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Fourth power mean of character sums

by

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1. Introduction. Let $q \geq 3$ be an integer and χ be a Dirichlet character modulo q. The character sums

$$\sum_{a=N+1}^{N+H} \chi(a)$$

play an important role in number theory. Pólya [5] and Vinogradov [6] proved the inequality

$$\Big|\sum_{a=1}^{x} \chi(a)\Big| \leq c\sqrt{p}\ln p$$

when $q = p$ is a prime. Actually, the above inequality holds with the constant $c = 1$. D. A. Burgess [1] obtained the mean value estimate

$$\sum_{n=1}^{k}\Big|\sum_{m=1}^{h} \chi(n+m)\Big|^2 < kh,$$

where h is any positive integer. For fourth power moments, he specified the problem to the case of $q = p$, and proved (see [2])

$$\sum_{\chi\neq\chi_0}\sum_{n=1}^{p}\Big|\sum_{m=1}^{h} \chi(n+m)\Big|^4 \leq 6p^2h^2.$$

For general modulus q, he summed the mean value over all primitive characters and obtained (see [3])

$$\sideset{}{^*}\sum_{\chi \bmod q}\sum_{n=1}^{p}\Big|\sum_{m=1}^{h} \chi(n+m)\Big|^4 \leq 8\tau^7(q)q^2h^2,$$

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where $\sum^*_{\chi \bmod q}$ denotes the summation over all primitive characters modulo q and $\tau(n)$ is the Dirichlet divisor function.

In order to obtain an asymptotic formula for higher moments of character sums, the authors [7] studied the $2k$th power mean of the even primitive character sums over the quarter interval $[1, q/4)$, and obtained the following sharper asymptotic formula:

$$\sum^*_{\chi(-1)=1} \Big| \sum_{a<q/4} \chi(a) \Big|^{2k} = \frac{J(q)q^k}{16} \left(\frac{\pi}{8}\right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p\nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O(q^{k+\varepsilon}),$$

where $\sum^*_{\chi(-1)=1}$ denotes the summation over all primitive characters modulo q such that $\chi(-1) = 1$, ε is any fixed positive number, $J(q)$ denotes the number of primitive characters modulo q, and $\prod_{p|q}$ denotes the product over all prime divisors p of q; finally $C_m^n = m!/n!(m-n)!$.

Unfortunately, the methods used in [7] only work for the case of primitive characters. For a general nonprincipal character modulo q, they are not efficient.

The present work deals mainly with the fourth power mean of the nonprincipal character sums over the interval $[1, q/4)$ by using the properties of Dedekind sums, Cochrane sums and Dirichlet L-functions, and obtains a sharper asymptotic formula for it. We prove the following:

THEOREM. *Let $q \geq 5$ be an odd integer. Then we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} \Big| \sum_{a<q/4} \chi(a) \Big|^4 = \frac{21\phi^4(q)}{256q} \prod_{p^\alpha \| q} \frac{\frac{(p+1)^3}{p(p^2+1)} - \frac{1}{p^{3\alpha-1}}}{1 + \frac{1}{p} + \frac{1}{p^2}} + O(q^{2+\varepsilon}),$$

where ε is any fixed positive number and $\prod_{p^\alpha \| q}$ denotes the product over all prime divisors p of q with $p^\alpha \mid q$ and $p^{\alpha+1} \nmid q$.

For $k \geq 3$, how to get an asymptotic formula for $\sum_{\chi \neq \chi_0} |\sum_{a<q/4} \chi(a)|^{2k}$ is an open problem.

2. Some lemmas. To prove the theorem, we need the following lemmas.

LEMMA 1 (see [4]). *Let $q \geq 3$ be an odd number. For any nonprincipal character χ modulo q, we have*

$$\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1 - 2\chi(2)} \sum_{a=1}^{(q-1)/2} \chi(a).$$

LEMMA 2 ([7, proof of Lemma 3]). *Let $q \geq 5$ be an odd integer and χ be a Dirichlet character modulo q such that $\chi(-1) = 1$. Then*

$$\sum_{a=1}^{[q/4]} \chi(a) = -\frac{\overline{\chi}(4)}{8q} \sum_{a=1}^{4q} a\chi\chi_4(a),$$

where χ_4 is the primitive Dirichlet character modulo 4.

LEMMA 3. *Let q be an odd number and χ be a primitive Dirichlet character modulo q such that $\chi(-1) = -1$. Then*

$$\sum_{a=1}^{[q/4]} \chi(a) = \frac{\overline{\chi}(4) - \overline{\chi}(2) - 2}{2q} \sum_{a=1}^{q} a\chi(a).$$

Proof. We only prove the lemma in the case of $q \equiv 1 \pmod 4$. A similar argument yields the same result for $q \equiv 3 \pmod 4$. From the properties of the Dirichlet character modulo q, we can write

$$\begin{aligned}
(1)\quad & 4\chi(4) \sum_{a=1}^{q-1} a\chi(a) \\
&= \sum_{a=1}^{(q-1)/4} 4a\chi(4a) + \sum_{a=(q+3)/4}^{(2q-2)/4} 4a\chi(4a) + \sum_{a=(2q+2)/4}^{(3q-3)/4} 4a\chi(4a) + \sum_{a=(3q+1)/4}^{q-1} 4a\chi(4a) \\
&= \sum_{a=1}^{(q-1)/4} 4a\chi(4a) + \sum_{a=1}^{(q-1)/4} (4a + q - 1)\chi(4a - 1) \\
&\quad + \sum_{a=1}^{(q-1)/4} (4a + 2q - 2)\chi(4a - 2) + \sum_{a=1}^{(q-1)/4} (4a + 3q - 3)\chi(4a - 3) \\
&= \sum_{a=1}^{q-1} a\chi(a) + \chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a - \overline{4}) \\
&\quad + 2\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a - 2 \cdot \overline{4}) + 3\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a - 3 \cdot \overline{4}).
\end{aligned}$$

Note that $\overline{4} \equiv \frac{3q+1}{4} \pmod q$ if $q \equiv 1 \pmod 4$. So from (1), we have

$$\begin{aligned}
(2)\qquad 4\chi(4) \sum_{a=1}^{q-1} a\chi(a) &= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=(2q+2)/4}^{(3q-3)/4} \chi(a) \\
&\quad - 2\chi(4)q \sum_{a=(q+3)/4}^{(2q-2)/4} \chi(a) - 3\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a)
\end{aligned}$$

$$= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=(q+3)/4}^{(2q-2)/4} \chi(a) - 3\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a)$$
$$= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=1}^{(q-1)/2} \chi(a) - 2\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a),$$

where we have used the equality $\chi(-1) = -1$ and

$$\sum_{a=(q+3)/4}^{(2q-2)/4} \chi(a) = - \sum_{a=(2q+2)/4}^{(3q-3)/4} \chi(a).$$

Now, from (2) and Lemma 1, we get

$$4\chi(4) \sum_{a=1}^{q-1} a\chi(a)$$
$$= \sum_{a=1}^{q-1} a\chi(a) - (\chi(2) - 2\chi(4)) \sum_{a=1}^{q-1} a\chi(a) - 2\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a).$$

That is,

$$\sum_{a=1}^{(q-1)/4} \chi(a) = \frac{\overline{\chi}(4) - \overline{\chi}(2) - 2}{2q} \sum_{a=1}^{q-1} a\chi(a) = \frac{\overline{\chi}(4) - \overline{\chi}(2) - 2}{2q} \sum_{a=1}^{q} a\chi(a).$$

This completes the proof of Lemma 3 in the case of $q \equiv 1 \pmod 4$.

LEMMA 4 (see [8]). *Let $q \geq 3$ be an integer with $(h, q) = 1$. Denote by $S(h, q)$ the Dedekind sum*

$$S(h, q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q}\right)\right) \left(\left(\frac{ha}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

Then

$$S(h, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) |L(1, \chi)|^2,$$

where the last sum is over all Dirichlet character modulo d with $\chi(-1) = -1$, and $L(s, \chi)$ is the Dirichlet L-function corresponding to χ.

LEMMA 5. *Let q be any odd integer with $q \geq 3$ and χ be a Dirichlet character modulo q. Then for any integer $m \geq 0$, we have the identity*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \overline{\chi}(2^m)\Big|\sideset{}{'}\sum_{r=1}^{q} r\chi(r)\Big|^4$$

$$= q^4\phi(q)\sum_{d|q}\sum_{l|q}\mu(d)\mu(l)\sideset{}{'}\sum_{r=1}^{q} S(2^m r, q/d)S(r, q/l),$$

where $\mu(n)$ is the Möbius function and the last sum is over all integers r with $1 \leq r \leq q$ and $(r,q)=1$.

Proof. First we define the Cochrane sum $C(h,q)$, which was first introduced by Professor Todd Cochrane during his visit to Xi'an in October 2000, as follows:

$$C(h,q) = \sideset{}{'}\sum_{a=1}^{q}\left(\left(\frac{\bar{a}}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right).$$

Note that

$$\sideset{}{'}\sum_{c=1}^{q}\chi(c)\left(\left(\frac{c}{q}\right)\right) = \sum_{c=1}^{q}\chi(c)\left(\frac{c}{q}-\frac{1}{2}\right) = 0 \quad \text{if } \chi(-1)=1.$$

Then from the orthogonality relation for Dirichlet characters modulo q, we can write

$$\begin{aligned}
(3)\quad C(a,q) &= \sideset{}{'}\sum_{r=1}^{q}\left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{ar}{q}\right)\right)\\
&= \frac{1}{\phi(q)}\sum_{\chi \bmod q}\left\{\sum_{r=1}^{q}\chi(r)\left(\left(\frac{r}{q}\right)\right)\right\}\times\left\{\sum_{s=1}^{q}\chi(s)\left(\left(\frac{as}{q}\right)\right)\right\}\\
&= \frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\left\{\sum_{r=1}^{q}\chi(r)\left(\left(\frac{r}{q}\right)\right)\right\}\times\left\{\sum_{s=1}^{q}\overline{\chi}(a)\chi(as)\left(\left(\frac{as}{q}\right)\right)\right\}\\
&= \frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(a)\left(\sum_{r=1}^{q}\chi(r)\left(\left(\frac{r}{q}\right)\right)\right)^2\\
&= \frac{1}{q^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(a)\Big(\sideset{}{'}\sum_{r=1}^{q} r\chi(r)\Big)^2,
\end{aligned}$$

where we have used the fact that $\sum_{r=1}^{q}\chi(r)=0$ if χ is not the principal character modulo q. Now the identity

$$\sum_{a=1}^{q}{}' C(2^m a, q) C(a, q) = \frac{1}{q^4\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(2^m) \Big| \sum_{r=1}^{q}{}' r\chi(r) \Big|^4 \tag{4}$$

follows from (3) immediately.

On the other hand, from the definition of $C(a, q)$ we have

$$\begin{aligned}
\sum_{a=1}^{q}{}' C(2^m a, q) C(a, q) &= \sum_{a=1}^{q}{}' \sum_{r=1}^{q}{}' \left(\left(\frac{\overline{r}}{q}\right)\right) \left(\left(\frac{2^m a r}{q}\right)\right) \sum_{s=1}^{q}{}' \left(\left(\frac{\overline{s}}{q}\right)\right) \left(\left(\frac{a s}{q}\right)\right) \\
&= \sum_{r=1}^{q}{}' \sum_{s=1}^{q}{}' \left(\left(\frac{\overline{r}}{q}\right)\right) \left(\left(\frac{\overline{s}}{q}\right)\right) \sum_{a=1}^{q}{}' \left(\left(\frac{2^m a r}{q}\right)\right) \left(\left(\frac{a s}{q}\right)\right) \\
&= \sum_{r=1}^{q}{}' \sum_{s=1}^{q}{}' \left(\left(\frac{\overline{r}}{q}\right)\right) \left(\left(\frac{\overline{s}}{q}\right)\right) \sum_{a=1}^{q}{}' \left(\left(\frac{2^m r \overline{s} a}{q}\right)\right) \left(\left(\frac{a}{q}\right)\right).
\end{aligned}$$

In the last step we have used the fact that if $(s, q) = 1$ and a runs through a reduced residue system modulo q, so does $\overline{s}a$. Therefore, from the definition of Dedekind sums $S(h, q)$ and the identities

$$\sum_{s=1}^{q}{}' = \sum_{d|q} \mu(d) \sum_{s=1}^{q/d} \quad \text{and} \quad S(r, q) = S(\overline{r}, q),$$

we have

$$\begin{aligned}
&\sum_{a=1}^{q}{}' C(2^m a, q) C(a, q) \\
&= \sum_{r=1}^{q}{}' \sum_{s=1}^{q}{}' \left(\left(\frac{\overline{r}}{q}\right)\right) \left(\left(\frac{\overline{s}}{q}\right)\right) \sum_{a=1}^{q}{}' \left(\left(\frac{2^m r \overline{s} a}{q}\right)\right) \left(\left(\frac{a}{q}\right)\right) \\
&= \sum_{r=1}^{q}{}' \sum_{s=1}^{q}{}' \left(\left(\frac{\overline{r}}{q}\right)\right) \left(\left(\frac{\overline{s}}{q}\right)\right) \sum_{l|q} \mu(l) \sum_{a=1}^{q/l} \left(\left(\frac{2^m r \overline{s} a}{q/l}\right)\right) \left(\left(\frac{a}{q/l}\right)\right) \\
&= \sum_{l|q} \mu(l) \sum_{r=1}^{q}{}' \sum_{s=1}^{q}{}' \left(\left(\frac{\overline{r}}{q}\right)\right) \left(\left(\frac{\overline{s}}{q}\right)\right) S(2^m r \overline{s}, q/l) \\
&= \sum_{l|q} \mu(l) \sum_{r=1}^{q}{}' \sum_{s=1}^{q}{}' \left(\left(\frac{\overline{r}}{q}\right)\right) \left(\left(\frac{s}{q}\right)\right) S(2^m r s, q/l) \\
&= \sum_{l|q} \mu(l) \sum_{d|q} \mu(d) \sum_{r=1}^{q}{}' \sum_{s=1}^{q/d} \left(\left(\frac{2^m r s}{q/d}\right)\right) \left(\left(\frac{s}{q/d}\right)\right) S(\overline{r}, q/l)
\end{aligned} \tag{5}$$

$$= \sum_{d|q} \sum_{l|q} \mu(d)\mu(l) \sideset{}{'}\sum_{r=1}^{q} S(2^m r, q/d) S(\overline{r}, q/l)$$

$$= \sum_{d|q} \sum_{l|q} \mu(d)\mu(l) \sideset{}{'}\sum_{r=1}^{q} S(2^m r, q/d) S(r, q/l).$$

Now Lemma 5 follows from (4) and (5). ∎

LEMMA 6. *Let u and v be odd integers with $(u, v) = d \geq 2$, and χ_u^0 and χ_v^0 be the principal characters modulo u and v, respectively. If $r(n) = \sum_{d|n} \chi_u^0(d)\chi_v^0(n/d)$, then for any integer $m \geq 0$ we have the identity*

$$\sum_{\substack{n=1\\(n,d)=1}}^{\infty} \frac{r(2^m n) r(n)}{n^2} = \frac{(3m+5)\pi^4}{72} \frac{\prod_{p|uv} \frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p|d} \frac{p^2}{p^2-1}}.$$

Proof. Noting that $r(n)$ is a multiplicative function, we can write

$$\sum_{\substack{n=1\\(n,d)=1}}^{\infty} \frac{r(2^m n) r(n)}{n^2} \tag{6}$$
$$= \left(r(2^m) + \sum_{j=1}^{\infty} \frac{(m+j+1)(j+1)}{4^j} \right) \sum_{\substack{n=1\\(n,d)=1\\2\nmid n}}^{\infty} \frac{r^2(n)}{n^2}.$$

After some simple calculation, we get

$$\sum_{j=1}^{\infty} \frac{(m+j+1)(j+1)}{4^j} = \frac{21m+53}{27}. \tag{7}$$

Moreover,

$$\sum_{\substack{n=1\\(n,d)=1\\2\nmid n}}^{\infty} \frac{r^2(n)}{n^2} = \prod_{\substack{p\nmid d\\p\neq 2}} \left(1 + \frac{r^2(p)}{p^2} + \frac{r^2(p^2)}{p^4} + \cdots \right)$$

by using the Euler product formula. Note that

$$r(p^\alpha) = \begin{cases} 1 & \text{if } p \mid u,\ p \nmid v, \\ \alpha + 1 & \text{if } p \nmid u,\ p \nmid v, \\ 1 & \text{if } p \nmid u,\ p \mid v, \\ 0 & \text{if } p \mid u,\ p \mid v, \end{cases}$$

for any positive integer α and prime p. Hence,

$$\sum_{\substack{n=1\\(n,d)=1\\2\nmid n}}^{\infty}\frac{r^2(n)}{n^2}=\prod_{\substack{p\nmid uv\\p\neq 2}}\left(1+\frac{2^2}{p^2}+\frac{3^2}{p^4}+\cdots\right)$$

$$\times\prod_{\substack{p\mid u\\p\nmid v}}\left(1+\frac{1}{p^2}+\frac{1}{p^4}+\cdots\right)\prod_{\substack{p\mid v\\p\nmid u}}\left(1+\frac{1}{p^2}+\frac{1}{p^4}+\cdots\right).$$

Let

$$S=1+\frac{2^2}{p^2}+\frac{3^2}{p^4}+\cdots.$$

It is clear that

$$S\left(1-\frac{1}{p^2}\right)^2=1+\frac{2}{p^2}\left(\frac{1}{1-\frac{1}{p^2}}\right).$$

Therefore,

$$\sum_{\substack{n=1\\(n,d)=1\\2\nmid n}}^{\infty}\frac{r^2(n)}{n^2}=\prod_{\substack{p\nmid uv\\p\neq 2}}\left(1-\frac{1}{p^2}\right)^{-3}\left(1+\frac{1}{p^2}\right)\prod_{p\mid uv}\frac{p^2}{p^2-1}\prod_{p\mid d}\left(\frac{p^2}{p^2-1}\right)^{-1}\tag{8}$$

$$=\frac{27\zeta^4(2)}{80\zeta(4)}\prod_{p\mid uv}\frac{(p^2-1)^2}{p^2(p^2+1)}\prod_{p\mid d}\left(\frac{p^2}{p^2-1}\right)^{-1}$$

$$=\frac{3\pi^4}{128}\prod_{p\mid uv}\frac{(p^2-1)^2}{p^2(p^2+1)}\prod_{p\mid d}\left(\frac{p^2}{p^2-1}\right)^{-1},$$

where we used the identities $\zeta(2)=\pi^2/6$ and $\zeta(4)=\pi^4/90$. Now from (6)–(8), we get the assertion.

LEMMA 7. *Let u and v be odd integers with $(u,v)=d\geq 2$, and χ_u^0 and χ_v^0 be the principal characters modulo u and v, respectively. Then for any integer $m\geq 0$ we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2^m)|L(1,\chi\chi_u^0)|^2|L(1,\chi\chi_v^0)|^2$$

$$=\frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}\phi(d)\frac{\prod_{p\mid uv}\frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p\mid d}\frac{p^2}{p^2-1}}+O_m(d^{\varepsilon}).$$

Proof. For convenience, we put

$$A(y,\chi) = \sum_{N<n\le y} \chi(n)r(n),$$

where N is a parameter with $d \le N < d^4$ and $r(n)$ was defined in Lemma 6. Then from Abel's identity we have

$$\begin{aligned}
L(1,\chi\chi_u^0)L(1,\chi\chi_v^0) &= \sum_{n=1}^{\infty} \frac{\chi(n)r(n)}{n} \\
&= \sum_{1\le n\le N} \frac{\chi(n)r(n)}{n} + \int_N^\infty \frac{A(y,\chi)}{y^2}\,dy.
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
&\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)|L(1,\chi\chi_u^0)|^2|L(1,\chi\chi_v^0)|^2 \\
&\quad= \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)\Bigg(\sum_{1\le n_1\le N} \frac{\overline{\chi}(n_1)r(n_1)}{n_1} + \int_N^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\Bigg) \\
&\quad\quad\times \Bigg(\sum_{1\le n_2\le N} \frac{\chi(n_2)r(n_2)}{n_2} + \int_N^\infty \frac{A(y,\chi)}{y^2}\,dy\Bigg) \\
&\quad= \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)\Bigg(\sum_{1\le n_1\le N} \frac{\overline{\chi}(n_1)r(n_1)}{n_1}\Bigg)\Bigg(\sum_{1\le n_2\le N} \frac{\chi(n_2)r(n_2)}{n_2}\Bigg) \\
&\quad\quad+ \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)\Bigg(\sum_{1\le n_1\le N} \frac{\overline{\chi}(n_1)r(n_1)}{n_1}\Bigg)\Bigg(\int_N^\infty \frac{A(y,\chi)}{y^2}\,dy\Bigg) \\
&\quad\quad+ \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)\Bigg(\sum_{1\le n_2\le N} \frac{\chi(n_2)r(n_2)}{n_2}\Bigg)\Bigg(\int_N^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\Bigg) \\
&\quad\quad+ \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)\Bigg(\int_N^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\Bigg)\Bigg(\int_N^\infty \frac{A(y,\chi)}{y^2}\,dy\Bigg) \\
&\quad:= M_1 + M_2 + M_3 + M_4.
\end{aligned} \tag{9}$$

Now we shall calculate each term in the expression (9).

(i) From the orthogonality of Dirichlet characters we can write

$$
(10)\quad M_1 = \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)\Big(\sum_{1\le n_1\le N} \frac{\overline{\chi}(n_1)r(n_1)}{n_1}\Big)\Big(\sum_{1\le n_2\le N} \frac{\chi(n_2)r(n_2)}{n_2}\Big)
$$

$$
= \frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_2\equiv n_1 \,(\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1n_2} - \frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_2\equiv n_1 \,(\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1n_2}.
$$

For convenience, we split the sum over n_1 or n_2 into the following cases: (a) $d \le n_1 \le N$, $d/2^m \le n_2 \le N$; (b) $d \le n_1 \le N$, $1 \le n_2 \le d/2^m - 1$; (c) $1 \le n_1 \le d-1$, $d/2^m \le n_2 \le N$; (d) $1 \le n_1 \le d-1$, $1 \le n_2 \le d/2^m - 1$. So we have

$$
\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{d\le n_1\le N\\ 2^m n_2\equiv n_1 \,(\bmod d)}} \sideset{}{'}\sum_{d/2^m\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1n_2}
$$

$$
\ll \phi(d) \sum_{1\le s_1\le N/d} \sum_{1\le s_2\le 2^m N/d} \sideset{}{'}\sum_{\substack{l_1=1\\ l_2\equiv l_1 \,(\bmod d)}}^{d-1} \sideset{}{'}\sum_{l_2=1}^{d-1} \frac{r(s_1d+l_1)r(s_2d+l_2)}{(s_1d+l_1)(s_2d+l_2)}
$$

$$
\ll \phi(d) \sum_{1\le s_1\le N/d} \sum_{1\le s_2\le 2^m N/d} \sideset{}{'}\sum_{l_1=1}^{d-1} \frac{[(s_1d+l_1)(s_2d+l_1)]^{\varepsilon}}{(s_1d+l_1)(s_2d+l_1)}
$$

$$
\ll \frac{\phi(d)}{d} \sum_{1\le s_1\le N/d} \sum_{1\le s_2\le 2^m N/d} \frac{[(s_1d+1)(s_2d+1)]^{\varepsilon}}{s_1s_2}
$$

$$
\ll_m d^{\varepsilon}
$$

and

$$
\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{d\le n_1\le N\\ 2^m n_2\equiv n_1 \,(\bmod d)}} \sideset{}{'}\sum_{1\le n_2\le d/2^m-1} \frac{r(n_1)r(n_2)}{n_1n_2}
$$

$$
\ll \phi(d) \sum_{1\le r\le N/d} \sum_{1\le n_2\le d/2^m-1} (rn_2d)^{\varepsilon-1} \ll d^{\varepsilon},
$$

and also

$$
\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le d-1\\ 2^m n_2\equiv n_1 \,(\bmod d)}} \sideset{}{'}\sum_{d/2^m\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1n_2} \ll d^{\varepsilon},
$$

where we have used the estimate $r(n) \ll n^{\varepsilon}$.

For the case $1 \le n_1 \le d-1$, $1 \le n_2 \le d/2^m - 1$, the solution of the congruence $2^m n_2 \equiv n_1 \pmod d$ is $2^m n_2 = n_1$. Hence,

$$\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le d-1 \\ 2^m n_2\equiv n_1 \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1\le n_2\le d/2^m-1} \frac{r(n_1)r(n_2)}{n_1 n_2}$$

$$= \frac{\phi(d)}{2^{m+1}} \sideset{}{'}\sum_{1\le n_2\le d/2^m-1} \frac{r(2^m n_2)r(n_2)}{n_2^2} = \frac{\phi(d)}{2^{m+1}} \sideset{}{'}\sum_{n=1}^{\infty} \frac{r(2^m n)r(n)}{n^2} + O_m(d^{\varepsilon}).$$

Now from Lemma 6, we immediately get

$$\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ 2^m n_2\equiv n_1 \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1 n_2} \tag{11}$$

$$= \frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}\,\phi(d)\,\frac{\prod_{p|uv}\frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p|d}\frac{p^2}{p^2-1}} + O_m(d^{\varepsilon}).$$

Similarly, we also get the estimate

$$\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ 2^m n_2\equiv -n_1 \,(\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1 n_2} \tag{12}$$

$$= \frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ 2^m n_2+n_1=d}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1 n_2} + \frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ 2^m n_2+n_1=ld,\, l\ge 2}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1 n_2}$$

$$\ll \phi(d) \sum_{1\le n\le d-1} \frac{2^m r(n) r\left(\frac{d-n}{2^m}\right)}{n(d-n)} + \phi(d) \sideset{}{'}\sum_{1\le n_1\le N} \sum_{l=[n_1/d]+2}^{[(N+n_1)/d]} \frac{2^m r(n_1) r\left(\frac{ld-n_1}{2^m}\right)}{ldn_1-n_1^2}$$

$$\ll_m \frac{\phi(d)}{d} \sum_{1\le n\le d-1} \frac{(n(d-n))^{\varepsilon}}{n} + \frac{\phi(d)}{d} \sideset{}{'}\sum_{1\le n_1\le N} \sum_{l=[n_1/d]+2}^{[(N+n_1)/d]} \frac{n_1^{\varepsilon}(ld-n_1)^{\varepsilon}}{ln_1-n_1^2/d}$$

$$\ll_m d^{\varepsilon} + \frac{\phi(d)d^{\varepsilon}}{d} \sum_{n_1=1}^{N} \sum_{l=1}^{N} \frac{n_1^{\varepsilon} l^{\varepsilon}}{ln_1} \ll_m d^{\varepsilon}.$$

Then from (10)–(12), we have

$$M_1 = \frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}\,\phi(d)\,\frac{\prod_{p|uv}\frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p|d}\frac{p^2}{p^2-1}} + O_m(d^{\varepsilon}). \tag{13}$$

(ii) Note the partition identity

$$\begin{aligned}
A(y,\chi) = &\sum_{n\le\sqrt{y}}\chi(n)\chi_u^0(n)\sum_{m\le y/n}\chi(m)\chi_v^0(m)\\
&+\sum_{m\le\sqrt{y}}\chi(m)\chi_v^0(m)\sum_{n\le y/m}\chi(n)\chi_u^0(n)\\
&-\sum_{n\le\sqrt{N}}\chi(n)\chi_u^0(n)\sum_{m\le N/n}\chi(m)\chi_v^0(m)\\
&-\sum_{m\le\sqrt{N}}\chi(m)\chi_v^0(m)\sum_{n\le N/m}\chi(n)\chi_u^0(n)\\
&-\Big(\sum_{n\le\sqrt{y}}\chi(n)\chi_u^0(n)\Big)\Big(\sum_{n\le\sqrt{y}}\chi(n)\chi_v^0(n)\Big)\\
&+\Big(\sum_{n\le\sqrt{N}}\chi(n)\chi_u^0(n)\Big)\Big(\sum_{n\le\sqrt{N}}\chi(n)\chi_v^0(n)\Big).
\end{aligned}$$

Applying the Cauchy inequality and the estimates for character sums

$$\begin{aligned}
\sum_{\chi\ne\chi_0}\Big|\sum_{N\le n\le M}\chi(n)\Big|^2 &= \sum_{\chi\ne\chi_0}\Big|\sum_{N\le n\le M\le N+d}\chi(n)\Big|^2\\
&= \phi(d)\sum_{N\le n\le M\le N+d}\chi_0(n) - \Big|\sum_{N\le n\le M\le N+d}\chi_0(n)\Big|^2 \le \frac{\phi^2(d)}{4}
\end{aligned}$$

and the identity

$$\sum_{N\le n\le M}\chi(n)\chi_u^0(n) = \sum_{d|u}\mu(d)\chi(d)\sum_{N/d\le n\le M/d}\chi(n),$$

we have

$$\begin{aligned}
\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|A(y,\chi)|^2 \ll{} &\sqrt{y}\sum_{n\le\sqrt{y}}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\Big|\sum_{m\le y/n}\chi(m)\chi_u^0(m)\Big|^2\\
&+\sqrt{y}\sum_{m\le\sqrt{y}}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\Big|\sum_{n\le y/m}\chi(n)\chi_v^0(n)\Big|^2\\
&+\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\Big|\sum_{n\le\sqrt{y}}\chi(n)\chi_u^0(n)\Big|^2\times\Big|\sum_{n\le\sqrt{y}}\chi(n)\chi_v^0(n)\Big|^2\\
\ll{} &yd^{2+\varepsilon}.
\end{aligned}\tag{14}$$

Then from the Cauchy inequality and (14) we can write

$$
\begin{aligned}
(15)\qquad M_2 &= \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)\Big(\sum_{1\le n_1\le N}\frac{\overline{\chi}(n_1)r(n_1)}{n_1}\Big)\Big(\int_N^\infty \frac{A(y,\chi)}{y^2}\,dy\Big)\\
&\ll \sum_{1\le n_1\le N} n_1^{\varepsilon-1}\int_N^\infty \frac{1}{y^2}\Big(\sum_{\chi(-1)=-1}|A(y,\chi)|\Big)\,dy\\
&\ll N^\varepsilon \int_N^\infty \frac{d^{3/2+\varepsilon}\sqrt{y}}{y^2}\,dy \ll \frac{d^{3/2+\varepsilon}}{N^{1/2-\varepsilon}}.
\end{aligned}
$$

(iii) Similar to (ii), we can also get

$$
(16)\qquad M_3 \ll \frac{d^{3/2+\varepsilon}}{N^{1/2-\varepsilon}}.
$$

(iv) By the same argument as in (ii), and noting the absolute convergence of the integrals, we can write

$$
\begin{aligned}
(17)\quad M_4 &= \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(2^m)\Big(\int_N^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy\Big)\Big(\int_N^\infty \frac{A(y,\chi)}{y^2}\,dy\Big)\\
&\le \int_N^\infty\int_N^\infty \frac{1}{y^2z^2}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} |A(y,\overline{\chi})|\,|A(z,\chi)|\,dy\,dz\\
&\ll \int_N^\infty \frac{1}{y^2}\int_N^\infty \frac{1}{z^2}\Big(\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|A(y,\overline{\chi})|^2\Big)^{1/2}\Big(\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|A(z,\chi)|^2\Big)^{1/2}\,dy\,dz\\
&\ll \Big(\int_N^\infty \frac{1}{y^2}\Big(\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|A(y,\chi)|^2\Big)^{1/2}\,dy\Big)^2\\
&\ll \Big(\int_N^\infty \frac{d^{1+\varepsilon}}{y^{3/2}}\,dy\Big)^2 \ll \frac{d^{2+\varepsilon}}{N}.
\end{aligned}
$$

Now, taking $N = d^3$, combining (9)–(17) we obtain the asymptotic formula of Lemma 7.

LEMMA 8 ([9, Lemma 5]). *Let p be a prime, and α and β be nonnegative integers with $\beta \ge \alpha$. Then*

$$\sum_{d_1|p^\beta}\sum_{d_2|p^\alpha}\frac{d_1^2d_2^2}{\phi(d_1)\phi(d_2)}\,\phi(d)\,\frac{\prod_{p|uv}\frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p|d}\frac{p^2}{p^2-1}}$$
$$= p^{3\alpha}\,\frac{\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}+\frac{(p^2-1)^2p^{2\alpha}(p^\beta-p^\alpha)}{(p-1)^2(p^2+1)},$$

where $d=(d_1,d_2)$ denotes the greatest common divisor of d_1 and d_2.

LEMMA 9. *Let q be any odd integer with $q\ge 3$, and χ be a Dirichlet character modulo q. Then for any integer $m\ge 0$, we have the asymptotic formulas*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(2^m)\Big|\sum_{r=1}^{q}r\chi(r)\Big|^4$$
$$=\frac{(3m+5)q^3\phi^4(q)}{72\cdot 2^{m+1}}\prod_{p^\alpha\|q}\frac{\frac{(p+1)^3}{p(p^2+1)}-\frac{1}{p^{3\alpha-1}}}{1+\frac{1}{p}+\frac{1}{p^2}}+O_m(q^{6+\varepsilon})$$

and

$$\sum_{\substack{\chi \bmod 4q\\ \chi(-1)=-1}}\Big|\sum_{r=1}^{4q}r\chi(r)\Big|^4=\frac{488}{9}\,q^3\phi^4(q)\prod_{p^\alpha\|q}\frac{\frac{(p+1)^3}{p(p^2+1)}-\frac{1}{p^{3\alpha-1}}}{1+\frac{1}{p}+\frac{1}{p^2}}+O_m(q^{6+\varepsilon}).$$

Proof. We only prove the first formula; the second one can be proved by the same method. From Lemmas 5 and 4 we have

$$(18)\qquad \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(2^m)\Big|\sum_{r=1}^{q}r\chi(r)\Big|^4$$
$$=q^4\phi(q)\sum_{d|q}\sum_{l|q}\mu(d)\mu(l){\sum_{a=1}^{q}}'S(2^ma,q/d)S(a,q/l)$$
$$=q^4\phi(q)\sum_{d|q}\sum_{l|q}\mu(d)\mu(l){\sum_{a=1}^{q}}'\Bigg(\frac{d}{\pi^2q}\sum_{u|q/d}\frac{u^2}{\phi(u)}\sum_{\substack{\chi \bmod u\\ \chi(-1)=-1}}\chi(2^ma)|L(1,\chi)|^2\Bigg)$$
$$\times\Bigg(\frac{l}{\pi^2q}\sum_{v|q/l}\frac{v^2}{\phi(v)}\sum_{\substack{\chi \bmod v\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2\Bigg)$$

$$= \frac{q^2\phi(q)}{\pi^4} \sum_{d|q} \sum_{l|q} \mu(d)\mu(l)dl \sum_{u|q/d} \sum_{v|q/l} \frac{u^2v^2}{\phi(u)\phi(v)}$$
$$\times \sum_{\substack{\chi_1 \bmod u \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod v \\ \chi_2(-1)=-1}} {\sum_{a=1}^{k}}' \chi_1(2^m a)\chi_2(a)|L(1,\chi_1)|^2|L(1,\chi_2)|^2.$$

For each character χ_1 modulo u, it is clear that there exists one and only one $q_1 \mid u$ with a unique primitive character $\chi^1_{q_1}$ modulo q_1 such that $\chi_1 = \chi^1_{q_1}\chi^0_u$. Similarly, we also have $\chi_2 = \chi^2_{q_2}\chi^0_v$, where $q_2 \mid v$, and $\chi^2_{q_2}$ is a primitive character modulo q_2. Noting that $u \mid q$ and $q \mid k$, from the orthogonality of Dirichlet characters we have

$$\text{(19)} \qquad {\sum_{a=1}^{q}}' \chi_1(a)\chi_2(a) = \sum_{a=1}^{q} [\chi^1_{q_1}(a)\chi^0_q(a)][\chi^2_{q_2}(a)\chi^0_q(a)]$$
$$= \begin{cases} \phi(q) & \text{if } q_1 = q_2 \text{ and } \chi^1_{q_1} = \overline{\chi^2_{q_2}}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $d_1 = (u,v)$. If $q_1 = q_2$ and $\chi^1_{q_1} = \overline{\chi^2_{q_2}}$, then $\chi^1_{q_1}\chi^0_{d_1}$ is also a character modulo d_1. So from (18), (19) and Lemma 7 we get

$$\text{(20)} \qquad \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(2^m) \Big| \sum_{r=1}^{q} r\chi(r) \Big|^4$$
$$= \frac{q^2\phi^2(q)}{\pi^4} \sum_{d|q} \sum_{l|q} \mu(d)\mu(l)dl \sum_{u|q/d} \sum_{v|q/l} \frac{u^2v^2}{\phi(u)\phi(v)}$$
$$\times \sum_{\substack{\chi \bmod (u,v) \\ \chi(-1)=-1}} \chi(2^m)|L(1,\chi\chi^0_u)|^2|L(1,\chi\chi^0_v)|^2$$
$$= \frac{q^2\phi^2(q)}{\pi^4} \sum_{d|q} \sum_{l|q} \sum_{u|q/d} \sum_{v|q/l} \frac{\mu(d)\mu(l)dlu^2v^2}{\phi(u)\phi(v)}$$
$$\times \left\{ \frac{(3m+5)\pi^4}{72 \cdot 2^{m+1}} \phi((u,v)) \frac{\prod_{p|uv} \frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p|(u,v)} \frac{p^2}{p^2-1}} + O_m((u,v)^\varepsilon) \right\}$$
$$= \frac{(3m+5)q^2\phi^2(q)}{72 \cdot 2^{m+1}} \sum_{d|q} \sum_{l|q} \sum_{u|q/d} \sum_{v|q/l} \frac{\mu(d)\mu(l)dlu^2v^2}{\phi(u)\phi(v)} \phi((u,v)) \frac{\prod_{p|uv} \frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p|(u,v)} \frac{p^2}{p^2-1}}$$
$$+ O_m(q^{6+\varepsilon}).$$

For any multiplicative functions $f(u)$ and $g(v)$,

$$\sum_{d|q}\sum_{l|q}\mu(d)\mu(l)dl\sum_{u|q/d}\sum_{v|q/l}f(u)g(v)$$
$$=\prod_{p^\alpha\|q}\Big[\sum_{u|p^\alpha}\sum_{v|p^\alpha}f(u)g(v)-2p\sum_{u|p^{\alpha-1}}\sum_{v|p^\alpha}f(u)g(v)+p^2\sum_{u|p^{\alpha-1}}\sum_{v|p^{\alpha-1}}f(u)g(v)\Big].$$

Now from Lemma 8 and the identity

$$p^{3\alpha}\,\frac{\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}+p^2\cdot p^{3\alpha-3}\,\frac{\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha-2}}}{1+\frac{1}{p}+\frac{1}{p^2}}$$
$$-2p\left[p^{3\alpha-3}\,\frac{\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha-2}}}{1+\frac{1}{p}+\frac{1}{p^2}}+\frac{(p^2-1)^2p^{2\alpha-2}(p^\alpha-p^{\alpha-1})}{(p-1)^2(p^2+1)}\right]$$
$$=\frac{p^{3\alpha+1}\left(1-\frac{1}{p}\right)^2}{1+\frac{1}{p}+\frac{1}{p^2}}\left(\frac{(p+1)^3}{(p^2+1)p^2}-\frac{1}{p^{3\alpha}}\right)$$

we get

$$(21)\quad \sum_{d|q}\sum_{l|q}\sum_{u|q/d}\sum_{v|q/l}\frac{\mu(d)\mu(l)dlu^2v^2}{\phi(u)\phi(v)}\,\phi((u,v))\,\frac{\prod_{p|uv}\frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p|(u,v)}\frac{p^2}{p^2-1}}$$
$$=\prod_{p^\alpha\|q}\left\{\sum_{d|p^\alpha}\sum_{l|p^\alpha}\mu(d)d\mu(l)l\sum_{u|p^\alpha/d}\sum_{v|p^\alpha/l}\frac{uv}{\phi(u)\phi(v)}\,\phi((u,v))\,\frac{\prod_{p_1|uv}\frac{(p_1^2-1)^2}{p_1^2(p_1^2+1)}}{\prod_{p_1|(u,v)}\frac{p_1^2}{p_1^2-1}}\right\}$$
$$=q\phi^2(q)\prod_{p^\alpha\|q}\frac{\frac{(p+1)^3}{p(p^2+1)}-\frac{1}{p^{3\alpha-1}}}{1+\frac{1}{p}+\frac{1}{p^2}}.$$

Now Lemma 9 follows immediately from (20) and (21).

LEMMA 10. *Let $q>2$ be an odd integer. Then*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1\\ \chi\neq\chi_0}}\Big|\sum_{b=1}^{4q}b\chi\chi_4(b)\Big|^4$$
$$=\sum_{\substack{\chi \bmod 4q\\ \chi(-1)=-1}}\Big|\sum_{b=1}^{4q}b\chi(b)\Big|^4-256\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|1-\chi(2)|^4\Big|\sum_{b=1}^{q}b\chi(b)\Big|^4.$$

Proof. Let χ_4^0 denote the principal character modulo 4. Then

$$
\sum_{\substack{\chi \bmod q\\ \chi(-1)=1\\ \chi\neq\chi_0}} \Big|\sum_{b=1}^{4q} b\chi\chi_4(b)\Big|^4 = \sum_{\substack{\chi \bmod 4q\\ \chi(-1)=-1}} \Big|\sum_{b=1}^{4q} b\chi(b)\Big|^4 - \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \Big|\sum_{b=1}^{4q} b\chi\chi_4^0(b)\Big|^4. \tag{22}
$$

For the inner summation in the second term, we have

$$
\begin{aligned}
\sum_{b=1}^{4q} b\chi\chi_4^0(b) &= \sum_{\substack{b=1\\ (b,2)=1}}^{4q} b\chi(b)\\
&= 2\sum_{\substack{b=1\\ (b,2)=1}}^{q} b\chi(b) + 2\sum_{\substack{b=1\\ 2\mid b}}^{q} b\chi(b) + 4q\sum_{\substack{b=1\\ 2\mid b}}^{q} \chi(b) + 2q\sum_{\substack{b=1\\ (b,2)=1}}^{q} \chi(b)\\
&= 2\sum_{b=1}^{q} b\chi(b) + 2q\sum_{\substack{b=1\\ 2\mid b}}^{q} \chi(b)\\
&= 2\sum_{b=1}^{q} b\chi(b) + 2\chi(2)\sum_{b=1}^{(q-1)/2} \chi(b).
\end{aligned}
$$

Now from Lemma 1, we have

$$
\sum_{b=1}^{4q} b\chi\chi_4^0(b) = (4-4\chi(2))\sum_{b=1}^{q} b\chi(b).
$$

Combining this with (22), we get the lemma.

3. Proof of the theorem. In this section we complete the proof of the theorem. From Lemmas 2, 3 and 10, we can write

$$
\begin{aligned}
\sum_{\chi\neq\chi_0} \Big|\sum_{a<q/4} \chi(a)\Big|^4 = \frac{1}{8^4q^4} &\sum_{\substack{\chi \bmod q\\ \chi(-1)=1\\ \chi\neq\chi_0}} \Big|\sum_{a=1}^{4q} a\chi\chi_4(a)\Big|^4\\
&+ \frac{1}{16q^4}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |2+\overline{\chi}(2)-\overline{\chi}(4)|^4\Big|\sum_{a=1}^{q} a\chi(a)\Big|^4
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{8^4q^4} \sum_{\substack{\chi \bmod 4q\\ \chi(-1)=-1}} \Big|\sum_{a=1}^{4q} a\chi(a)\Big|^4 - \frac{1}{16q^4} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |1-\chi(2)|^4\Big|\sum_{a=1}^{q} a\chi(a)\Big|^4 \\
&\quad + \frac{1}{16q^4} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |2+\overline{\chi}(2)-\overline{\chi}(4)|^4\Big|\sum_{a=1}^{q} a\chi(a)\Big|^4 \\
&= \frac{1}{8^4q^4} \sum_{\substack{\chi \bmod 4q\\ \chi(-1)=-1}} \Big|\sum_{a=1}^{4q} a\chi(a)\Big|^4 \\
&\quad + \frac{1}{16q^4} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} [40+12\chi(2)+12\overline{\chi}(2)-24\chi(4) \\
&\quad - 24\overline{\chi}(4)-4\chi(8)-4\overline{\chi}(8)4\chi(16)+4\overline{\chi}(16)]\Big|\sum_{a=1}^{q} a\chi(a)\Big|^4.
\end{aligned}
$$

Noting that

$$
\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \overline{\chi}(2^m)\Big|\sum_{r=1}^{q} r\chi(r)\Big|^4 = \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^m)\Big|\sum_{r=1}^{q} r\chi(r)\Big|^4,
$$

from Lemma 9 we get

$$
\sum_{\chi\neq\chi_0}\Big|\sum_{a<q/4}\chi(a)\Big|^4 = \frac{21\phi^4(q)}{256q}\prod_{p^\alpha\| q}\frac{\frac{(p+1)^3}{p(p^2+1)}-\frac{1}{p^{3\alpha-1}}}{1+\frac{1}{p}+\frac{1}{p^2}} + O(q^{2+\varepsilon}).
$$

This completes the proof of the theorem.

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