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Representations by sextenary quadratic forms whose coefficients are 1, 2 and 4

by

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1. Introduction. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers respectively. For $n \in \mathbb{N}_0$, and $a_1, \ldots, a_6 \in \mathbb{N}$ we define

$$N(a_1, \ldots, a_6; n) := \operatorname{card}\{(x_1, \ldots, x_6) \in \mathbb{Z}^6 \mid n = a_1x_1^2 + \cdots + a_6x_6^2\}. \tag{1.1}$$

Clearly,

$$N(a_1, \ldots, a_6; 0) = 1. \tag{1.2}$$

As $N(a_1, \ldots, a_6; n)$ is invariant under permutations of $a_1, \ldots, a_6$, we may suppose that

$$a_1 \leq \cdots \leq a_6.$$

There are 21 sextuples $(a_1, \ldots, a_6)$ satisfying

$$a_1, \ldots, a_6 \in \{1, 2, 4\}, \qquad 1 = a_1 \leq a_2 \leq \cdots \leq a_6. \tag{1.3}$$

For the 4 sextuples

$$(a_1, \ldots, a_6) = (1,1,1,1,1,1), (1,1,1,1,2,2), (1,1,2,2,2,2), (1,2,2,2,2,4)$$

it is known that $N(a_1, \ldots, a_6; n)$ $(n \in \mathbb{N})$ can be expressed in terms of the two sums

$$G_4(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{n/d}\right) d^2, \quad n \in \mathbb{N}, \tag{1.4}$$

$$H_4(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\frac{-4}{d}\right) d^2, \quad n \in \mathbb{N}, \tag{1.5}$$

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where $\left(\frac{-4}{d}\right)$ $(d \in \mathbb{N})$ is the Legendre–Jacobi–Kronecker symbol for discriminant -4 given by

$$\left(\frac{-4}{d}\right) = \begin{cases} +1 & \text{if } d \equiv 1 \pmod 4, \\ -1 & \text{if } d \equiv 3 \pmod 4, \\ 0 & \text{if } d \equiv 0 \pmod 2. \end{cases} \tag{1.6}$$

For $n \in \mathbb{N}$ we have

$$N(1,1,1,1,1,1;n) = 16G_4(n) - 4H_4(n), \tag{1.7}$$
$$N(1,1,1,1,2,2;n) = 8G_4(n) - 2(1+(-1)^n)H_4(n), \tag{1.8}$$
$$N(1,1,2,2,2,2;n) = 4G_4(n) - 2(1+(-1)^n)H_4(n), \tag{1.9}$$
$$N(1,2,2,2,2,4;n) = 2G_4(n) - 4H_4(n/4) \tag{1.10}$$

(see [2, Theorem 2.3, p. 551]). (If $f : \mathbb{N} \to \mathbb{C}$ and $m \notin \mathbb{N}$ we set $f(m) = 0$. Thus $H_4(n/4) = 0$ if $n \not\equiv 0 \pmod 4$.) Formula (1.7) is Jacobi's classical formula for the number of representations of a positive integer n as a sum of six squares (see for example [1]).

For the 8 sextuples

$$\begin{aligned}(a_1,\dots,a_6) = {}&(1,1,1,1,1,4),(1,1,1,1,4,4),(1,1,1,2,2,4),(1,1,1,4,4,4),\\ &(1,1,2,2,4,4),(1,1,4,4,4,4),(1,2,2,4,4,4),(1,4,4,4,4,4),\end{aligned}$$

$N(a_1,\dots,a_6;n)$ $(n \in \mathbb{N})$ can be expressed in terms of $G_4(n)$ and $H_4(n)$ when $n \not\equiv 1 \pmod 4$; however, when $n \equiv 1 \pmod 4$ the additional sum

$$I(n) := \sum_{\substack{(x,y)\in\mathbb{Z}^2\\ n=x^2+4y^2}} (x^2-4y^2), \qquad n \in \mathbb{N},\ n \equiv 1 \pmod 4, \tag{1.11}$$

is required. Let $q \in \mathbb{C}$ be such that $|q| < 1$. A basic property of $I(n)$ is

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} I(n)q^n = 2q\prod_{n=1}^{\infty}(1-q^{4n})^6 \tag{1.12}$$

(see [11, Vol. II, p. 377] and [17, p. 122]). We have

$$N(1,1,1,1,1,4;n) = \begin{cases} 6G_4(n) + 2I(n) & \text{if } n \equiv 1 \pmod 4, \\ 10G_4(n) & \text{if } n \equiv 2,3 \pmod 4, \\ 6G_4(n) - 4H_4(n) & \text{if } n \equiv 0 \pmod 4, \end{cases} \tag{1.13}$$

$$N(1,1,1,1,4,4;n) = \begin{cases} 4G_4(n) + 2I(n) & \text{if } n \equiv 1 \pmod 4, \\ 4G_4(n) & \text{if } n \equiv 3 \pmod 4, \\ 6G_4(n) & \text{if } n \equiv 2 \pmod 4, \\ 2G_4(n) - 4H_4(n) & \text{if } n \equiv 0 \pmod 4, \end{cases} \tag{1.14}$$

$$(1.15)\quad N(1,1,1,4,4,4;n)=\begin{cases} 3G_4(n)+\frac{3}{2}I(n) & \text{if } n\equiv 1 \pmod 4,\\ G_4(n) & \text{if } n\equiv 3 \pmod 4,\\ 3G_4(n) & \text{if } n\equiv 2 \pmod 4,\\ G_4(n)-4H_4(n) & \text{if } n\equiv 0 \pmod 4,\end{cases}$$

$$(1.16)\quad N(1,1,4,4,4,4;n)=\begin{cases} 2G_4(n)+I(n) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 4,\\ G_4(n) & \text{if } n\equiv 2 \pmod 4,\\ G_4(n)-4H_4(n) & \text{if } n\equiv 0 \pmod 4,\end{cases}$$

$$(1.17)\quad N(1,4,4,4,4,4;n)=\begin{cases} G_4(n)+\frac{1}{2}I(n) & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 2,3 \pmod 4,\\ G_4(n)-4H_4(n) & \text{if } n\equiv 0 \pmod 4,\end{cases}$$

$$(1.18)\quad N(1,1,1,2,2,4;n)=\begin{cases} 4G_4(n)+I(n) & \text{if } n\equiv 1 \pmod 4,\\ 4G_4(n) & \text{if } n\equiv 2,3 \pmod 4,\\ 4G_4(n)-4H_4(n) & \text{if } n\equiv 0 \pmod 4,\end{cases}$$

$$(1.19)\quad N(1,1,2,2,4,4;n)=\begin{cases} 2G_4(n)+I(n) & \text{if } n\equiv 1 \pmod 4,\\ 2G_4(n) & \text{if } n\equiv 2,3 \pmod 4,\\ 2G_4(n)-4H_4(n) & \text{if } n\equiv 0 \pmod 4,\end{cases}$$

$$(1.20)\quad N(1,2,2,4,4,4;n)=\begin{cases} G_4(n)+\frac{1}{2}I(n) & \text{if } n\equiv 1 \pmod 4,\\ G_4(n) & \text{if } n\equiv 2,3 \pmod 4,\\ G_4(n)-4H_4(n) & \text{if } n\equiv 0 \pmod 4\end{cases}$$

(see [3, Theorem 1.1]).

In this paper we consider the remaining $21-4-8=9$ sextuples. For the two sextuples

$$(a_1,\dots,a_6)=(1,1,1,1,1,2),(1,2,2,2,2,2),$$

we show that $N(a_1,\dots,a_6;n)$ ($n\in\mathbb{N}$) can be given in terms of the sums

$$(1.21)\qquad G_8(n):=\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-8}{n/d}\right)d^2,\quad n\in\mathbb{N},$$

$$(1.22)\qquad H_8(n):=\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-8}{d}\right)d^2,\quad n\in\mathbb{N},$$

where $\left(\frac{-8}{d}\right)$ ($d\in\mathbb{N}$) is the Legendre–Jacobi–Kronecker symbol for discriminant -8, namely

$$(1.23)\qquad \left(\frac{-8}{d}\right)=\begin{cases} +1 & \text{if } d\equiv 1,3 \pmod 8,\\ -1 & \text{if } d\equiv 5,7 \pmod 8,\\ 0 & \text{if } d\equiv 0 \pmod 2.\end{cases}$$

In Section 5 we prove the following result.

THEOREM 1. *Let $n \in \mathbb{N}$. Then*

(i) $$N(1,1,1,1,1,2;n) = \tfrac{32}{3}G_8(n) - \tfrac{2}{3}H_8(n),$$

(ii) $$N(1,2,2,2,2,2;n) = \tfrac{8}{3}G_8(n) - \tfrac{2}{3}H_8(n).$$

The formulas of Theorem 1 were stated but not proved by Liouville [14], [15]. For the remaining $9 - 2 = 7$ sextuples, we require for the evaluation of $N(a_1, \ldots, a_6; n)$ $(n \in \mathbb{N})$, in addition to $G_8(n)$ and $H_8(n)$, the integers $c(n)$ $(n \in \mathbb{N})$ defined by

(1.24) $$\sum_{n=1}^{\infty} c(n)q^n := q \prod_{n=1}^{\infty} (1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2.$$

It is known from the work of Martin [16, Table 1, p. 4853] that $c(n)$ is a multiplicative function of n. We prove the following result in Section 6.

THEOREM 2. *Let $n \in \mathbb{N}$. Then*

(i) $N(1,1,1,1,2,4;n)$
$$= \begin{cases} \frac{16}{3}G_8(n) + \frac{8}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{16}{3}G_8(n) - \frac{2}{3}H_8(n) - \frac{8}{3}c(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

(ii) $$N(1,1,1,2,2,2;n) = \tfrac{16}{3}G_8(n) - \tfrac{2}{3}H_8(n) + \tfrac{4}{3}c(n),$$

(iii) $N(1,1,1,2,4,4;n)$
$$= \begin{cases} \frac{8}{3}G_8(n) + \frac{10}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{8}{3}G_8(n) - \frac{2}{3}H_8(n) - 2c(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

(iv) $N(1,1,2,2,2,4;n)$
$$= \begin{cases} \frac{8}{3}G_8(n) + \frac{4}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{8}{3}G_8(n) - \frac{2}{3}H_8(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

(v) $N(1,1,2,4,4,4;n)$
$$= \begin{cases} \frac{4}{3}G_8(n) + \frac{8}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{4}{3}G_8(n) - \frac{2}{3}H_8(n) - \frac{2}{3}c(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

(vi) $N(1,2,2,2,4,4;n)$
$$= \begin{cases} \frac{4}{3}G_8(n) + \frac{2}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{4}{3}G_8(n) - \frac{2}{3}H_8(n) - \frac{2}{3}c(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

(vii) $N(1,2,4,4,4,4;n)$
$$= \begin{cases} \frac{2}{3}G_8(n) + \frac{4}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{2}{3}G_8(n) - \frac{2}{3}H_8(n) & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Following Ramanujan (see for example [4, p. 6]), we define the theta functions $\varphi(q)$ and $\psi(q)$ by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{1.25}$$

The basic properties of $\varphi(q)$ are

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{1.26}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{1.27}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2) \tag{1.28}$$

(see [4, pp. 71, 72, 15]). As simple consequences of Jacobi's triple product identity, we have

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}, \qquad \varphi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{1-q^{2n}}, \tag{1.29}$$

and

$$\psi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{1-q^n} \tag{1.30}$$

(see for example [10, pp. 282–284]). Setting

$$E_k := E_k(q) = \prod_{n=1}^{\infty} (1-q^{kn}), \qquad k \in \mathbb{N}, \tag{1.31}$$

we see from (1.29)–(1.31) that

$$\varphi(q) = \frac{E_2^5}{E_1^2 E_4^2}, \qquad \varphi(-q) = \frac{E_1^2}{E_2}, \qquad \psi(q) = \frac{E_2^2}{E_1}, \qquad \varphi(q^2) = \frac{E_4^5}{E_2^2 E_8^2}, \tag{1.32}$$

so that

$$\varphi(q)\varphi(q^2) = \frac{E_2^3 E_4^3}{E_1^2 E_8^2}. \tag{1.33}$$

From (1.1) and (1.25) we deduce that

$$\sum_{n=0}^{\infty} N(a_1, \ldots, a_6; n) q^n = \varphi(q^{a_1}) \cdots \varphi(q^{a_6}). \tag{1.34}$$

The proofs of Theorems 1 and 2 depend upon the following three results.

THEOREM 3. *Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then*

$$\sum_{n=1}^{\infty} G_8(n) q^n = \frac{1}{8}\varphi^5(q)\varphi(q^2) - \frac{1}{8}\varphi(q)\varphi^5(q^2).$$

THEOREM 4. *Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then*

$$1 - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)q^n = -\frac{1}{3}\,\varphi^5(q)\varphi(q^2) + \frac{4}{3}\,\varphi(q)\varphi^5(q^2).$$

THEOREM 5. *Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then*

$$\sum_{n=1}^{\infty} c(n)q^n = -\frac{1}{4}\,\varphi^5(q)\varphi(q^2) + \frac{3}{4}\,\varphi^3(q)\varphi^3(q^2) - \frac{1}{2}\,\varphi(q)\varphi^5(q^2).$$

Theorem 3 is proved in Section 2, Theorem 4 in Section 3 and Theorem 5 in Section 4.

In order to prove Theorems 3 and 4 we need the following results.

THEOREM 6. *Let $q \in \mathbb{C}$ be such that $|q| < 1$. Let $a \in \mathbb{C} \setminus \{0, 1, -1\}$ be such that $a \neq q^n$ for all $n \in \mathbb{Z}$. Then*

$$\prod_{n=1}^{\infty} \frac{(1-a^2q^n)(1-a^{-2}q^n)(1-q^n)^6}{(1-aq^n)^4(1-a^{-1}q^n)^4} = 1 + \frac{(1-a)^3}{a(1+a)} \sum_{n=1}^{\infty} \Big(\sum_{d\mid n}(a^d - a^{-d})d^2\Big)q^n.$$

Carlitz [5, eq. (1.3), p. 168] derived this theorem from a well-known formula in the theory of elliptic functions for the derivative of the Weierstrass $\wp$-function [20, Question 24, p. 459]. An elementary proof has been given by Dobbie [8].

THEOREM 7. *Let $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{C} \setminus \{0\}$ be such that $a_i \neq q^n a_j$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$ and all $n \in \mathbb{Z}$, and $a_1a_2a_3 = b_1b_2b_3$. Then*

$$\begin{aligned}
&\frac{(1-a_1b_1^{-1})(1-a_1b_2^{-1})(1-a_1b_3^{-1})}{(1-a_1a_2^{-1})(1-a_1a_3^{-1})} \prod_{j=1}^{\infty}\bigg(\frac{(1-a_1b_1^{-1}q^j)(1-a_1^{-1}b_1q^j)(1-a_1b_2^{-1}q^j)}{(1-a_1a_2^{-1}q^j)(1-a_1^{-1}a_2q^j)} \\
&\quad\times \frac{(1-a_1^{-1}b_2q^j)(1-a_1b_3^{-1}q^j)(1-a_1^{-1}b_3q^j)}{(1-a_1a_3^{-1}q^j)(1-a_1^{-1}a_3q^j)}\bigg) + \frac{(1-a_2b_1^{-1})(1-a_2b_2^{-1})(1-a_2b_3^{-1})}{(1-a_2a_1^{-1})(1-a_2a_3^{-1})} \\
&\quad\times \prod_{j=1}^{\infty} \frac{(1-a_2b_1^{-1}q^j)(1-a_2^{-1}b_1q^j)(1-a_2b_2^{-1}q^j)(1-a_2^{-1}b_2q^j)(1-a_2b_3^{-1}q^j)(1-a_2^{-1}b_3q^j)}{(1-a_2a_1^{-1}q^j)(1-a_2^{-1}a_1q^j)(1-a_2a_3^{-1}q^j)(1-a_2^{-1}a_3q^j)} \\
&\quad+ \frac{(1-a_3b_1^{-1})(1-a_3b_2^{-1})(1-a_3b_3^{-1})}{(1-a_3a_1^{-1})(1-a_3a_2^{-1})} \prod_{j=1}^{\infty}\bigg(\frac{(1-a_3b_1^{-1}q^j)(1-a_3^{-1}b_1q^j)(1-a_3b_2^{-1}q^j)}{(1-a_3a_1^{-1}q^j)(1-a_3^{-1}a_1q^j)} \\
&\qquad\qquad\times \frac{(1-a_3^{-1}b_2q^j)(1-a_3b_3^{-1}q^j)(1-a_3^{-1}b_3q^j)}{(1-a_3a_2^{-1}q^j)(1-a_3^{-1}a_2q^j)}\bigg) = 0.
\end{aligned}$$

Theorem 7 is a special case of a result about sigma functions, which was probably known to Weierstrass (see [20, Example 3, p. 451]). It can also be found in or deduced from [18], [19, eq. 7.4.3], [9, Ex. 5.23, p. 138] and [13].

Theorem 5 is proved using Berndt's catalogue of theta functions in terms of the parameters $x = 1-\varphi^4(-q)/\varphi^4(q)$ and $z = \varphi^2(q)$ (see [4, pp. 122, 123]).

Finally, in Section 7 we show that $c(n)$ defined in (1.24) satisfies the relation $c(2n) = -2c(n),\ n \in \mathbb{N}$.

2. Proof of Theorem 3. Let $q \in \mathbb{C}$ be such that $|q| < 1$. For $a, b \in \mathbb{N}$ with $a \le b$ we define

$$E_{a,b} = E_{a,b}(q) := \prod_{n=0}^{\infty}(1 - q^{bn+a}). \tag{2.1}$$

From (1.30) and (2.1) we see that

$$E_b = \prod_{n=1}^{\infty}(1 - q^{bn}) = \prod_{n=0}^{\infty}(1 - q^{bn+b}) = E_{b,b}. \tag{2.2}$$

If $b \equiv 0 \pmod 2$, then $b/2 \in \mathbb{N},\ b/2 \le b$ and

$$\begin{aligned} E_b E_{b/2,b} &= \prod_{n=0}^{\infty}(1 - q^{bn+b}) \prod_{n=0}^{\infty}(1 - q^{bn+b/2}) \\ &= \prod_{n=0}^{\infty}(1 - q^{b(2n+2)/2})(1 - q^{b(2n+1)/2}) = \prod_{n=1}^{\infty}(1 - q^{bn/2}) \end{aligned}$$

so that

$$E_{b/2,b} = E_{b/2}/E_b, \qquad b \in \mathbb{N},\ b \equiv 0 \pmod 2. \tag{2.3}$$

Thus, taking $b = 2, 4, 8$ in (2.3), we obtain

$$E_{1,2} = E_1/E_2, \qquad E_{2,4} = E_2/E_4, \qquad E_{4,8} = E_4/E_8. \tag{2.4}$$

Let $c \in \mathbb{N}$. We have

$$b\mathbb{N}_0 + a = \bigcup_{d=0}^{c-1}(b(c\mathbb{N}_0 + d) + a) = \bigcup_{d=0}^{c-1}(bc\mathbb{N}_0 + (bd + a))$$

and

$$(bc\mathbb{N}_0 + (bd + a)) \cap (bc\mathbb{N}_0 + (bd' + a)) = \emptyset \quad \text{ if } 0 \le d, d' < c,\ d \ne d'.$$

Note that $1 \le bd + a \le bc$. Thus (2.1) gives

$$E_{a,b} = \prod_{d=0}^{c-1} E_{a+bd,bc}. \tag{2.5}$$

Taking $a = 1, b = 2, c = 4$ in (2.5), by (2.4) we have

$$E_{1,8}E_{3,8}E_{5,8}E_{7,8} = E_{1,2} = E_1/E_2. \tag{2.6}$$

Cooper and Hirschhorn ([6, eqs. (1) and (2)] or [7, Theorem 1]) have shown that

$$\varphi(q)+\varphi(q^2)=\frac{2E_{3,8}E_{5,8}E_8^2}{E_{1,8}E_{7,8}E_4}, \tag{2.7}$$

$$\varphi(q)-\varphi(q^2)=2q\,\frac{E_{1,8}E_{7,8}E_8^2}{E_{3,8}E_{5,8}E_4}. \tag{2.8}$$

Multiplying (2.7) and (2.8) together, we obtain

$$\varphi^2(q)-\varphi^2(q^2)=4q\,\frac{E_8^4}{E_4^2}\,. \tag{2.9}$$

From the equations (2.6), (2.7) and (2.8), we have

$$\frac{1}{E_{1,8}^2E_{7,8}^2}=\frac{1}{2}\left(\varphi(q)+\varphi(q^2)\right)\frac{E_2E_4}{E_1E_8^2}, \tag{2.10}$$

$$\frac{q}{E_{3,8}^2E_{5,8}^2}=\frac{1}{2}\left(\varphi(q)-\varphi(q^2)\right)\frac{E_2E_4}{E_1E_8^2}. \tag{2.11}$$

Thus, appealing to (2.10), (2.11), (1.33) and (2.9), we have

$$\begin{aligned} q\,\frac{E_2E_8^6}{E_4}&\left(\frac{1}{E_{1,8}^4E_{7,8}^4}+\frac{q^2}{E_{3,8}^4E_{5,8}^4}\right)\\ &=\frac{1}{2}\,q\,\frac{E_2^3E_4E_8^2}{E_1^2}\left(\varphi^2(q)+\varphi^2(q^2)\right)\\ &=\frac{1}{2}\,\varphi(q)\varphi(q^2)(\varphi^2(q)-\varphi^2(q^2))(\varphi^2(q)+\varphi^2(q^2)) \end{aligned}$$

so that

$$q\,\frac{E_2E_8^6}{E_4E_{1,8}^4E_{7,8}^4}+q^3\,\frac{E_2E_8^6}{E_4E_{3,8}^4E_{5,8}^4}=\frac{1}{8}\,\varphi^5(q)\varphi(q^2)-\frac{1}{8}\,\varphi(q)\varphi^5(q^2). \tag{2.12}$$

For $a\in\{1,3\}$ we have

$$\sum_{\substack{e=1\\ e\equiv a \,(\mathrm{mod}\, 8)}}^{\infty}\sum_{d=1}^{\infty}d^2q^{de}=\sum_{f=0}^{\infty}\sum_{d=1}^{\infty}d^2q^{8df+ad}=\sum_{d=1}^{\infty}d^2q^{ad}+\sum_{f=1}^{\infty}\sum_{d=1}^{\infty}d^2q^{8df+ad},$$

that is,

$$\sum_{\substack{e=1\\ e\equiv a \,(\mathrm{mod}\, 8)}}^{\infty}\sum_{d=1}^{\infty}d^2q^{de}=\frac{q^a(1+q^a)}{(1-q^a)^3}+\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}d^2q^{ad}\Big)q^{8n}. \tag{2.13}$$

By a similar calculation we have

$$\sum_{\substack{e=1\\ e\equiv -a \,(\mathrm{mod}\, 8)}}^{\infty}\sum_{d=1}^{\infty}d^2q^{de}=\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}d^2q^{-ad}\Big)q^{8n}. \tag{2.14}$$

Then, by (1.21), (2.13) and (2.14), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} G_8(n)q^n &= \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{-8}{n/d}\Big)d^2\Big)q^n = \sum_{d,e=1}^{\infty}\Big(\frac{-8}{e}\Big)d^2q^{de}\\ &= \sum_{a\in\{1,3\}}\sum_{\substack{e=1\\ e\equiv a \,(\mathrm{mod}\, 8)}}^{\infty}\sum_{d=1}^{\infty} d^2q^{de} - \sum_{a\in\{1,3\}}\sum_{\substack{e=1\\ e\equiv -a \,(\mathrm{mod}\, 8)}}^{\infty}\sum_{d=1}^{\infty} d^2q^{de}\\ &= \sum_{a\in\{1,3\}}\Big(\frac{q^a(1+q^a)}{(1-q^a)^3} + \sum_{n=1}^{\infty}\Big(\sum_{d\mid n} d^2q^{ad}\Big)q^{8n}\Big)\\ &\quad - \sum_{a\in\{1,3\}}\sum_{n=1}^{\infty}\Big(\sum_{d\mid n} d^2q^{-ad}\Big)q^{8n},\end{aligned}$$

that is,

$$\begin{aligned}\sum_{n=1}^{\infty} G_8(n)q^n &= \sum_{n=1}^{\infty}\sum_{d\mid n} d^2(q^d+q^{3d}-q^{-d}-q^{-3d})q^{8n}\\ &\quad + \frac{q(1+q)}{(1-q)^3} + \frac{q^3(1+q^3)}{(1-q^3)^3}.\end{aligned} \tag{2.15}$$

Replacing q by q^8 in Carlitz's theorem (Theorem 6) and then taking $a=q$, we obtain

$$\begin{aligned}1 &+ \frac{(1-q)^3}{q(1+q)}\sum_{n=1}^{\infty}\Big(\sum_{d\mid n} d^2(q^d-q^{-d})\Big)q^{8n}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{8n+2})(1-q^{8n-2})(1-q^{8n})^6}{(1-q^{8n+1})^4(1-q^{8n-1})^4} = \frac{(1-q)^4}{(1-q^2)}\,\frac{E_{2,4}E_8^6}{E_{1,8}^4E_{7,8}^4},\end{aligned}$$

so that by (2.4),

$$\frac{q(1+q)}{(1-q)^3} + \sum_{n=1}^{\infty}\Big(\sum_{d\mid n} d^2(q^d-q^{-d})\Big)q^{8n} = q\,\frac{E_2E_8^6}{E_4E_{1,8}^4E_{7,8}^4}. \tag{2.16}$$

Similarly, by taking $a=q^3$, we obtain

$$\frac{q^3(1+q^3)}{(1-q^3)^3} + \sum_{n=1}^{\infty}\Big(\sum_{d\mid n} d^2(q^{3d}-q^{-3d})\Big)q^{8n} = q^3\,\frac{E_2E_8^6}{E_4E_{3,8}^4E_{5,8}^4}. \tag{2.17}$$

Adding (2.16) and (2.17), and appealing to (2.15), we deduce

$$\sum_{n=1}^{\infty} G_8(n)q^n = q\,\frac{E_2E_8^6}{E_4E_{1,8}^4E_{7,8}^4} + q^3\,\frac{E_2E_8^6}{E_4E_{3,8}^4E_{5,8}^4}. \tag{2.18}$$

The required result now follows from (2.12) and (2.18). ∎

3. Proof of Theorem 4. Let $q \in \mathbb{C}$ be such that $|q| < 1$. Substituting the values of $\varphi(q)$, $\varphi(-q)$ and $\varphi(q^2)$ from (1.32) into (1.27), and multiplying by $E_1^4 E_2^4 E_4^4 E_8^4$, we obtain

$$E_2^2(E_2^{12} + E_1^8 E_4^4)E_8^4 - 2E_1^4 E_4^{14} = 0. \tag{3.1}$$

Multiplying both sides of this equation by

$$E_2^2(E_2^{12} - E_1^8 E_4^4)E_8^4 + 2E_1^4 E_4^{14},$$

we obtain

$$E_2^4(E_2^{24} - E_1^{16} E_4^8)E_8^8 + 4E_1^{12} E_2^2 E_4^{18} E_8^4 - 4E_1^8 E_4^{28} = 0. \tag{3.2}$$

Rearranging (3.2) slightly, we have

$$4E_1^8 E_4^{28} - E_2^{28} E_8^8 = 4E_1^{12} E_2^2 E_4^{18} E_8^4 - E_1^{16} E_2^4 E_4^8 E_8^8. \tag{3.3}$$

Dividing both sides of (3.3) by $E_1^{16} E_2^4 E_4^{10} E_8^6$, and appealing to (1.32), we deduce

$$\frac{E_2^4 E_8^2}{E_1^8 E_4^2} \left(4\varphi^4(q^2) - \varphi^4(q)\right) = 4\,\frac{E_4^8}{E_1^4 E_2^2 E_8^2} - \frac{E_8^2}{E_4^2}. \tag{3.4}$$

Now let $\omega = e^{2\pi i/8}$. We define

$$\mathit{\Pi}_{1,7} := \prod_{n=1}^{\infty} (1 - \omega q^n)(1 - \omega^7 q^n), \tag{3.5}$$

$$\mathit{\Pi}_{3,5} := \prod_{n=1}^{\infty} (1 - \omega^3 q^n)(1 - \omega^5 q^n). \tag{3.6}$$

Thus,

$$\begin{aligned} \mathit{\Pi}_{1,7}\mathit{\Pi}_{3,5} &= \prod_{n=1}^{\infty} (1 - \omega q^n)(1 - \omega^3 q^n)(1 - \omega^5 q^n)(1 - \omega^7 q^n) \\ &= \prod_{n=1}^{\infty} (1 + q^{4n}) = \prod_{n=1}^{\infty} \frac{(1 - q^{8n})}{(1 - q^{4n})} \end{aligned}$$

so that

$$\mathit{\Pi}_{1,7}\mathit{\Pi}_{3,5} = E_8/E_4. \tag{3.7}$$

Choosing $a_1 = 1$, $a_2 = \omega$, $a_3 = \omega^3$, $b_1 = \omega^2$, $b_2 = \omega^4$ and $b_3 = \omega^6$ in Theorem 7, we obtain

$$(3.8)\qquad \frac{2(1+\omega^2)(1-\omega^2)}{(1+\omega)(1+\omega^3)}\prod_{n=1}^{\infty}\frac{(1+q^n)^2(1+\omega^2q^n)^2(1-\omega^2q^n)^2}{(1-\omega q^n)(1-\omega^3q^n)(1-\omega^5q^n)(1-\omega^7q^n)}$$
$$+\frac{(1+\omega)(1+\omega^3)(1-\omega^3)}{(1-\omega)(1+\omega^2)}\prod_{n=1}^{\infty}\frac{(1-\omega^3q^n)^2(1-\omega^5q^n)^2}{(1+\omega^2q^n)(1-\omega^2q^n)}$$
$$+\frac{(1+\omega)(1-\omega)(1+\omega^3)}{(1-\omega^2)(1-\omega^3)}\prod_{n=1}^{\infty}\frac{(1-\omega q^n)^2(1-\omega^7q^n)^2}{(1+\omega^2q^n)(1-\omega^2q^n)}=0.$$

Straightforward calculations show that

$$\frac{2(1+\omega^2)(1-\omega^2)}{(1+\omega)(1+\omega^3)}=-2i\sqrt{2},$$
$$\frac{(1+\omega)(1+\omega^3)(1-\omega^3)}{(1-\omega)(1+\omega^2)}=(\sqrt{2}+1)i,$$
$$\frac{(1+\omega)(1-\omega)(1+\omega^3)}{(1-\omega^2)(1-\omega^3)}=(\sqrt{2}-1)i.$$

Appealing to (3.5)–(3.7), we deduce

$$\prod_{n=1}^{\infty}\frac{(1+q^n)^2(1+\omega^2q^n)^2(1-\omega^2q^n)^2}{(1-\omega q^n)(1-\omega^3q^n)(1-\omega^5q^n)(1-\omega^7q^n)}$$
$$=\frac{E_4}{E_8}\prod_{n=1}^{\infty}(1+q^n)^2(1+q^{2n})^2=\frac{E_4^3}{E_1^2E_8},$$

$$\prod_{n=1}^{\infty}\frac{(1-\omega^3q^n)^2(1-\omega^5q^n)^2}{(1+\omega^2q^n)(1-\omega^2q^n)}=\Pi_{3,5}^2\prod_{n=1}^{\infty}\frac{1}{1+q^{2n}}=\frac{E_2}{E_4}\,\Pi_{3,5}^2,$$

and

$$\prod_{n=1}^{\infty}\frac{(1-\omega q^n)^2(1-\omega^7q^n)^2}{(1+\omega^2q^n)(1-\omega^2q^n)}=\Pi_{1,7}^2\prod_{n=1}^{\infty}\frac{1}{1+q^{2n}}=\frac{E_2}{E_4}\,\Pi_{1,7}^2.$$

Thus (3.8) becomes, after dividing by iE_2/E_4,

$$(3.9)\qquad (\sqrt{2}-1)\Pi_{1,7}^2+(\sqrt{2}+1)\Pi_{3,5}^2=2\sqrt{2}\,\frac{E_4^4}{E_1^2E_2E_8}.$$

Squaring both sides of (3.9), and appealing to (3.7), we obtain

$$(3.10)\qquad (3-2\sqrt{2})\Pi_{1,7}^4+(3+2\sqrt{2})\Pi_{3,5}^4=\frac{8E_4^8}{E_1^4E_2^2E_8^2}-\frac{2E_8^2}{E_4^2}.$$

Appealing to (3.4), we see that (3.10) becomes

$$(3.11)\qquad (3-2\sqrt{2})\Pi_{1,7}^4+(3+2\sqrt{2})\Pi_{3,5}^4=\frac{2E_2^4E_8^2}{E_1^8E_4^2}\,(4\varphi^4(q^2)-\varphi^4(q)).$$

Then, by (3.7), (3.11) and (1.33), we obtain

$$\begin{aligned}\frac{(3+2\sqrt{2})}{6}\,\frac{E_1^6E_4}{E_2}\,\Pi_{1,7}^{-4}+\frac{(3-2\sqrt{2})}{6}\,\frac{E_1^6E_4}{E_2}\,\Pi_{3,5}^{-4}&\\ &=\frac{1}{6}\,\frac{E_1^6E_4}{E_2}\,(\Pi_{1,7}\Pi_{3,5})^{-4}((3+2\sqrt{2})\Pi_{3,5}^4+(3-2\sqrt{2})\Pi_{1,7}^4)\\ &=\frac{1}{6}\,\frac{E_1^6E_4^5}{E_2E_8^4}\,\frac{2E_2^4E_8^2}{E_1^8E_4^2}\,(4\varphi^4(q^2)-\varphi^4(q))\\ &=\frac{1}{3}\,\varphi(q)\varphi(q^2)(4\varphi^4(q^2)-\varphi^4(q))\end{aligned}$$

so that

$$\begin{aligned}\frac{(3+2\sqrt{2})}{6}\,\frac{E_1^6E_4}{E_2}\,\Pi_{1,7}^{-4}+\frac{(3-2\sqrt{2})}{6}\,\frac{E_1^6E_4}{E_2}\,\Pi_{3,5}^{-4}&\\ &=\frac{4}{3}\,\varphi(q)\varphi^5(q^2)-\frac{1}{3}\,\varphi^5(q)\varphi(q^2).\end{aligned}\tag{3.12}$$

The Gaussian sum for discriminant -8 gives

$$\omega^d+\omega^{3d}-\omega^{5d}-\omega^{7d}=\left(\frac{-8}{d}\right)\sqrt{-8},\qquad d\in\mathbb{N}$$

(see [12, Theorem 215, p. 221]). Hence

$$(\omega^d-\omega^{-d})-(\omega^{5d}-\omega^{-5d})=\left(\frac{-8}{d}\right)2i\sqrt{2},\quad d\in\mathbb{N}.\tag{3.13}$$

Taking $a=\omega$ in Carlitz's theorem (Theorem 6), we obtain after a little rearrangement

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}(\omega^d-\omega^{-d})d^2\Big)q^n&\\ &=\frac{\omega(1+\omega)}{(1-\omega)^3}\prod_{n=1}^{\infty}\frac{(1+q^{2n})(1-q^n)^6}{(1-\omega q^n)^4(1-\omega^7q^n)^4}-\frac{\omega(1+\omega)}{(1-\omega)^3},\end{aligned}\tag{3.14}$$

and taking $a=\omega^5$ we get

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}(\omega^{5d}-\omega^{-5d})d^2\Big)q^n&\\ &=\frac{\omega^5(1+\omega^5)}{(1-\omega^5)^3}\prod_{n=1}^{\infty}\frac{(1+q^{2n})(1-q^n)^6}{(1-\omega^3q^n)^4(1-\omega^5q^n)^4}-\frac{\omega^5(1+\omega^5)}{(1-\omega^5)^3}.\end{aligned}\tag{3.15}$$

Hence, from (3.13)–(3.15), we obtain

$$
\begin{aligned}
2i\sqrt{2}\sum_{n=1}^{\infty} H_8(n)q^n &= 2i\sqrt{2}\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\Big(\frac{-8}{d}\Big)d^2\Big)q^n \\
&= \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}((\omega^d-\omega^{-d})-(\omega^{5d}-\omega^{-5d}))d^2\Big)q^n \\
&= \frac{\omega(1+\omega)}{(1-\omega)^3}\,\frac{E_1^6E_4}{E_2}\,\Pi_{1,7}^{-4} \\
&\quad - \frac{\omega^5(1+\omega^5)}{(1-\omega^5)^3}\,\frac{E_1^6E_4}{E_2}\,\Pi_{3,5}^{-4} - \Big(\frac{\omega(1+\omega)}{(1-\omega)^3}-\frac{\omega^5(1+\omega^5)}{(1-\omega^5)^3}\Big) \\
&= \frac{-4i-3\sqrt{2}\,i}{2}\,\frac{E_1^6E_4}{E_2}\,\Pi_{1,7}^{-4} - \frac{-4i+3\sqrt{2}\,i}{2}\,\frac{E_1^6E_4}{E_2}\,\Pi_{3,5}^{-4} + 3\sqrt{2}\,i.
\end{aligned}
$$

Dividing both sides by $2\sqrt{2}\,i$, we deduce

$$
\sum_{n=1}^{\infty} H_8(n)q^n = \frac{-3-2\sqrt{2}}{4}\,\frac{E_1^6E_4}{E_2}\,\Pi_{1,7}^{-4} + \frac{-3+2\sqrt{2}}{4}\,\frac{E_1^6E_4}{E_2}\,\Pi_{3,5}^{-4} + \frac{3}{2}.
$$

Thus, by (3.12), we obtain

$$
\begin{aligned}
1-\frac{2}{3}\sum_{n=1}^{\infty} H_8(n)q^n &= \frac{3+2\sqrt{2}}{6}\,\frac{E_1^6E_4}{E_2}\,\Pi_{1,7}^{-4} + \frac{3-2\sqrt{2}}{6}\,\frac{E_1^6E_4}{E_2}\,\Pi_{3,5}^{-4} \\
&= \frac{4}{3}\,\varphi(q)\varphi^5(q^2) - \frac{1}{3}\,\varphi^5(q)\varphi(q^2),
\end{aligned}
$$

as asserted. ∎

4. Proof of Theorem 5. Let

$$
x = 1-\frac{\varphi^4(-q)}{\varphi^4(q)}, \qquad z=\varphi^2(q). \tag{4.1}
$$

From Berndt's catalogue of formulas for theta functions [4, pp. 122, 123], we have

$$
\varphi(q) = z^{1/2}, \tag{4.2}
$$

$$
\varphi(-q) = (1-x)^{1/4}z^{1/2}, \tag{4.3}
$$

$$
\varphi(q^2) = 2^{-1/2}(1+(1-x)^{1/2})^{1/2}z^{1/2}, \tag{4.4}
$$

$$
\psi(q) = 2^{-1/2}q^{-1/8}x^{1/8}z^{1/2}. \tag{4.5}
$$

Appealing to (4.2)–(4.5) and (1.32), we obtain

$$\frac{E_2^5}{E_1^2E_4^2} = z^{1/2}, \tag{4.6}$$

$$\frac{E_1^2}{E_2} = (1-x)^{1/4}z^{1/2}, \tag{4.7}$$

$$\frac{E_4^5}{E_2^2E_8^2} = 2^{-1/2}(1+(1-x)^{1/2})^{1/2}z^{1/2}, \tag{4.8}$$

$$\frac{E_2^2}{E_1} = 2^{-1/2}q^{-1/8}x^{1/8}z^{1/2}. \tag{4.9}$$

Solving (4.7) and (4.9) for E_1 and E_2, then (4.6) for E_4, and finally (4.8) for E_8, we obtain

$$E_1 = 2^{-1/6}q^{-1/24}x^{1/24}(1-x)^{1/6}z^{1/2}, \tag{4.10}$$

$$E_2 = 2^{-1/3}q^{-1/12}x^{1/12}(1-x)^{1/12}z^{1/2}, \tag{4.11}$$

$$E_4 = 2^{-2/3}q^{-1/6}x^{1/6}(1-x)^{1/24}z^{1/2}, \tag{4.12}$$

$$E_8 = 2^{-13/12}q^{-1/3}x^{1/12}(1-x)^{1/48}(1-(1-x)^{1/2})^{1/4}z^{1/2}. \tag{4.13}$$

Then, using (4.10)–(4.13), we deduce

$$\begin{aligned}
&E_2^{28}E_8^8 + 2E_1^8E_4^{28} + 4qE_1^{12}E_2^6E_4^6E_8^{12} - 3E_1^4E_2^{14}E_4^{14}E_8^4 \\
&\quad = 2^{-18}q^{-5}x^3(1-x)^{5/2}(1-(1-x)^{1/2})^2z^{18} + 2^{-19}q^{-5}x^5(1-x)^{5/2}z^{18} \\
&\qquad + 2^{-19}q^{-5}x^3(1-x)^3(1-(1-x)^{1/2})^3z^{18} \\
&\qquad - 3\cdot 2^{-19}q^{-5}x^4(1-x)^{5/2}(1-(1-x)^{1/2})z^{18} \\
&\quad = 2^{-19}q^{-5}x^3(1-x)^{5/2}z^{18}(2(1-(1-x)^{1/2})^2 \\
&\qquad + x^2 + (1-x)^{1/2}(1-(1-x)^{1/2})^3 - 3x(1-(1-x)^{1/2})) = 0.
\end{aligned}$$

Thus

$$qE_1^{12}E_2^6E_4^6E_8^{12} = -\frac{1}{4}\,E_2^{28}E_8^8 + \frac{3}{4}\,E_1^4E_2^{14}E_4^{14}E_8^4 - \frac{1}{2}\,E_1^8E_4^{28}. \tag{4.14}$$

Dividing both sides of (4.14) by $E_1^{10}E_2^5E_4^5E_8^{10}$, we obtain

$$qE_1^2E_2E_4E_8^2 = -\frac{1}{4}\,\frac{E_2^{23}}{E_1^{10}E_4^5E_8^2} + \frac{3}{4}\,\frac{E_2^9E_4^9}{E_1^6E_8^6} - \frac{1}{2}\,\frac{E_4^{23}}{E_1^2E_2^5E_8^{10}}. \tag{4.15}$$

Appealing to (1.24), (1.31), (1.32) and (4.15), we deduce

$$\sum_{n=1}^{\infty} c(n)q^n = -\frac{1}{4}\,\varphi^5(q)\varphi(q^2) + \frac{3}{4}\,\varphi^3(q)\varphi^3(q^2) - \frac{1}{2}\,\varphi(q)\varphi^5(q^2)$$

as required. ■

5. Proof of Theorem 1. (i) We have, by Theorems 3 and 4,

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,1,1,1,2;n)q^n &= \varphi^5(q)\varphi(q^2)\\ &= \left(-\frac{1}{3}\,\varphi^5(q)\varphi(q^2) + \frac{4}{3}\,\varphi(q)\varphi^5(q^2)\right) + \frac{4}{3}\,(\varphi^5(q)\varphi(q^2) - \varphi(q)\varphi^5(q^2))\\ &= 1 - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)q^n + \frac{32}{3}\sum_{n=1}^{\infty} G_8(n)q^n\end{aligned}$$

so that for $n \in \mathbb{N}$ we get $N(1,1,1,1,1,2;n) = \frac{32}{3}G_8(n) - \frac{2}{3}H_8(n)$, as claimed.

(ii) By Theorems 3 and 4 we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,2,2,2,2,2;n)q^n &= \varphi(q)\varphi^5(q^2)\\ &= \left(-\frac{1}{3}\,\varphi^5(q)\varphi(q^2) + \frac{4}{3}\,\varphi(q)\varphi^5(q^2)\right) + \left(\frac{1}{3}\,\varphi^5(q)\varphi(q^2) - \frac{1}{3}\,\varphi(q)\varphi^5(q^2)\right)\\ &= 1 - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)q^n + \frac{8}{3}\sum_{n=1}^{\infty} G_8(n)q^n\end{aligned}$$

so that for $n \in \mathbb{N}$ we obtain $N(1,2,2,2,2,2;n) = \frac{8}{3}G_8(n) - \frac{2}{3}H_8(n)$, as asserted. ∎

6. Proof of Theorem 2. In order to prove the formulas of Theorem 2, we introduce the shorthand notation

$$a := \varphi(q), \qquad b := \varphi(-q), \qquad c := \varphi(q^2). \tag{6.1}$$

Then, by (1.26), (1.27) and (6.1), we obtain

$$\varphi(q^4) = \frac{a+b}{2}, \qquad c^2 = \frac{a^2+b^2}{2}. \tag{6.2}$$

By Theorems 3–5, and equations (6.1) and (6.2), we have

$$\sum_{n=1}^{\infty} G_8(n)q^n = \left(\frac{3}{32}\,a^5 - \frac{1}{16}\,a^3b^2 - \frac{1}{32}\,ab^4\right)c, \tag{6.3}$$

$$1 - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)q^n = \left(\frac{2}{3}\,a^3b^2 + \frac{1}{3}\,ab^4\right)c, \tag{6.4}$$

$$\sum_{n=1}^{\infty} c(n)q^n = \left(\frac{1}{8}\,a^3b^2 - \frac{1}{8}\,ab^4\right)c. \tag{6.5}$$

From (6.3)–(6.5) we deduce

$$a^5c = 1 + \frac{32}{3}\sum_{n=1}^{\infty} G_8(n)q^n - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)q^n, \tag{6.6}$$

$$a^3b^2c = 1 - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)q^n + \frac{8}{3}\sum_{n=1}^{\infty} c(n)q^n, \tag{6.7}$$

$$ab^4c = 1 - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)q^n - \frac{16}{3}\sum_{n=1}^{\infty} c(n)q^n. \tag{6.8}$$

Under the mapping $q \mapsto -q$, we have $a \mapsto b$, $b \mapsto a$ and $c \mapsto c$. Thus (6.6)–(6.8) yield

$$a^4bc = 1 - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)(-q)^n - \frac{16}{3}\sum_{n=1}^{\infty} c(n)(-q)^n, \tag{6.9}$$

$$a^2b^3c = 1 - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)(-q)^n + \frac{8}{3}\sum_{n=1}^{\infty} c(n)(-q)^n, \tag{6.10}$$

$$b^5c = 1 + \frac{32}{3}\sum_{n=1}^{\infty} G_8(n)(-q)^n - \frac{2}{3}\sum_{n=1}^{\infty} H_8(n)(-q)^n. \tag{6.11}$$

(i) *Evaluation of* $N(1,1,1,1,2,4;n)$. Appealing to (1.34), (6.1), (6.6) and (6.9), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} & N(1,1,1,1,2,4;n)q^n \\ &= \varphi^4(q)\varphi(q^2)\varphi(q^4) = \tfrac{1}{2}a^5c + \tfrac{1}{2}a^4bc \\ &= 1 + \sum_{n=1}^{\infty}\big(\tfrac{16}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) - \tfrac{8}{3}(-1)^n c(n)\big)q^n\end{aligned}$$

so that, for $n \in \mathbb{N}$,

$$\begin{aligned}N(1,1,1,1,2,4;n) &= \tfrac{16}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) - \tfrac{8}{3}(-1)^n c(n) \\ &= \begin{cases}\tfrac{16}{3}G_8(n) + \tfrac{8}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \tfrac{16}{3}G_8(n) - \tfrac{2}{3}H_8(n) - \tfrac{8}{3}c(n) & \text{if } n \equiv 0 \pmod 2,\end{cases}\end{aligned}$$

as asserted.

(ii) *Evaluation of* $N(1,1,1,2,2,2;n)$. Appealing to (1.34), (6.1), (6.2), (6.6) and (6.7), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,1,2,2,2;n)q^n &= \varphi^3(q)\varphi^3(q^2) = \tfrac{1}{2}a^5c + \tfrac{1}{2}a^3b^2c \\ &= 1 + \sum_{n=1}^{\infty}\big(\tfrac{16}{3}G_8(n) - \tfrac{2}{3}H_8(n) + \tfrac{4}{3}c(n)\big)q^n\end{aligned}$$

so that $N(1,1,1,2,2,2;n) = \frac{16}{3}G_8(n) - \frac{2}{3}H_8(n) + \frac{4}{3}c(n)$ for $n \in \mathbb{N}$, as asserted.

(iii) *Evaluation of* $N(1,1,1,2,4,4;n)$. Appealing to (1.34), (6.1), (6.2), (6.6), (6.7) and (6.9), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} & N(1,1,1,2,4,4;n)q^n \\ &= \varphi^3(q)\varphi(q^2)\varphi^2(q^4) = \tfrac{1}{4}a^5c + \tfrac{1}{2}a^4bc + \tfrac{1}{4}a^3b^2c \\ &= 1 + \sum_{n=1}^{\infty} \big(\tfrac{8}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) + \tfrac{2}{3}(1-4(-1)^n)c(n)\big)q^n\end{aligned}$$

so that, for $n \in \mathbb{N}$,

$$\begin{aligned}N(1,1,1,2,4,4;n) &= \tfrac{8}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) + \tfrac{2}{3}(1-4(-1)^n)c(n) \\ &= \begin{cases} \frac{8}{3}G_8(n) + \frac{10}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{8}{3}G_8(n) - \frac{2}{3}H_8(n) - 2c(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}\end{aligned}$$

as asserted.

(iv) *Evaluation of* $N(1,1,2,2,2,4;n)$. Appealing to (1.34), (6.1), (6.2), (6.6), (6.7), (6.9) and (6.10), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} & N(1,1,2,2,2,4;n)q^n \\ &= \varphi^2(q)\varphi^3(q^2)\varphi(q^4) = \tfrac{1}{4}a^5c + \tfrac{1}{4}a^4bc + \tfrac{1}{4}a^3b^2c + \tfrac{1}{4}a^2b^3c \\ &= 1 + \sum_{n=1}^{\infty} \big(\tfrac{8}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) + \tfrac{2}{3}(1-(-1)^n)c(n)\big)q^n\end{aligned}$$

so that, for $n \in \mathbb{N}$,

$$\begin{aligned}N(1,1,2,2,2,4;n) &= \tfrac{8}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) + \tfrac{2}{3}(1-(-1)^n)c(n) \\ &= \begin{cases} \frac{8}{3}G_8(n) + \frac{4}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \frac{8}{3}G_8(n) - \frac{2}{3}H_8(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}\end{aligned}$$

as asserted.

(v) *Evaluation of* $N(1,1,2,4,4,4;n)$. Appealing to (1.34), (6.1), (6.2), (6.6), (6.7), (6.9) and (6.10), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} & N(1,1,2,4,4,4;n)q^n \\ &= \varphi^2(q)\varphi(q^2)\varphi^3(q^4) = \tfrac{1}{8}a^5c + \tfrac{3}{8}a^4bc + \tfrac{3}{8}a^3b^2c + \tfrac{1}{8}a^2b^3c \\ &= 1 + \sum_{n=1}^{\infty} \big(\tfrac{4}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) + \tfrac{1}{3}(3-5(-1)^n)c(n)\big)q^n\end{aligned}$$

so that, for $n \in \mathbb{N}$,

$$N(1,1,2,4,4,4;n) = \tfrac{4}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) + \tfrac{1}{3}(3-5(-1)^n)c(n)$$
$$= \begin{cases} \tfrac{4}{3}G_8(n) + \tfrac{8}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \tfrac{4}{3}G_8(n) - \tfrac{2}{3}H_8(n) - \tfrac{2}{3}c(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

as asserted.

(vi) *Evaluation of* $N(1,2,2,2,4,4;n)$. Appealing to (1.34), (6.1), (6.2) and (6.6)–(6.10), we obtain

$$\sum_{n=0}^{\infty} N(1,2,2,2,4,4;n)q^n$$
$$= \varphi(q)\varphi^3(q^2)\varphi^2(q^4) = \tfrac{1}{8}a^5c + \tfrac{1}{4}a^4bc + \tfrac{1}{4}a^3b^2c + \tfrac{1}{4}a^2b^3c + \tfrac{1}{8}ab^4c$$
$$= 1 + \sum_{n=1}^{\infty}\big(\tfrac{4}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) - \tfrac{2}{3}(-1)^n c(n)\big)q^n$$

so that, for $n \in \mathbb{N}$,

$$N(1,2,2,2,4,4;n) = \tfrac{4}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) - \tfrac{2}{3}(-1)^n c(n)$$
$$= \begin{cases} \tfrac{4}{3}G_8(n) + \tfrac{2}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \tfrac{4}{3}G_8(n) - \tfrac{2}{3}H_8(n) - \tfrac{2}{3}c(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

as asserted.

(vii) *Evaluation of* $N(1,2,4,4,4,4;n)$. Appealing to (1.34), (6.1), (6.2) and (6.6)–(6.10), we obtain

$$\sum_{n=0}^{\infty} N(1,2,4,4,4,4;n)q^n$$
$$= \varphi(q)\varphi(q^2)\varphi^4(q^4) = \tfrac{1}{16}a^5c + \tfrac{1}{4}a^4bc + \tfrac{3}{8}a^3b^2c + \tfrac{1}{4}a^2b^3c + \tfrac{1}{16}\,ab^4c$$
$$= 1 + \sum_{n=1}^{\infty}\big(\tfrac{2}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) + \tfrac{2}{3}(1-(-1)^n)c(n)\big)q^n$$

so that, for $n \in \mathbb{N}$,

$$N(1,2,4,4,4,4;n) = \tfrac{2}{3}G_8(n) - \tfrac{1}{3}(1+(-1)^n)H_8(n) + \tfrac{2}{3}(1-(-1)^n)c(n)$$
$$= \begin{cases} \tfrac{2}{3}G_8(n) + \tfrac{4}{3}c(n) & \text{if } n \equiv 1 \pmod 2, \\ \tfrac{2}{3}G_8(n) - \tfrac{2}{3}H_8(n) & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

as asserted. ∎

7. A property of $c(n)$. We close by proving the following property of $c(n)$:

$$c(2n) = -2c(n), \qquad n \in \mathbb{N}. \tag{7.1}$$

As a consequence of (7.1) we have (as $c(1) = 1$)

$$c(2^k) = (-1)^k 2^k, \qquad k \in \mathbb{N}_0. \tag{7.2}$$

Proof of (7.1). From (1.31) we have

$$\begin{aligned} E_1(-q) &= \prod_{n=1}^{\infty}(1-(-q)^n) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}) \\ &= E_2 \prod_{n=1}^{\infty} \frac{1+q^n}{1+q^{2n}} = E_2 \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^n}\,\frac{1-q^{2n}}{1-q^{4n}} \end{aligned}$$

so that

$$E_1(-q) = \frac{E_2^3}{E_1 E_4}. \tag{7.3}$$

Now, by (1.24) and (1.31), we have

$$\sum_{n=1}^{\infty} c(n) q^n = q E_1^2 E_2 E_4 E_8^2 \tag{7.4}$$

so that, by (7.3),

$$\sum_{n=1}^{\infty} c(n)(-q)^n = -q\,\frac{E_2^7 E_8^2}{E_1^2 E_4}. \tag{7.5}$$

Thus, by (7.4) and (7.5), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} c(2n) q^{2n} &= \frac{1}{2}\sum_{n=1}^{\infty} c(n) q^n + \frac{1}{2}\sum_{n=1}^{\infty} c(n)(-q)^n \\ &= -\frac{1}{2}\, q E_2^2 E_4 E_8^2 \left(\frac{E_2^5}{E_1^2 E_4^2} - \frac{E_1^2}{E_2}\right). \end{aligned}$$

Appealing to (1.32), we deduce that

$$\sum_{n=1}^{\infty} c(2n) q^{2n} = -\frac{1}{2}\, q E_2^2 E_4 E_8^2 (\varphi(q) - \varphi(-q)).$$

Recalling that

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8)$$

(see for example [4, p. 71]) and that

$$\psi(q^8) = \frac{E_{16}^2}{E_8}$$

from (1.32) with q replaced by q^8, we obtain (by (7.4))

$$\sum_{n=1}^{\infty} c(2n)q^{2n} = -2q^2E_2^2E_4E_8E_{16}^2 = -2\sum_{n=1}^{\infty} c(n)q^{2n},$$

from which we deduce (7.1) on equating coefficients of q^{2n} $(n \in \mathbb{N})$. ∎

Numerical evidence suggests that for an odd prime p and $n \in \mathbb{N}$, we have

$$c(p^n) = c(p)c(p^{n-1}) - p^2c(p^{n-2}) \quad \text{if } \left(\frac{-8}{p}\right) = 1 \text{ and } n \geq 2, \tag{7.6}$$

and

$$c(p^n) = \frac{1}{2}(1 + (-1)^n)p^n \quad \text{if } \left(\frac{-8}{p}\right) = -1. \tag{7.7}$$

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