^*(2,2,1,1,1), \zeta^*(2,1,1,2,1)$	$6\zeta(7)$	1
	5	$(2,1,2,1)$	$\zeta^*(2,1,2,1,1)$	$3\zeta(7)$	2
	6	$(2,1,1,1,1)$	$\zeta^*(2,1,1,1,1,1)$	$6\zeta(7)$	1

Together with other known results (for example, by the formula in [2]), we get the following relations.

EXAMPLES. (a) For any $n > 1$, we have

$$\sum_{i=0}^{n-2} \zeta^*(\underbrace{2,\ldots,2}_{i}, 3, \underbrace{2,\ldots,2}_{n-2-i}, 1) = (2^{2-2n} + 2n - 3)\zeta(2n).$$

Indeed, we use Theorem 1 for $\alpha = \{(3, \underbrace{2,\ldots,2}_{n-2})\}$, and replace $\zeta^*(\underbrace{2,\ldots,2}_{n})$ by using $\zeta^*(\underbrace{2,\ldots,2}_{n}) = 2(1-2^{1-2n})\zeta(2n)$, which is a special case ($k = 2n = 2s$) of T. Aoki and the first author's result ([2])

$$\sum_{n=s}^{k-s} \sum_{\mathbf{k}\in I_0(k,n,s)} \zeta^*(\mathbf{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})\zeta(k).$$

(b) For any positive integers l and m, we have

$$\zeta^*(\underbrace{2, \underbrace{1,\ldots,1}_{m-1}, 2, \underbrace{1,\ldots,1}_{m-1}, \ldots, 2, \underbrace{1,\ldots,1}_{m-1}}_{lm}, 1) = (m+1)\zeta(l(m+1)+1).$$

For example, if we put $m = 2$, we have

$$\zeta^*(\underbrace{2,1,2,1,\ldots,2,1}_{2l}, 1) = 3\zeta(3l+1).$$

3. Proof of Theorem 1 and the sum formula. After proving the main result, we will give an elementary proof of the sum formula by using Theorem 1.

In the proof of Theorem 1, we need a key fact on an infinite series C defined as follows. For any positive integers $n, k_1, \ldots, k_n$ with $k_1 + \cdots + k_n > n$ (i.e., at least one of the k_i's is > 1), define $C(k_1, \ldots, k_n)$ as the convergent series

$$C(k_1,\ldots,k_n) = \sum_{\substack{a_1\ge\cdots\ge a_n\ge a_{n+1}\ge 1\\ a_1\ne a_{n+1}}} \frac{1}{a_1^{k_1}\cdots a_n^{k_n}(a_1 - a_{n+1})}.$$

KEY LEMMA 1. *For any positive integers n and $k_1, \ldots, k_n$ with $k_i > 1$ for some i, we have*

$$C(k_1,k_2,\ldots,k_n) - C(k_2,k_3,\ldots,k_n,k_1) = k_1\zeta(k+1) - \sum_{i=0}^{k_1-2} \zeta^*(k_1 - i, k_2, k_3, \ldots, k_n, i+1),$$

where we put $k = k_1 + \cdots + k_n$ and the sum on the right is understood to be 0 if $k_1 = 1$.

Proof. For any non-negative integer $i \le k_1 - 2$, we have

$$\sum_{\substack{a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge 1 \\ a_1 \ne a_{n+1}}} \frac{1}{a_1^{k_1-i} a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{i} (a_1 - a_{n+1})}$$

$$= \sum_{\substack{a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge 1 \\ a_1 \ne a_{n+1}}} \frac{1}{a_1^{k_1-i-1} a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{i+1}} \left(\frac{1}{a_1 - a_{n+1}} - \frac{1}{a_1} \right)$$

$$= \sum_{\substack{a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge 1 \\ a_1 \ne a_{n+1}}} \frac{1}{a_1^{k_1-i-1} a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{i+1} (a_1 - a_{n+1})}$$

$$- \sum_{\substack{a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge 1 \\ a_1 \ne a_{n+1}}} \frac{1}{a_1^{k_1-i} a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{i+1}}$$

$$= \sum_{\substack{a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge 1 \\ a_1 \ne a_{n+1}}} \frac{1}{a_1^{k_1-i-1} a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{i+1} (a_1 - a_{n+1})}$$

$$- \sum_{a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge 1} \frac{1}{a_1^{k_1-i} a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{i+1}} + \sum_{a \ge 1} \frac{1}{a^{k_1-i+k_2+k_3+\cdots+k_n+i+1}}$$

$$= \sum_{\substack{a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge 1 \\ a_1 \ne a_{n+1}}} \frac{1}{a_1^{k_1-i-1} a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{i+1} (a_1 - a_{n+1})}$$

$$- \zeta^*(k_1 - i, k_2, \ldots, k_n, i+1) + \zeta(k+1).$$

Adding up the above equality for $i = 0, 1, \ldots, k_1 - 2$, we obtain

$$C(k_1, k_2, \ldots, k_n) = \sum_{\substack{a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge 1 \\ a_1 \ne a_{n+1}}} \frac{1}{a_1 a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1-1} (a_1 - a_{n+1})}$$

$$- \sum_{i=0}^{k_1-2} \zeta^*(k_1 - i, k_2, \ldots, k_n, i+1) + (k_1 - 1)\zeta(k+1).$$

The first sum on the right-hand side can be written as

$$\sum_{\substack{a_1 \ge a_2 \ge \cdots \ge a_{n+1} \ge 1 \\ a_1 \ne a_{n+1}}} \frac{1}{a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1}} \left(\frac{1}{a_1 - a_{n+1}} - \frac{1}{a_1} \right)$$

$$= \sum_{\substack{a_2 \ge a_3 \ge \cdots \ge a_{n+1} \ge 1 \\ a_2 \ne a_{n+1}}} \frac{1}{a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1}} \sum_{a_1 \ge a_2} \left(\frac{1}{a_1 - a_{n+1}} - \frac{1}{a_1} \right)$$

$$+ \sum_{a_2 = a_3 = \cdots = a_{n+1} \ge 1} \frac{1}{a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1}} \sum_{a_1 > a_2} \left(\frac{1}{a_1 - a_{n+1}} - \frac{1}{a_1} \right)$$

$$= \sum_{\substack{a_2 \geq a_3 \geq \cdots \geq a_{n+1} \geq 1 \\ a_2 \neq a_{n+1}}} \frac{1}{a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1}} \sum_{a_{n+2}=1}^{a_{n+1}} \frac{1}{a_2 - a_{n+2}}$$

$$+ \sum_{a_2 = a_3 = \cdots = a_{n+1} \geq 1} \frac{1}{a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1}} \sum_{a_{n+2}=0}^{a_{n+1}-1} \frac{1}{a_2 - a_{n+2}}$$

$$= \sum_{\substack{a_2 \geq a_3 \geq \cdots \geq a_{n+1} \geq 1 \\ a_2 \neq a_{n+1}}} \frac{1}{a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1}} \sum_{a_{n+2}=1}^{a_{n+1}} \frac{1}{a_2 - a_{n+2}}$$

$$+ \sum_{a_2 = a_3 = \cdots = a_{n+1} \geq 1} \frac{1}{a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1}} \sum_{a_{n+2}=1}^{a_{n+1}-1} \frac{1}{a_2 - a_{n+2}} + \sum_{a \geq 1} \frac{1}{a^{k_2 + \cdots + k_n + k_1 + 1}}$$

$$= \sum_{\substack{a_2 \geq a_3 \geq \cdots \geq a_{n+2} \geq 1 \\ a_2 \neq a_{n+2}}} \frac{1}{a_2^{k_2} \cdots a_n^{k_n} a_{n+1}^{k_1} (a_2 - a_{n+2})} + \sum_{a \geq 1} \frac{1}{a^{k+1}}$$

$$= C(k_2, \ldots, k_n, k_1) + \zeta(k+1).$$

Thus we get the equality of Key Lemma 1. ∎

Proof of Theorem 1. The formula is readily obtained when we apply the Key Lemma 1 for all cyclic permutations of $(k_1, \ldots, k_n) \in \alpha$ and add them up. ∎

It is worth pointing out that Theorem 1 provides another proof of the sum formula.

Proof of the sum formula via Theorem 1. For each $\alpha \in \Pi(k, n)$, the number of MZSVs in the corresponding class is $(k-n)|\alpha|/n$, and so the mean value of each MZSVs can be computed as

$$\frac{k|\alpha|}{n} \zeta(k+1) \cdot \frac{n}{(k-n)|\alpha|} = \frac{k}{k-n} \zeta(k+1),$$

by using Theorem 1. It is well known that the number of MZSVs with weight $k+1$ and depth $n+1$ is $\binom{k-1}{n}$. So multiplying these values yields

$$\frac{k}{k-n} \zeta(k+1) \cdot \binom{k-1}{n} = \binom{k}{n} \zeta(k+1)$$

as the sum of all MZSVs of weight $k+1$ and depth $n+1$, and finally we get the sum formula for MZSVs:

$$\sum_{(k_1, k_2, \ldots, k_{n+1}) \in I(k, n+1)} \zeta^*(k_1 + 1, k_2, \ldots, k_{n+1}) = \binom{k}{n} \zeta(k+1). \ ∎$$

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