ACTA ARITHMETICA
155.3 (2012)

On Waring's problem for two cubes and two small cubes

by

JÖRG BRÜDERN (Göttingen)

1. Introduction. It is expected that all large natural numbers are the sum of four positive cubes, and Davenport [7] confirmed this for all but $O(N^{29/30+\varepsilon})$ of the integers n with $1 \le n \le N$. The exponent 29/30 has now been reduced to a number slightly smaller than 37/42 (see [11, 2, 3, 17, 18]). No such result can be valid with fewer cubes because a sum of three cubes is incongruent to 4 modulo 9. However, the fourth cube is probably almost redundant. In particular, it should be possible to choose the fourth cube quite small in terms of the number to be represented. In this spirit, Wooley and the author [6] established that almost all natural numbers n are the sum of four cubes, one of which is bounded in size by $n^{5/36}$.

It is then natural to ask what happens if another cube is restricted in size. The theme was taken up by Lee [9] in a very recent work. For fixed positive real θ, he considers the number $r_\theta(n)$ of solutions to

$$x_1^3 + x_2^3 + z_1^3 + z_2^3 = n$$

in natural numbers x_j, z_j with $z_j \le n^\theta$ $(j = 1, 2)$. Lee's main result [9, Theorem 1.1], is that whenever $\theta \ge \frac{192}{869}$, then $r_\theta(n) \ge 1$ for almost all n. Inter alia, a little more is proved. The heuristics underpinning the Hardy–Littlewood method suggest that the asymptotic formula

$$r_\theta(n) \sim \frac{\Gamma(4/3)^2}{\Gamma(2/3)} \mathfrak{S}(n) n^{2\theta-1/3} \quad (n \to \infty) \tag{1}$$

should hold for $0 < \theta < 1/3$. Here $\mathfrak{S}(n)$ is the familiar singular series associated with sums of four cubes, and it is useful to recall that

$$\mathfrak{S}(n) \gg 1 \tag{2}$$

(see [16, §4.6, exercise 3]). An examination of Lee's argument reveals that

2010 *Mathematics Subject Classification*: Primary 11P55.
Key words and phrases: Waring's problem, sums of cubes, diminishing ranges.

DOI: 10.4064/aa155-3-4

whenever $\theta \geq \frac{192}{869}$, then the lower bound

$$r_\theta(n) \gg \mathfrak{S}(n) n^{2\theta - 1/3} \tag{3}$$

is valid for almost all n, and his aformentioned result is a corollary.

To appreciate the strength of Lee's result, note that for $\theta > 1/6$, the asymptotic formula (1), if true, would imply that $r_\theta(n) \geq 1$ for large n. On the other hand, a simple lattice point count shows that the mean

$$N^{-1} \sum_{n \leq N} r_{1/6}(n)$$

tends to a limit strictly less than 1, so that $r_{1/6}(n) = 0$ for a positive proportion of the natural numbers. Thus, Lee's result cannot be valid for $\theta \leq 1/6$.

Lee's approach depends on minor arc estimates imported from Brüdern and Wooley [6], and hence on analytic descriptions of the iterative schemes dominating much recent work on Waring's problem. Implicitly, Lee's treatment rests on the p-adic iteration developed in [11, 14, 5]. However, when two small cubes are present in a sum of four, then the representation problem lends itself to much older routines sometimes referred to as diminishing ranges. In this method, one restricts variables in size to enforce diagonal behaviour in certain auxiliary symmetric diophantine equations. The idea goes back to Hardy and Littlewood [8] and was developed further by Davenport ([7] and Chapter 6 of Vaughan [16]) and Vaughan [10, 12]. The arrival of the new iterative method of Vaughan [15] has diminished interest in the traditional procedures over the past two decades, but for problems involving small variables these are ideally suited, and as we shall see momentarily, they provide a sizeable improvement over Lee's result. Yet, one should note a change of paradigm: we wish to choose variables as small as we can whereas in earlier work emphasis was on diagonal behaviour with size constraints as mild as possible.

Some notation is required before our main result can be announced. Let P be a large number (the main parameter), and let $0 < \theta < 1/3$. Let

$$Q = P^{5/6}, \qquad R = \tfrac{1}{2} P^{3\theta}. \tag{4}$$

Let $\varrho_\theta(n, P)$ denote the number of solutions of

$$n = x^3 + y^3 + z_1^3 + z_2^3 \tag{5}$$

with

$$P < x \leq 2P, \qquad Q < y \leq 2Q, \qquad R < z_j \leq 2R. \tag{6}$$

Theorem. *Let $7/36 < \theta \leq 2/9$. Then there is a positive number δ such that*

$$\sum_{P^3 < n \leq 8P^3} \big(\varrho_\theta(n, P) - \tfrac{1}{3}\mathfrak{S}(n) Q R^2 n^{-2/3}\big)^2 \ll P^{12\theta + 2/3 - \delta}.$$

An inspection of conditions (4)–(6) reveals that whenever $P^3 < n \le 8P^3$, then $r_\theta(n) \ge \varrho_\theta(n,P)$. Moreover, by (2), the Theorem implies that the inequality $\varrho_\theta(n,P) < \frac{1}{20}\mathfrak{S}(n)n^{2\theta-7/18}$ can hold for at most $O(P^{3-\delta})$ of these values n. A familiar dyadic dissection argument now suffices to conclude as follows.

COROLLARY. *Let $7/36 < \theta \le 2/9$. Then there is a positive number δ such that for all but $O(N^{1-\delta})$ of the integers n with $1 \le n \le N$ one has*

$$r_\theta(n) \gg \mathfrak{S}(n)n^{2\theta-7/18}. \tag{7}$$

The upper bound on θ in these results is of no importance, and has been introduced for technical convenience only. For larger values of θ Lee has stronger results anyway. For comparison with Lee's Theorem 1.1 and the limiting value $\theta = 1/6$, note that

$$\tfrac{192}{869} = 0.2209\ldots, \qquad \tfrac{7}{36} = 0.1944\ldots, \qquad \tfrac{1}{6} = 0.1666\ldots.$$

Thus, our method covers about 51 percent of the uncertain range $1/6 < \theta < \frac{192}{869}$ in Lee's approach, but we no longer obtain the essentially best possible lower bound (3). The weaker estimate (7) has the interesting feature that the exponent $2\theta - 7/18$ approaches 0 as θ tends to $7/36$. Thus, we work with a singular integral that only just is large. These observations reflect the diminished range for y in (5). Lee [9] suggests to view representations counted by $r_\theta(n)$ as sums of $6\theta + 2$ cubes. In this sense, the Corollary shows almost all n to be the sum of $3 + 1/6 + \varepsilon$ cubes, for any fixed $\varepsilon > 0$. One can take this point of view a step further and consider the representations counted by $\varrho_\theta(n,P)$ as sums of $11/6 + 2\theta$ cubes. The Theorem then asserts that almost all natural numbers are the sum of $3 + \varepsilon$ cubes, a result that is essentially best possible. The Theorem also contains a first result concerning sums of four cubes with three small variables: almost all n are the sum of four cubes, three of which are of size at most $n^{5/18}$. However, it is not difficult to obtain better estimates in this direction by other methods. Limitations of space do not permit a discussion of this matter here.

2. The method. We follow a strategy that is the dispersion method in disguise. For a direct application of the latter, let $\varrho^*_\theta(n,P) = \frac{1}{3}\mathfrak{S}(n)QR^2n^{-2/3}$ for $P^3 < n \le 8P^3$, and $\varrho^*_\theta(n,P) = 0$ otherwise, and then square out the variance

$$V = \sum_n (\varrho_\theta(n,P) - \varrho^*_\theta(n,P))^2. \tag{8}$$

One of the three terms that then arise is the sum

$$S = \sum_n \varrho_\theta(n,P)^2,$$

and it is usually the one that presents the main difficulties. The argument that we describe below yields an asymptotic formula for S whenever $\theta > 7/36$, at least in principle. Companion formulae for the other two terms would exhibit cancellation, leading to a successful estimation of V. It is, however, simpler to perform all this after taking Fourier transforms. The idea is not new, it occurs in work of Vaughan [13], and in a different context in [4], but has hardly been used elsewhere. Note that S equals the number of solutions of the symmetric diophantine equation

$$x_1^3 - x_2^3 = y_1^3 - y_2^3 + z_1^3 + z_2^3 - z_3^3 - z_4^3 \tag{9}$$

with

$$P < x_j \le 2P, \quad Q < y_j \le 2Q, \quad R < z_j \le 2R. \tag{10}$$

We evaluate S by the Hardy–Littlewood method, in two ways.

The standard approach involves the exponential sums

$$f(\alpha) = \sum_{P<x\le 2P} e(\alpha x^3), \quad g(\alpha) = \sum_{Q<y\le 2Q} e(\alpha y^3), \quad h(\alpha) = \sum_{R<z\le 2R} e(\alpha z^3).$$

By orthogonality, one then has

$$S = \int_0^1 |f(\alpha)g(\alpha)h(\alpha)^2|^2 \, d\alpha. \tag{11}$$

Alternatively, one may difference the left hand side of (9) by $k = x_1 - x_2$. Then (9) transforms into

$$k(3x_2^2 + 3x_2k + k^2) = y_1^3 - y_2^3 + z_1^3 + z_2^3 - z_3^3 - z_4^3. \tag{12}$$

By (10), the modulus of the right hand side here does not exceed $9Q^3$, at least when P is large. Also, by (10),

$$3x_2^2 + 3x_2k + k^2 = \left(\tfrac{3}{2}x_2 + k\right)^2 + \tfrac{3}{4}x_2^2 \ge \tfrac{3}{4}P^2.$$

Hence, (12) implies that $|k| \le K$ where $K = 12P^{1/2}$. Now let

$$F(\alpha) = \sum_{|k|\le K} \sum_{P<x\le 2P} e(\alpha k(3x^2 + 3xk + k^2)). \tag{13}$$

Then, again by orthogonality,

$$S = \int_0^1 F(\alpha)|g(\alpha)h(\alpha)^2|^2 \, d\alpha. \tag{14}$$

The integral representations (11) and (14) will play a fundamental role in the derivation of a minor arc estimate in Section 6.

3. A first asymptotic evaluation. From now on we suppose that the real number θ is fixed and chosen in the interval $7/36 < \theta \le 2/9$. The main parameter P is supposed to exceed a suitable positive number P_0 that may increase from one such requirement to the next. Also, we apply the following convention concerning the letter ε. Whenever ε occurs in a statement, it denotes a real number, and it is asserted that the statement is true for any fixed positive value assigned to ε.

In this section we apply the Hardy–Littlewood method to the integral (14). Let $1 \le X \le P$, and let $\mathfrak{M}(X)$ denote the union of the intervals

$$\{\alpha \in [0,1] : |q\alpha - a| \le XQ^{-3}\} \tag{15}$$

with $0 \le a \le q$, $1 \le q \le X$ and $(a,q) = 1$. The observant connoisseur will notice that the arcs (15) are shaped to suit the exponential sum $g(\alpha)$, but are not a perfect fit for the longer sum $f(\alpha)$ which is absent in (14).

The current hypotheses ensure that $\nu = \frac{1}{9}\big(\theta - \frac{7}{36}\big)$ is positive, and we put

$$\mathfrak{M} = \mathfrak{M}(P^\nu), \qquad \mathfrak{m} = [0,1] \setminus \mathfrak{M}.$$

The main estimate in this section is that

$$\int_{\mathfrak{m}} F(\alpha)|g(\alpha)h(\alpha)^2|^2\, d\alpha \ll P^{-1-\frac{1}{7}\nu}Q^2R^4. \tag{16}$$

Once this is established, one deduces from (14) the following result.

Lemma 1. *In the notation introduced above, one has*

$$S = \int_{\mathfrak{M}} F(\alpha)|g(\alpha)h(\alpha)^2|^2\, d\alpha + O(P^{-1-\frac{1}{7}\nu}Q^2R^4).$$

The proof of (16) is a pruning exercise for which an estimate for $F(\alpha)$ is pivotal. Let $\Upsilon : [0,1] \to [0,1]$ be the function that is 0 outside $\mathfrak{M}(P)$, and for $\alpha \in \mathfrak{M}(P)$ is defined by

$$\Upsilon(\alpha) = (q + Q^3|q\alpha - a|)^{-1},$$

where a, q is the uniquely defined pair that occurs in (15).

Lemma 2. *For $\alpha \in [0,1]$ one has*

$$F(\alpha) \ll P^{1+\varepsilon} + P^{3/2+\varepsilon}\Upsilon(\alpha)^{1/2}.$$

Proof. Let $a \in \mathbb{Z}$, $q \in \mathbb{N}$ be coprime with $\alpha = a/q + \beta$ and $|\beta| \le q^{-2}$. Then, by usual Weyl differencing (see the Lemma in Vaughan [10]),

$$|F(\alpha)|^2 \ll K^2P^{2+\varepsilon}\bigg(\frac{1}{q} + \frac{1}{P} + \frac{q}{KP^2}\bigg).$$

Note that $KP^2 = 12Q^3$. Hence, a familiar principle (see Exercise 2 in §2.8 of Vaughan [16]) transforms the previous bound into

$$|F(\alpha)|^2 \ll P^{3+\varepsilon}\big((q + Q^3q|\beta|)^{-1} + P^{-1} + Q^{-3}(q + Q^3q|\beta|)\big).$$

By Dirichlet's theorem on diophantine approximation, one may chose the coprime pair a, q with $1 \le q \le P^{3/2}$ and $q|\beta| \le P^{-3/2}$, so that

$$|F(\alpha)|^2 \ll P^{3+\varepsilon}(q + Q^3 q|\beta|)^{-1} + P^{2+\varepsilon}.$$

In particular, if $q > P$, then $F(\alpha) \ll P^{1+\varepsilon}$. If $q \le P$, then $\alpha \in \mathfrak{M}(P)$, and the lemma follows.

Lemma 3. *Let $R \le Q^{4/5}$. Then*

$$\int_0^1 |g(\alpha)h(\alpha)^2|^2 \, d\alpha \ll Q^{1+\varepsilon}R^2.$$

Davenport [7, Lemma 1] gives a proof when $R = Q^{4/5}$, but an inspection of his argument shows that his method applies to smaller values of R as well. By working along the lines of Vaughan [10], the hypothesis in Lemma 3 may be relaxed to $R \le Q^{5/6}$, but in the range $\theta \le 2/9$ this is not needed.

We are ready to embark on the deduction of (16). By Lemmas 2 and 3,

$$\int_{\mathfrak{m}} F|gh^2|^2 \, d\alpha \ll P^{1+\varepsilon}QR^2 + P^{3/2+\varepsilon}M \tag{17}$$

where

$$M = \int_{\mathfrak{M}(P)\setminus\mathfrak{M}} \Upsilon^{1/2}|gh^2|^2 \, d\alpha. \tag{18}$$

The estimation of M proceeds through the Ramanujan sum technique. In Lemma 2 of Brüdern [1], we choose $\Psi(\alpha) = |h(\alpha)|^4$. Then, since the inequality

$$\int_0^1 |h(\alpha)|^4 \, d\alpha \ll R^{2+\varepsilon} \tag{19}$$

is merely a special case of Hua's lemma ([16, Lemma 2.5]), we infer that

$$\int_{\mathfrak{M}(P)} \Upsilon|h|^4 \, d\alpha \ll Q^{\varepsilon-3}(PR^2 + R^4) \ll Q^{\varepsilon-3}R^4. \tag{20}$$

If $\alpha \in \mathfrak{M}(P) \setminus \mathfrak{M}$, then $\Upsilon(\alpha) \ll P^{-\nu}$. We further deduce that

$$\int_{\mathfrak{M}(P)\setminus\mathfrak{M}} \Upsilon^{7/6}|h|^4 \, d\alpha \ll Q^{\varepsilon-3}R^4P^{-\nu/6}. \tag{21}$$

We combine these estimates with a major arc upper bound for $g(\alpha)$. By Theorem 4.1 and Lemma 6.3 of [16], when $\alpha \in \mathfrak{M}(P)$, one has

$$|g(\alpha)| \ll Q\Upsilon(\alpha)^{1/3} + P^{1/2+\varepsilon}. \tag{22}$$

Consequently, by (18),

$$M \ll Q \int_{\mathfrak{M}(P)\setminus\mathfrak{M}} \Upsilon^{5/6}|gh^4| \, d\alpha + P^{1/2+\varepsilon} \int_{\mathfrak{M}(P)} \Upsilon^{1/2}|gh^4| \, d\alpha. \tag{23}$$

By Schwarz's inequality, Lemma 3 and (20),

$$\int_{\mathfrak{M}(P)} \Upsilon^{1/2}|gh^4|\,d\alpha \ll (Q^{1+\varepsilon}R^2)^{1/2}(Q^{\varepsilon-3}R^4)^{1/2} \ll Q^{\varepsilon-1}R^3. \tag{24}$$

Another use of (22) gives

$$\int_{\mathfrak{M}(P)\setminus\mathfrak{M}} \Upsilon^{5/6}|gh^4|\,d\alpha \ll Q \int_{\mathfrak{M}(P)\setminus\mathfrak{M}} \Upsilon^{7/6}|h^4|\,d\alpha + P^{1/2+\varepsilon} \int_{\mathfrak{M}(P)} \Upsilon^{5/6}|h^4|\,d\alpha.$$

The first integral on the right hand side was estimated in (21), and for the second, one may use Hölder's inequality and then apply (19) and (20) to deduce that

$$\int_{\mathfrak{M}(P)} \Upsilon^{5/6}|h^4|\,d\alpha \ll (R^{2+\varepsilon})^{1/6}(Q^{\varepsilon-3}R^4)^{5/6}.$$

This shows that

$$\int_{\mathfrak{M}(P)\setminus\mathfrak{M}} \Upsilon^{5/6}|gh^4|\,d\alpha \ll Q^{\varepsilon-2}R^4P^{-\nu/6} + P^{1/2+\varepsilon}Q^{-5/2}R^{11/3}.$$

On combining this with (23), (24) and (17), we now have

$$\begin{aligned}\int_{\mathfrak{m}} F|gh^2|^2\,d\alpha \ll P^{1+\varepsilon}QR^2 + P^{2+\varepsilon}Q^{-1}R^3 + P^{3/2+\varepsilon}Q^{-1}R^4P^{-\nu/6}\\ + P^{2+\varepsilon}Q^{-3/2}R^{11/3}.\end{aligned}$$

This establishes (16), as one readily checks.

4. A major arc analysis. In this section we consider the integral that occurs in Lemma 1. For $m \in \mathbb{Z}$ let

$$K(m) = \int_{\mathfrak{M}} |g(\alpha)h(\alpha)^2|^2 e(\alpha m)\,d\alpha.$$

Standard major arc analysis yields an asymptotic formula for $K(m)$. Let

$$S(q,a) = \sum_{r=1}^{q} e(ar^3/q)$$

and

$$v(\beta) = \frac{1}{3}\sum_{P^3<l\le 8P^3} l^{-2/3}e(\beta l), \qquad w(\beta) = \frac{1}{3}\sum_{Q^3<l\le 8Q^3} l^{-2/3}e(\beta l).$$

When $\alpha \in \mathfrak{M}(P)$, there is a unique pair a, q of integers with $1 \le q \le P$ and $(a,q) = 1$ such that $|q\alpha - a| \le PQ^{-3}$, and we define functions $f^*, g^* : \mathfrak{M}(P) \to \mathbb{C}$ by

$$f^*(\alpha) = q^{-1}S(q,a)v(\alpha - a/q), \qquad g^*(\alpha) = q^{-1}S(q,a)w(\alpha - a/q). \tag{25}$$

Note that $P^\nu Q^{-3} \le \frac{1}{6}P^{-2}$, so that a crude application of Theorem 4.1 of Vaughan [16] shows that whenever $\alpha \in \mathfrak{M}$, one has

$$f(\alpha) = f^*(\alpha) + O(P^\nu), \qquad g(\alpha) = g^*(\alpha) + O(P^\nu). \tag{26}$$

There is a similar but simpler formula for $h(\alpha)$. One may first use partial summation to compare $h(\alpha)$ with $h(a/q)$, and then apply Theorem 4.1 of Vaughan [16] to $h(a/q)$. Then, in the same notation as before, one finds that whenever $\alpha \in \mathfrak{M}$, one has

$$h(\alpha) = h^*(\alpha) + O(P^\nu) \tag{27}$$

where now $h^* : \mathfrak{M}(P) \to \mathbb{C}$ is defined by

$$h^*(\alpha) = q^{-1}S(q,a)R. \tag{28}$$

For $\alpha \in \mathfrak{M}$, the approximations (26) and (27) combine to

$$|g(\alpha)h(\alpha)^2|^2 = |g^*(\alpha)h^*(\alpha)^2|^2 + O(Q^2R^3P^\nu). \tag{29}$$

We multiply with $e(\alpha m)$ and integrate. The measure of $\mathfrak{M}$ is $O(P^{2\nu}Q^{-3})$, whence

$$K(m) = \int_{\mathfrak{M}} |g^*(\alpha)h^*(\alpha)^2|^2 e(\alpha m)\,d\alpha + O(Q^{-1}R^3P^{3\nu}). \tag{30}$$

We write

$$B(q,m) = q^{-6} \sum_{\substack{a=1\\(a,q)=1}}^{q} |S(q,a)|^6 e(am/q). \tag{31}$$

By (25) and (28),

$$\int_{\mathfrak{M}} |g^*(\alpha)h^*(\alpha)^2|^2 e(\alpha m)\,d\alpha = R^4 \sum_{q\le P^\nu} B(q,m) \int_{-P^\nu/(qQ^3)}^{P^\nu/(qQ^3)} |w(\beta)|^2 e(\beta m)\,d\beta. \tag{32}$$

Here we wish to replace the integral on the right hand side by the complete integral

$$J(m) = \int_{-1/2}^{1/2} |w(\beta)|^2 e(\beta m)\,d\beta. \tag{33}$$

By Lemma 6.2 of Vaughan [16], for $|\beta|\le 1/2$, one has $w(\beta) \ll Q(1+Q^3|\beta|)^{-1}$, and consequently

$$\int_{-P^\nu/(qQ^3)}^{P^\nu/(qQ^3)} |w(\beta)|^2 e(\beta m)\,d\beta = J(m) + O(Q^{-1}P^{-\nu}q).$$

By (32),

$$\int_{\mathfrak{M}} |g^*(\alpha)h^*(\alpha)^2|^2 e(\alpha m)\,d\alpha = R^4\mathfrak{T}(m,P^\nu)J(m)+E$$

where

$$\mathfrak{T}(m,X)=\sum_{q\le X}B(q,m) \quad\text{and}\quad E\ll R^4Q^{-1}P^{-\nu}\sum_{q\le P^\nu}q|B(q,m)|.$$

By (31), one has $|B(q,m)|\le B(q,0)$. Much as in the proof of Lemma 2.11 in Vaughan [16], one finds that $B(q,0)$ is multiplicative. The natural number q factors uniquely as $q=ru^3$ where r is cube-free. Then, by Lemmas 4.3 and 4.4 of [16], one finds that $B(q,0)\ll q^{1+\varepsilon}r^{-3}u^{-6}$. It follows that

$$\sum_{q\le P^\nu} qB(q,0)\ll P^\varepsilon\sum_{ru^3\le P^\nu} r^{-1}\ll P^{\nu/3+\varepsilon}. \tag{34}$$

On collecting together the above, we now infer from (30) that

$$K(m)=\mathfrak{T}(m,P^\nu)J(m)R^4+O(P^{-\nu/2}Q^{-1}R^4)$$

because the error in (30) is considerably smaller. In this expansion, we take

$$m=m(k,x)=k(3x^2+3kx+k^2) \tag{35}$$

and sum over k and x as in (13). This yields

$$\int_{\mathfrak{M}} F(\alpha)|g(\alpha)h(\alpha)^2|^2\,d\alpha = R^4\sum_{|k|\le K}\sum_{P<x\le 2P}\mathfrak{T}(m,P^\nu)J(m)+O(P^{(3-\nu)/2}Q^{-1}R^4),$$

and this final formula may be injected into Lemma 1 to conclude as follows.

Lemma 4. *One has*

$$S=R^4\sum_{|k|\le K}\sum_{P<x\le 2P}\mathfrak{T}(m(k,x),P^\nu)J(m(k,x))+O(P^{1-\frac{1}{7}\nu}Q^2R^4).$$

5. Another major arc analysis. In this section we compute the major arc contribution to the alternative integral (11) for S. Thus, we now examine

$$S_0=\int_{\mathfrak{M}}|f(\alpha)g(\alpha)h(\alpha)^2|^2\,d\alpha. \tag{36}$$

By (29), we have

$$S_0=\int_{\mathfrak{M}}|fg^*h^{*2}|^2\,d\alpha+O\Big(Q^2R^3P^\nu\int_{\mathfrak{M}}|f|^2\,d\alpha\Big).$$

By (26), $|f|^2\ll|f^*|^2+P^{2\nu}$, and the measure of $\mathfrak{M}$ is $O(P^{2\nu}Q^{-3})$. Hence,

$$\int_{\mathfrak{M}}|f|^2\,d\alpha\ll\int_{\mathfrak{M}}|f^*|^2\,d\alpha+P^{4\nu}Q^{-3}.$$

By Lemma 6.3 of Vaughan [16], when $\alpha \in \mathfrak{M}$ and a/q is the centre of the interval to which α belongs, then

$$f^*(\alpha) \ll q^{-1/3} P(1 + P^3|\alpha - a/q|)^{-1}. \tag{37}$$

A straightforward and crude estimation therefore shows that

$$\int_{\mathfrak{M}} |f^*|^2 \, d\alpha \ll P^{2\nu-1}.$$

This combines to

$$S_0 = \int_{\mathfrak{M}} |fg^*h^{*2}|^2 \, d\alpha + O(P^{3\nu-1}Q^2R^3). \tag{38}$$

We now mimic the argument leading to Lemma 4. Rather more care is needed, however. We can no longer sum the integral (32) trivially, because we should now sum over about P^2 values $m = x_1^3 - x_2^3$, and not only over $O(P^{3/2})$ terms $m = k(3x^2 + 3kx + k^2)$.

Let $\mathfrak{N}$ denote the union of the intervals

$$\{\alpha \in [0,1] : |q\alpha - a| \le P^{-7/4}\}$$

with $0 \le a \le q$, $1 \le q \le P^\nu$ and $(a,q) = 1$. Note that $\mathfrak{N} \subset \mathfrak{M}(P)$ so that f^*, g^*, h^* are defined on $\mathfrak{N}$. We proceed to show that

$$\int_{\mathfrak{N}\setminus\mathfrak{M}} |fg^*h^{*2}|^2 \, d\alpha \ll P^{\varepsilon-5/4}Q^2R^4. \tag{39}$$

To see this, we apply Theorem 4.1 of Vaughan [16] to deduce that for $\alpha \in \mathfrak{N}$ one has $f(\alpha) = f^*(\alpha) + O(P^{5/8+\varepsilon})$. Consequently,

$$|f(\alpha)|^2 \ll |f^*(\alpha)|^2 + P^{5/4+\varepsilon}. \tag{40}$$

Also, much as in (32), and observing positivity, one finds that

$$\int_{\mathfrak{N}} |g^*h^{*2}|^2 \, d\alpha \ll R^4 \sum_{q \le P^\nu} B(q,0)J(0).$$

By (33) and orthogonality, $J(0) \ll Q^{-1}$. The argument in (34) now gives

$$\int_{\mathfrak{N}} |g^*h^{*2}|^2 \, d\alpha \ll Q^{-1}R^4. \tag{41}$$

By (40) and (41),

$$\int_{\mathfrak{N}\setminus\mathfrak{M}} |fg^*h^{*2}|^2 \, d\alpha \ll \int_{\mathfrak{N}\setminus\mathfrak{M}} |f^*g^*h^{*2}|^2 \, d\alpha + P^{5/4+\varepsilon}Q^{\varepsilon-1}R^4,$$

and since $P^{5/4}Q^{-1} = P^{-5/4}Q^2$, the last term on the right is acceptable. Moreover, much as in (32) again, one finds from (37) that

$$\int_{\mathfrak{N}\setminus\mathfrak{M}} |f^*g^*h^{*2}|^2 \, d\alpha \ll P^2Q^2R^4 \sum_{q \le P^\nu} q^{-2/3} B(q,0) \int_{P^\nu/(qQ^3)}^{\infty} (1 + P^3|\beta|)^{-2} \, d\beta.$$

A straightforward and now familar estimation yields (39) immediately.

The expansions (38) and (39) combine to

$$S_0 = \int_{\mathfrak{N}} |fg^*h^{*2}|^2\,d\alpha + O(P^{\varepsilon-5/4}Q^2R^4), \tag{42}$$

and one has

$$\int_{\mathfrak{N}} |fg^*h^{*2}|^2\,d\alpha = \sum_{P<x_1,x_2\le 2P} \int_{\mathfrak{N}} |g^*(\alpha)h^*(\alpha)^2|^2 e(\alpha(x_1^3-x_2^3))\,d\alpha.$$

For the wider arcs $\mathfrak{N}$, it is easy to recycle the argument of the previous section. Following the estimations as begun in (30), one finds that

$$\begin{aligned}\int_{\mathfrak{N}} &|g^*(\alpha)h^*(\alpha)^2|^2 e(\alpha m)\,d\alpha \\ &= R^4 \sum_{q\le P^\nu} B(q,m)\Big(J(m) + O\Big(\int_{P^{-7/4}q^{-1}}^{\infty} \frac{Q^2}{(1+Q^3|\beta|)^2}\,d\beta\Big)\Big),\end{aligned}$$

and again as in the previous section this leads to

$$\int_{\mathfrak{N}} |g^*(\alpha)h^*(\alpha)^2|^2 e(\alpha m)\,d\alpha = R^4\mathfrak{T}(m,P^\nu)J(m) + O(R^4Q^{-4}P^{7/4+\nu}).$$

It remains to sum over $m = x_1^3 - x_2^3$ to conclude that

$$\int_{\mathfrak{N}} |fg^*h^{*2}|^2\,d\alpha = R^4 \sum_{P<x_1,x_2\le 2P} \mathfrak{T}(x_1^3-x_2^3,P^\nu)J(x_1^3-x_2^3) + O(R^4Q^{-4}P^{15/4+\nu}).$$

We rewrite the error term via $P^5 = Q^6$ and deduce from (42) the following result.

Lemma 5. *One has*

$$S_0 = R^4 \sum_{P<x_1,x_2\le 2P} \mathfrak{T}(x_1^3-x_2^3,P^\nu)J(x_1^3-x_2^3) + O(P^{\nu-5/4}Q^2R^4).$$

6. The principal minor arcs estimate. By (33) and orthogonality,

$$J(m) = \frac{1}{9} \sum_{\substack{l_1-l_2=m \\ Q^3<l_j\le 8Q^3}} (l_1l_2)^{-2/3}.$$

In particular, the sum is empty when $|m| > 8Q^3$. Hence, in Lemma 5, we may restrict the sum over x_1, x_2 to pairs with $|x_1^3 - x_2^3| \le 8Q^3$. Then, as in the argument leading from (12) to (14), we may write $k = x_1 - x_2$ and $x = x_2$ to obtain

$$S_0 = R^4 \sum_{|k|\le K} \sum_{P<x\le 2P} \mathfrak{T}(m(k,x),P^\nu)J(m(k,x)) + O(P^{\nu-5/4}Q^2R^4)$$

where $m(k,x)$ is given by (35). We now subtract this from the expansion for S obtained in Lemma 4. Since (11) and (36) show that

$$S - S_0 = \int_{\mathfrak{m}} |fgh^2|^2\, d\alpha,$$

this yields

$$\int_{\mathfrak{m}} |fgh^2|^2\, d\alpha \ll P^{-1-\frac{1}{7}\nu} Q^2 R^4. \tag{43}$$

The main difficulty has now been overcome, the required minor arc estimate is available. However, the endgame becomes easier if we proceed by pruning to the root. Let $\mathfrak{P}$ denote the union of the intervals

$$\{\alpha \in [0,1] : |\alpha - a/q| \le P^{-11/4}\} \tag{44}$$

with $0 \le a \le q$, $1 \le q \le P^\nu$ and $(a,q)=1$. Then $\mathfrak{P} \subset \mathfrak{M}$. By (26), we have $|f|^2 \ll |f^*|^2 + P^{2\nu}$ on $\mathfrak{M}$, and Lemma 3 then gives

$$\int_{\mathfrak{M}\setminus\mathfrak{P}} |fgh^2|^2\, d\alpha \ll \int_{\mathfrak{M}\setminus\mathfrak{P}} |f^*gh^2|^2\, d\alpha + P^{2\nu} Q^{1+\varepsilon} R^2.$$

Also, by trivial estimates and (37),

$$\int_{\mathfrak{M}\setminus\mathfrak{P}} |f^*gh^2|^2\, d\alpha \ll Q^2R^4 \int_{\mathfrak{M}\setminus\mathfrak{P}} |f^*|^2\, d\alpha$$

$$\ll P^2Q^2R^4 \sum_{q\le P^\nu} q^{1/3} \int_{P^{-11/4}}^{\infty} (1+P^3|\beta|)^{-2}\, d\beta \ll P^{2\nu-5/4}Q^2R^4.$$

The minor arcs $\mathfrak{p} = [0,1]\setminus\mathfrak{P}$ are the union of $\mathfrak{m}$ and $\mathfrak{M}\setminus\mathfrak{P}$, so that the preceding bounds combine with (43) to give

$$\int_{\mathfrak{p}} |fgh^2|^2\, d\alpha \ll P^{1-\frac{1}{7}\nu} Q^2 R^4. \tag{45}$$

7. Endgame. We complete the proof of the Theorem by a straightforward but unorthodox treatment of the major arc contribution, featuring a single variable singular integral. It will be convenient to write

$$\varrho_{\mathfrak{P}}(n,P) = \int_{\mathfrak{P}} f(\alpha)g(\alpha)h(\alpha)^2 e(-\alpha n)\, d\alpha,$$

and $\varrho_{\mathfrak{p}}$ for the same expression with $\mathfrak{P}$ replaced by $\mathfrak{p}$. For the latter, Bessel's inequality and (45) yield

$$\sum_n |\varrho_{\mathfrak{p}}(n,P)|^2 \ll P^{1-\frac{1}{7}\nu} Q^2 R^4. \tag{46}$$

Furthermore, by orthogonality,

$$\varrho_\theta(n,P) = \varrho_{\mathfrak{P}}(n,P) + \varrho_{\mathfrak{p}}(n,P). \tag{47}$$

In view of (46) and (47), it remains to compare $\varrho_{\mathfrak{P}}(n,P)$ with $\varrho^*_\theta(n,P)$ in mean square. We do this in three steps.

For $\alpha\in\mathfrak{P}$, let a/q be the centre of the interval (44) to which α belongs. Then, similarly to the deduction of (27), one finds that

$$g(\alpha)=Qq^{-1}S(q,a)+O(P^\nu).$$

Combined with (26) and (27), this gives

$$f(\alpha)g(\alpha)h(\alpha)^2=\mathcal{F}(\alpha)+O(P^{1+\nu}QR) \tag{48}$$

where the function $\mathcal{F}:\mathfrak{P}\to\mathbb{C}$ is defined by

$$\mathcal{F}(\alpha)=QR^2q^{-3}S(q,a)^3f^*(\alpha). \tag{49}$$

Now let

$$\varrho^\dagger(n)=\int_{\mathfrak{P}}\mathcal{F}(\alpha)e(-\alpha n)\,d\alpha.$$

Then the numbers $\varrho_{\mathfrak{P}}(n,P)-\varrho^\dagger(n)$ are the Fourier coefficients of the function that is $f^*gh^2-\mathcal{F}$ on $\mathfrak{P}$, and is 0 elsewhere in $[0,1]$. Hence, by Bessel's inequality,

$$\sum_n|\varrho_{\mathfrak{P}}(n,P)-\varrho^\dagger(n)|^2\le\int_{\mathfrak{P}}|f^*gh^2-\mathcal{F}|^2\,d\alpha.$$

The measure of $\mathfrak{P}$ is $O(P^{2\nu-11/4})$. Invoking (48), we infer that

$$\sum_n|\varrho_{\mathfrak{P}}(n,P)-\varrho^\dagger(n)|^2\ll P^{4\nu-3/4}Q^2R^2. \tag{50}$$

For the next step, write

$$A(q,n)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\frac{S(q,a)}{q}\right)^4e\Big(-\frac{an}{q}\Big),\qquad\mathfrak{S}(n,X)=\sum_{q\le X}A(q,n).$$

Lemma 4.8 of Vaughan [16] asserts that

$$\sum_{q\le X}q^{1/3}|A(q,n)|\ll(nX)^\varepsilon.$$

By partial summation, this implies the bound

$$\sum_{q>X}|A(q,n)|\ll n^\varepsilon X^{\varepsilon-1/3}.$$

In particular, the singular series

$$\mathfrak{S}(n)=\lim_{X\to\infty}\mathfrak{S}(n,X)$$

converges absolutely and satisfies

$$\mathfrak{S}(n)\ll n^\varepsilon,\qquad\mathfrak{S}(n,X)=\mathfrak{S}(n)+O(n^\varepsilon X^{\varepsilon-1/3}). \tag{51}$$

In a similar vein, let

$$I(n) = \int_{-1/2}^{1/2} v(\beta)e(-\beta n)\,d\beta, \qquad E(n) = \int_{P^{-11/4}<|\beta|\le 1/2} v(\beta)e(-\beta n)\,d\beta.$$

Then, by orthogonality, one has $I(n) = \frac{1}{3}n^{-2/3}$ in the range $P^3 < n \le 8P^3$, and $I(n) = 0$ otherwise. Consequently, we may write the function $\varrho_\theta^*(n,P)$ in the simple form

$$\varrho_\theta^*(n,P) = QR^2\mathfrak{S}(n)I(n). \tag{52}$$

Before we compare this with $\varrho^\dagger(n)$ in mean, we estimate $I(n)$ and $E(n)$. By Lemma 6.2 of Vaughan [16], one has $v(\beta) \ll P(1+P^3|\beta|)^{-1}$. Hence, one finds that

$$E(n) \ll P^{-2}\log P \tag{53}$$

uniformly in n, whereas the explicit formula for $I(n)$ yields the slightly better bound

$$I(n) \ll P^{-2}. \tag{54}$$

Also, by Bessel's inequality and the aforementioned bound for $v(\beta)$, one deduces the estimate

$$\sum_n |E(n)|^2 \ll \int_{P^{-11/4}<|\beta|\le 1/2} |v(\beta)|^2\,d\beta \ll P^{-5/4}. \tag{55}$$

We enter the final phase by recalling (49) and (51) to find that

$$\begin{aligned}\varrho^\dagger(n) &= \mathfrak{S}(n,P^\nu)QR^2 \int_{-P^{-11/4}}^{P^{11/4}} v(\beta)\,d\beta \\ &= (\mathfrak{S}(n) + O(P^{-\nu/4}n^\varepsilon))QR^2(I(n)-E(n)).\end{aligned}$$

By (51)–(54), this can be estimated further to yield

$$\varrho^\dagger(n) = \varrho_\theta^*(n,P) + O(n^\varepsilon P^{-2-\nu/6}QR^2 + n^\varepsilon QR^2|E(n)|).$$

We sum the consequential inequality

$$|\varrho^\dagger(n) - \varrho_\theta^*(n,P)|^2 \ll n^\varepsilon P^{-4-\nu/3}Q^2R^4 + n^\varepsilon Q^2R^4|E(n)|^2$$

over $1 \le n \le 9P^3$ to infer from (55) that

$$\sum_{n\le 9P^3} |\varrho^\dagger(n) - \varrho_\theta^*(n,P)|^2 \ll P^{-1-\nu/3}Q^2R^4.$$

This combines with (50) to yield

$$\sum_{n\le 9P^3} |\varrho_{\mathfrak{P}}(n,P) - \varrho_\theta^*(n,P)|^2 \ll P^{-1-\nu/3}Q^2R^4.$$

Note that $\varrho_\theta(n, P) = 0$ when $n > 9P^3$, at least when P is large. Hence, the sum defining V in (8) extends over $1 \le n \le 9P^3$ only. Therefore, by (47), the last estimate and (46) furnish the bound

$$V \ll P^{-1-\frac{1}{7}\nu} Q^2 R^4.$$

This completes the proof of the Theorem, with $\delta = \frac{1}{7}\nu$.

References

[1] J. Brüdern, *A problem in additive number theory*, Math. Proc. Cambridge Philos. Soc. 103 (1988), 27–33.

[2] J. Brüdern, *Sums of four cubes*, Monatsh. Math. 107 (1989), 179–188.

[3] J. Brüdern, *On Waring's problem for cubes*, Math. Proc. Cambridge Philos. Soc. 109 (1991), 229–256.

[4] J. Brüdern, *Cubic Diophantine inequalities III*, Period. Math. Hungar. 42 (2001), 211–226.

[5] J. Brüdern and T. D. Wooley, *On Waring's problem: three cubes and a sixth power*, Nagoya Math. J. 163 (2001), 13–53.

[6] J. Brüdern and T. D. Wooley, *On Waring's problem: three cubes and a minicube*, Nagoya Math. 200 (2010), 59–91.

[7] H. Davenport, *On Waring's problem for cubes*, Acta Math. 71 (1939), 123–143.

[8] G. H. Hardy and J. E. Littlewood, *Some problems of "Partitio Numerorum" (VI): Further researches in Waring's Problem*, Math. Z. 23 (1925), 1–37.

[9] S.-L. A. Lee, *On Waring's problem: two cubes and two minicubes*, Acta Arith. 151 (2012), 377–399.

[10] R. C. Vaughan, *Sums of three cubes*, Bull. London Math. Soc. 17 (1985), 17–20.

[11] R. C. Vaughan, *On Waring's problem for cubes*, J. Reine Angew. Math. 365 (1986) 121–170.

[12] R. C. Vaughan, *On Waring's problem for smaller exponents*, Proc. London Math. Soc. (3) 52 (1986), 445–463.

[13] R. C. Vaughan, *On Waring's problem for sixth powers*, J. London Math. Soc. (2) 33 (1986), 227–236.

[14] R. C. Vaughan, *On Waring's problem for cubes. II*, J. London Math. Soc. 39 (1989), 205–218.

[15] R. C. Vaughan, *A new iterative method in Waring's problem*, Acta Math. 162 (1989), 1–71.

[16] R. C. Vaughan, *The Hardy–Littlewood Method*, 2nd ed., Cambridge Univ. Press, Cambridge, 1997.

[17] T. D. Wooley, *Breaking classical convexity in Waring's problem: sums of cubes and quasi-diagonal behaviour*, Invent. Math. 122 (1995), 421–451.

[18] T. D. Wooley, *Sums of three cubes*, Mathematika 47 (2000), 53–61.

Jörg Brüdern
Mathematisches Institut
Bunsenstrasse 3–5
D-37085 Göttingen, Germany
E-mail: bruedern@uni-math.gwdg.de

Received on 31.8.2011 (6814)