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From explicit estimates for primes to explicit estimates for the Möbius function

by

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1. Introduction. There is a vast literature concerning explicit estimates for the summatory function of the Möbius function: we cite for instance [21], [1], [4], [3], [6], [7], [10], [11]. The paper [5] proposes a very useful annoted bibliography covering relevant items up to 1983. It has been known since the beginning of the 20th century at least (see for instance [13]) that showing that $M(x) = \sum_{n\le x} \mu(n)$ is $o(x)$ is equivalent to showing that the Chebyshev function $\psi(x) = \sum_{n\le x} \Lambda(n)$ is asymptotic to x. We have good explicit estimates for $\psi(x)-x$ (see for instance [19], [22] and [9]). This is due to the fact that we can use analytic tools in this problem since the residues at the poles of the Dirichlet generating series (here $-\zeta'(s)/\zeta(s)$) are known. However, this situation has no counterpart in the Möbius function case. It would thus be highly valuable to deduce estimates for $M(x)$ from estimates for $\psi(x) - x$, but a precise quantitative link is missing. I proposed some years back the following conjecture:

Conjecture (Strong form of Landau's equivalence Theorem, II). *There exist positive constants c_1 and c_2 such that*

$$|M(x)|/x \le c_1 \max_{c_2 x < y \le x/c_2} |\psi(y) - y|/y + c_1 x^{-1/4}.$$

This conjecture is trivially true under the Riemann Hypothesis. In this connection, we note that [23] proves that in the case of Beurling's generalized integers, one can have $M_{\mathcal{P}}(x) = o(x)$ without having $\psi(x) \sim x$. This reference has been kindly shown to me by Harold Diamond whom I warmly thank here.

We have not been able to prove such a strong estimate, but we are still able to derive an estimate for $M(x)$ from estimates for $\psi(x)-x$. Our process can be seen as a generalization of the initial idea of [21], also used in [10].

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We describe it in Section 3, after a combinatorial preparation. Here is our main theorem.

THEOREM 1.1. *For $D \geq 1\,078\,853$, we have*

$$\Big|\sum_{d\leq D}\mu(d)\Big| \leq \frac{0.0130\log D - 0.118}{(\log D)^2}D.$$

The last result of this shape is from [10] and has 0.10917 (starting from $D = 695$) instead of 0.0130.

Following an idea of [11] which we recall in the last section, we deduce from the above the following estimate.

COROLLARY 1.2. *For $D \geq 60\,298$, we have*

$$\Big|\sum_{d\leq D}\mu(d)/d\Big| \leq \frac{0.0260\log D - 0.118}{(\log D)^2}.$$

The last result of this shape is from [11] and has 0.2185 (starting from $x = 33$) instead of 0.0260. Here are two results that are easier to remember:

COROLLARY 1.3. *For $D \geq 60\,200$, we have*

$$\Big|\sum_{d\leq D}\mu(d)/d\Big| \leq \frac{\log D - 4}{40(\log D)^2}.$$

If we replace the -4 by 0, the resulting bound is valid from $24\,270$ onward.

COROLLARY 1.4. *For $D \geq 50\,000$, we have*

$$\Big|\sum_{d\leq D}\mu(d)/d\Big| \leq \frac{3\log D - 10}{100(\log D)^2}.$$

If we replace the -10 by 0, the resulting bound is valid from $11\,815$ onward.

We will meet another problem in between, which is to relate quantitatively the error term $\psi(x) - x$ with the error term concerning the approximation of $\tilde{\psi}(x) = \sum_{n\leq x}\Lambda(n)/n$ by $\log x - \gamma$. This problem is surprisingly difficult but [16] offers a good enough solution.

Notation. We write $R(x) = \psi(x) - x$ and $r(x) = \tilde{\psi}(x) - \log x + \gamma$, where we recall that

$$\tilde{\psi}(x) = \sum_{n\leq x}\Lambda(n)/n. \tag{1.1}$$

We shall use square brackets to denote the integer part and curly parentheses to denote the fractional part, so that $D = [D] + \{D\}$. But since this notation is used seldom, we shall also use square brackets in their usual function.

2. A combinatorial tool. In this section we prove a certain formal identity. Let F be a function and $Z=-F'/F$ the opposite of its logarithmic derivative. We look at

$$F[1/F]^{(k)}=P_k.$$

It is immediate to compute the first values and we find that

$$P_0=F,\quad P_1=Z,\quad P_2=Z'+Z^2,\quad P_3=Z''+3ZZ'+Z^3. \tag{2.1}$$

In general, the following recursion formula holds:

$$P_k=F(P_{k-1}/F)'=P'_{k-1}+ZP_{k-1}. \tag{2.2}$$

Here is the result this leads to:

THEOREM 2.1. *We have*

$$F[1/F]^{(k)}=\sum_{\sum_{i\ge1}ik_i=k}\frac{k!}{k_1!k_2!\cdots(1!)^{k_1}(2!)^{k_2}\cdots}\prod_{k_i}Z^{(i-1)k_i}.$$

We can prove it by using the recursion formula given above. We now present a different argument. Let us expand $1/F(s+X)$ in a Taylor series around $X=0$:

$$\frac{1}{F(s+X)}=\sum_{k\ge0}[1/F(s)]^{(k)}\frac{X^k}{k!}.$$

We do the same for $-F'(s+X)/F(s+X)$:

$$\frac{-F'(s+X)}{F(s+X)}=\sum_{k\ge0}[Z(s)]^{(k)}\frac{X^k}{k!}.$$

Integrating formally this expression, we get

$$-\log(F(s+X)/F(s))=\sum_{k\ge1}[Z(s)]^{(k-1)}\frac{X^k}{k!}$$

where the constant term is chosen so that the constant term is indeed 0. We then apply the exponential formula

$$\exp\Big(\sum_{k\ge1}x_kX^k/k!\Big)=\sum_{m\ge0}Y_m(x_1,x_2,\ldots)\frac{X^m}{m!}$$

where the $Y_m(x_1,x_2,\ldots)$ are the complete exponential Bell polynomials whose expression yields the theorem above.

3. The general argument. Let us specialize $F=\zeta$ in Theorem 2.1. The left hand side therein has a simple pole at $s=1$ with residue being $k!$ times the kth Taylor coefficient of $1/\zeta(s)$ at $s=1$. Let us denote by $\mathfrak{R}_k$ this residue. By a routine argument, we get

$$\sum_{\ell \le L} \mathbb{1} \star (\mu \log^k)(\ell) = \mathfrak{R}_k L + o(L). \tag{3.1}$$

Note that, thanks to Theorem 2.1, the error term is quantified in terms of the error term in the approximations of both $\psi(x)-x$ and $\tilde{\psi}(x) - \log x + \gamma$. Getting this error term in fact requires using a good enough error term for both these quantities (see for instance [12]). We then continue

$$\sum_{\ell \le L} \mu(\ell) \log^k \ell = \sum_{d \le L} \mu(d) \biggl(\mathfrak{R}_k \frac{L}{d} + o(L/d)\biggr), \tag{3.2}$$

which ensures that $\sum_{\ell \le L} \mu(\ell) \log^k \ell$ is $o(L \log L)$.

The case $k = 2$ is most enlightening. In this case, our method consists in writing

$$\sum_{\ell \le L} \mu(\ell) \log^2 \ell = \sum_{d\ell \le L} \mu(\ell) \bigl(\Lambda \star \Lambda(d) - \Lambda(d) \log d\bigr). \tag{3.3}$$

It turns out that the main term of the summatory function of $\Lambda \log$ (namely $L \log L$) cancels the one of $\Lambda \star \Lambda$. This requires the prime number theorem. In deriving the prime number theorem from Selberg's formula $\mu \star \log^2 = \Lambda \log + \Lambda \star \Lambda$, it is a well known difficulty to show that both summands indeed contribute and this is another show-up of the parity principle. We modify (3.3) as follows:

$$2\gamma + \sum_{\ell \le L} \mu(\ell) \log^2 \ell = \sum_{d\ell \le L} \mu(\ell) \bigl(\Lambda \star \Lambda(d) - \Lambda(d) \log d + 2\gamma\bigr). \tag{3.4}$$

The case $k = 1$ is classical, but it is interesting to note that this is the starting point of [21].

4. Some known estimates and straightforward consequences.

Lemma 4.1 ([18]). $\max_{t \ge 1} \psi(t)/t = \psi(113)/113 \le 1.04$.

Concerning small values, we quote from [17] the following result:

$$|\psi(x) - x| \le \sqrt{x} \quad (8 \le x \le 10^{10}). \tag{4.1}$$

If we change $\sqrt{x}$ to $\sqrt{2x}$, this is valid from $x = 1$ onwards. Furthermore

$$|\psi(x) - x| \le 0.8\sqrt{x} \quad (1\,500 \le x \le 10^{10}). \tag{4.2}$$

Lemma 4.2.

$$|\psi(x) - x| \le 0.0065\, x/\log x \quad (x \ge 1\,514\,928).$$

Proof. By [8, Théorème 1.3] improving on [22, Theorem 7], we have

$$|\psi(x) - x| \le 0.0065\, x/\log x \quad (x \ge \exp(22)). \tag{4.3}$$

We readily extend this estimate to $x \geq 3\,430\,190$ by using (4.2). We then use the function `WalkPsi` from the script `IntR.gp` (with the proper `model` function). ∎

Lemma 4.3. *For $x \geq 7\,105\,266$, we have*

$$|\psi(x) - x|/x \leq 0.000213.$$

Proof. We start with the estimate from [20, (4.1)]

$$|\psi(x) - x|/x \leq 0.000213 \qquad (x \geq 10^{10}). \tag{4.4}$$

We extend it to $x \geq 14\,500\,000$ by using (4.2). We complete the proof by using the following Pari/Gp script (see [15]):

```
{CalculeLambdas(Taille)=
   my(pk, Lambdas);
   Lambdas = vector(Taille);
   forprime(p = 2,Taille,
            pk = p;
            while(pk <= Taille, Lambdas[pk] = p; pk*=p));
   return(Lambdas);}

{model(n)=n}

{WalkPsi(zmin, zmax)=
   my(res = 0.0, mo, maxi, psiaux = 0.0, Lambdas);
   Lambdas = CalculeLambdas(zmax);
   for(y = 2, zmin,
      if(Lambdas[y]!=0, psiaux += log(Lambdas[y]),));
   maxi = abs(psiaux-zmin)/model(zmin);
   for(y = zmin+1, zmax,
      mo = 1/model(y);
      maxi = max(maxi, abs(psiaux-y)*mo);
      if(Lambdas[y]!=0, psiaux += log(Lambdas[y]),);
      maxi = max(maxi, abs(psiaux-y)*mo));
   print("|psi(x)-x|/model(x) <= ", maxi, " pour ",
         zmin, " <= x <= ", zmax);
   return(maxi);}
```

∎

Lemma 4.4. *For $x \geq 32\,054$, we have*

$$|\psi(x) - x|/x \leq 0.003.$$

Proof. The preceding lemma proves this for $x \geq 7\,105\,266$. By using (4.2), we extend it to $x \geq 102\,500$. We complete the proof by using the same script as in the proof of Lemma 4.3. ∎

We quote from [16] the following lemma.

Lemma 4.5. *When $x \geq 23$, we have*

$$\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*\left(\frac{0.0067}{\log x}\right).$$

Let us turn our attention to the summatory function of the Möbius function. In [6], we find the bound

$$|M(x)| \le 0.571\sqrt{x} \quad (33 \le x \le 10^{12}). \tag{4.5}$$

In [7], we find

$$|M(x)| \le x/2360 \quad (x \ge 617\,973) \tag{4.6}$$

(see also [4]) which [2] (published also in [3]) improves to

$$|M(x)| \le x/4345 \quad (x \ge 2\,160\,535). \tag{4.7}$$

Bounds for squarefree numbers

Lemma 4.6. *For $D \ge 1$ we have*

$$\sum_{d \le D} \mu^2(d) = \frac{6}{\pi^2} D + \mathcal{O}^*(0.7\sqrt{D}).$$

For $D \ge 10$, we can replace 0.7 by 0.5.

Proof. [1] (see also [2]) proves that

$$\sum_{d \le D} \mu^2(d) = \frac{6}{\pi^2} D + \mathcal{O}^*(0.1333\sqrt{D}) \quad (D \ge 1\,664),$$

and we use direct inspection using Pari/Gp to conclude. ∎

Lemma 4.7. *Let $D/K \ge 1$. Let f be a non-negative non-decreasing C^1 function. Then*

$$\sum_{D/L < d \le D/K} \mu^2(d) f(D/d) \le 1.31\, f(L) + \frac{6D}{\pi^2} \int_K^L \frac{f(t)\,dt}{t^2} + 0.35\sqrt{D} \int_K^L \frac{f(t)\,dt}{t^{3/2}}.$$

Proof. We use a simple integration by parts to write

$$\begin{aligned}\sum_{D/L < d \le D/K} \mu^2(d) f(D/d) &= \sum_{D/L < d \le D/K} \mu^2(d)\Big(f(K) + \int_K^{D/d} f'(t)\,dt\Big) \\ &= \sum_{D/L < d \le D/K} \mu^2(d) f(K) + \int_K^L \Big(\sum_{D/L < d \le D/t} \mu^2(d)\Big) f'(t)\,dt.\end{aligned}$$

We then employ Lemma 4.6 to get the bound

$$\frac{6D}{\pi^2 K} f(K) + \int_K^L \frac{6D}{\pi^2 t} f'(t)\,dt + 0.7\sqrt{\frac{D}{K}}\, f(K) + 0.7 \int_K^L \sqrt{\frac{D}{t}} f'(t)\,dt.$$

Two integrations by parts give the expression

$$\frac{6}{\pi^2} f(L) + \int_K^L \frac{6D}{\pi^2 t^2} f(t)\,dt + 0.7\, f(L) + 0.35\sqrt{D} \int_K^L \frac{f(t)\,dt}{t^{3/2}}.$$

The lemma follows readily. ∎

5. A preliminary estimate on primes. Our aim here is to evaluate

$$R_4(D)=\sum_{d_1\le\sqrt{D}}\Lambda(d_1)R(D/d_1). \tag{5.1}$$

This remainder term is crucial in the final analysis and will be numerically one of the dominant terms.

LEMMA 5.1. *When $D\ge 1$, and $\sqrt{D}\ge T\ge 1$, we have*

$$\sum_{d\le T}\frac{\Lambda(d)}{d\log(D/d)}\le 1.04\log\frac{\log D}{\log(D/T)}+\frac{1.04}{\log D}.$$

Proof. Let $f(t)=1/(t\log(D/t))$. By a classical summation by parts we have

$$\begin{aligned}\sum_{d\le T}\Lambda(d)f(d)&=\sum_{d\le T}\Lambda(d)f(T)-\sum_{d\le T}\Lambda(d)\int_d^T f'(t)\,dt\\ &\le\frac{1.04}{\log(D/T)}-1.04\int_1^T tf'(t)\,dt\\ &\le\frac{1.04}{\log(D/T)}-1.04\,[tf(t)]_1^T+1.04\int_1^T f(t)\,dt\\ &\le\frac{1.04}{\log D}+1.04\int_{D/T}^{D}\frac{dt}{t\log t}\le\frac{1.04}{\log D}+1.04\log\frac{\log D}{\log(D/T)}\end{aligned}$$

as required. ∎

LEMMA 5.2. *We have $|R_4(D)|/D\le 0.0065$ when $D\ge 10^{10}$. When $D\ge$ $1\,300\,000\,000$, we have $|R_4(D)|/D\le 0.0073$.*

The proof that follows is somewhat clumsy due to the fact that we have not been able to compute $R_4(D)$ for D up to 10^{10}. By inspecting the expression defining R_4 and the proof below, the reader will see one could try to get a better bound for

$$\sum_{D^{1/4}<d\le\sqrt{D}}\Lambda(d)R(D/d).$$

Indeed, one can compute the exact values of $R(D/d)$ and try to approximate them properly so as not to loose the sign changes in the expression. A proper model is even given by the explicit formula for $\psi(x)$. We have however tried to use the resulting polynomial, namely $x-\sum_{|\gamma|\le G}x^{1/2+i\gamma}/(1/2+i\gamma)$ with $G=20$, $G=30$ and $G=200$, but the approximation was very weak. It may be better to find directly a numerical fit for $R(x)$ in this limited range. It should be noted that the function $R(x)$ is highly erratical. Such a process would be important since the value 0.0065 that we get here decides a large part of the final value in Theorem 1.1.

Proof of Lemma 5.2. When $D \geq 1514928^2$, by Lemmas 4.2 and 5.1 we have

$$|R_4(D)|/D \leq 0.0065 \sum_{d\leq\sqrt{D}} \frac{\Lambda(d)}{d\log(D/d)} \leq 0.0065\cdot\left(0.73+\frac{1.04}{\log D}\right).$$

This implies that $|R_4(D)|/D \leq 0.00499$ in the given range. When $10^{10} \leq D \leq 1514928^2$, we set $T = D/10^{10}$ and write

$$\begin{aligned}|R_4(D)|/D &\leq 0.000213\sum_{d\leq T}\frac{\Lambda(d)}{d}+\frac{1}{D^{1/2}}\sum_{T<d\leq\sqrt{D}}\frac{\Lambda(d)}{\sqrt{d}}\\ &\leq 0.000213\,\tilde{\psi}(T)\\ &\quad+\frac{1}{D^{1/2}}\left(\frac{\psi(\sqrt{D})-\psi(T)}{D^{1/4}}+\frac{1}{2}\int_T^{\sqrt{D}}\frac{\psi(u)-\psi(T)}{u^{3/2}}\,du\right),\end{aligned}$$

i.e. on using $\psi(u) \leq u + \sqrt{u}$,

$$\begin{aligned}|R_4(D)|/D &\leq 0.000213\,\tilde{\psi}(T)\\ &\quad+\frac{1}{D^{1/2}}\left(\frac{\psi(\sqrt{D})}{D^{1/4}}-\frac{\psi(T)}{T^{1/2}}+\frac{1}{2}\int_T^{\sqrt{D}}\frac{\psi(u)}{u^{3/2}}\,du\right)\\ &\leq 0.000213\,\tilde{\psi}(T)\\ &\quad+\frac{1}{D^{1/2}}\left(\frac{\sqrt{D}+D^{1/4}}{D^{1/4}}-\frac{T-\sqrt{T}}{T^{1/2}}+D^{1/4}-\sqrt{T}+\log\frac{\sqrt{D}}{T}\right),\end{aligned}$$

i.e. since $\tilde{\psi}(x) \leq \log x$ when $x \geq 1$,

$$\begin{aligned}|R_4(D)|/D &\leq 0.000213\log T\\ &\quad+\frac{1}{D^{1/2}}\left(2D^{1/4}-2\sqrt{T}+2+\log\frac{\sqrt{D}}{T}\right).\end{aligned}$$

We deduce that $|R_4(D)|/D \leq 0.0065$ when $D \geq 10^{10}$. When now $10^9 \leq D \leq 10^{10}$, we proceed as follows:

$$\begin{aligned}|R_4(D)|/D &\leq \frac{1}{D^{1/2}}\left(\frac{\psi(1500)}{1500^{1/2}}+\frac{1}{2}\int_1^{1500}\frac{\psi(u)}{u^{3/2}}\,du\right)\\ &\quad+\frac{0.8}{D^{1/2}}\left(\frac{\psi(\sqrt{D})-\psi(1500)}{D^{1/4}}+\frac{1}{2}\int_{1500}^{\sqrt{D}}\frac{\psi(u)-\psi(1500)}{u^{3/2}}\,du\right).\end{aligned}$$

We readily compute that $\psi(1500) = 1509.27 + \mathcal{O}^*(0.01)$, so that

$$|R_4(D)|/D^{1/2} \le (0.2 - 0.8)\frac{1509.3}{1500^{1/2}} + 0.642 + 0.8 \cdot 1.04\,(2D^{1/4} - 1500^{1/2}).$$

The right hand side is not more than 0.0073 when $D \ge 1\,300\,000\,000$. ∎

6. The relevant error term for the primes. The main actor of this section is the remainder term R_2^* defined by

$$\sum_{d\le D}(\Lambda\star\Lambda(d) - \Lambda(d)\log d) = -2[D]\gamma + R_2^*(D). \tag{6.1}$$

The object of this section is to derive explicit estimate for R_2^* from explicit estimates for ψ. Most of the work has already been done in the previous section, and we essentially put things in shape. Here is our result.

Lemma 6.1. *When $D \ge 1\,435\,319$, we have $|R_2^*(D)|/D \le 0.0213$.*

We start by an expression for R_2^*.

Lemma 6.2.

$$|R_2^*(D)| \le 2D|r(\sqrt{D})| + 2D^{1/2}R(\sqrt{D}) + R(\sqrt{D})^2 + R(D)\log D$$
$$+ 1 + 2\gamma + 2R_4(D) + \left|\int_1^D R(t)\frac{dt}{t}\right|$$

where R_4 is defined in (5.1).

Proof. The proof is fully pedestrian. We have

$$\sum_{d\le D}\Lambda(d)\log d = \psi(D)\log D - \int_1^D \psi(t)\,dt/t$$
$$= D\log D - D + 1 + R(D)\log D - \int_1^D R(t)\,dt/t.$$

Concerning the other summand, the Dirichlet hyperbola formula yields

$$\sum_{d_1d_2\le D}\Lambda(d_1)\Lambda(d_2) = 2\sum_{d_1\le\sqrt{D}}\Lambda(d_1)\psi(D/d_1) - \psi(\sqrt{D})^2$$
$$= 2D\sum_{d_1\le\sqrt{D}}\frac{\Lambda(d_1)}{d_1} - D$$
$$- 2\sqrt{D}R(\sqrt{D}) - R(\sqrt{D})^2 + 2\sum_{d_1\le\sqrt{D}}\Lambda(d_1)R(D/d_1)$$
$$= D\log D - 2D\gamma - D$$
$$+ 2Dr(\sqrt{D}) - 2\sqrt{D}R(\sqrt{D}) - R(\sqrt{D})^2 + 2R_4(D).$$

We arrive at $R_2^*(D) = R_3(D) - 1 + 2R_4(D) - R(D)\log D + \int_1^D R(t)\,dt/t$, where

$$R_3(D) = 2Dr(\sqrt{D}) - 2\gamma\{D\} - 2\sqrt{D}R(\sqrt{D}) - R(\sqrt{D})^2. \tag{6.2}$$

The lemma follows readily. ∎

Lemma 6.3. *For the real number D satisfying $3 \le D \le 110\,000\,000$, we have*

$$|R_2^*(D)| \le 1.80\sqrt{D}\log D.$$

When $110\,000\,000 \le D \le 1\,800\,000\,000$, *we have*

$$|R_2^*(D)| \le 1.93\sqrt{D}\log D.$$

We used a Pari/Gp script. The only non-obvious point is that we have precomputed the values of $\Lambda \star \Lambda - \Lambda \star \log$ on intervals of length $2\cdot 10^6$. On letting this script run longer (about twenty days), I would most probably be able to show that the bound $|R_2^*(D)| \le 2\sqrt{D}\log D$ holds when $D \le 10^{10}$. This would improve a bit on the final result.

Lemma 6.4.

$$\int_1^{10^8} R(t)\,dt/t = -129.559 + \mathcal{O}^*(0.01).$$

We used a Pari/Gp script as above, but the running time was much shorter.

Proof of Lemma 6.1. Assume that $D \ge 1.3\cdot 10^9$. We start with Lemma 6.2. We bound $r(\sqrt{D})$ via Lemma 4.5 (this requires $D \ge 23^2$), then $R(\sqrt{D})$ by Lemma 4.4 (this requires $D \ge 32054^2$), and $R(D)\log D$ by using Lemma 4.2 (this requires $D \ge 1\,514\,928$). We bound R_4 by appealing to Lemma 5.2. We conclude by appealing to Lemma 4.3. All of that amounts to the bound

$$|R_2^*(D)| \le \frac{4\cdot 0.0067\,D}{\log D} + 0.006\,D + (0.003)^2 D + 0.0065\,D + 0.0073\,D + 132 + 0.000213D - 0.000213\cdot 10^8.$$

We arrive at

$$|R_2^*(D)|/D \le 0.0213 \tag{6.3}$$

when $D \ge 1.3\cdot 10^9$. Thanks to Lemma 6.3, we extend this bound to $D \ge 1\,435\,319$. ∎

7. Estimating $M(D)$. We appeal to (3.4) and use the Dirichlet hyperbola formula. In this manner we get our starting equation:

$$\sum_{d\le D}\mu(d)\log^2 d = 2\gamma + \sum_{d\le D/K}\mu(d)R_2^*(D/d) + \sum_{k\le K}R_2^*(k)\sum_{D/(k+1)<d\le D/k}\mu(d). \tag{7.1}$$

This equation is much more important than it looks since a bound for $R_2^*(k)$ that is $\ll k/(\log k)^2$ shows that the second sum converges. A more usual treatment would consist in writing

$$\sum_{d\le D}\mu(d)\log^2 d = 2\gamma + \sum_{d\le D/K}\mu(d)R_2^*(D/d) + \sum_{k\le K}(\Lambda\star\Lambda-\Lambda\log+2\gamma)(k)\sum_{D/K<d\le D/k}\mu(d),$$

as in [21] for instance. However, when we bound $M(D/k)-M(D/(k+1))$ roughly by $D/(k(k+1))$ in (7.1), we get $D\sum_{k\le K}|R_2^*(k)|/(k(k+1))$, which is expected to be $\mathcal{O}(D)$. On bounding $M(D/k)-M(D/K)$ by D/k in the second expression, we only get $D\sum_{k\le K}|\Lambda\star\Lambda-\Lambda\log-2\gamma|(k)/k$, which is of size $D\log^2 K$. Practically, if we want to use a bound of the shape $|M(x)|\le x/4345$, we will loose the differentiating aspect and will bound $|M(D/k)-M(D/(k+1))|$ by $2D/(4345\,k)$ and not by $D/(4345\,k^2)$. It is thus better to use differentiation-difference on the variable $R_2^*(k)$ when k is fairly small. It turns out that small is large enough! We write

$$\sum_{k\le K}R_2^*(k)\big(M(D/k)-M(D/(k+1))\big) = \sum_{k\le K}(\Lambda\star\Lambda-\Lambda\log+2\gamma)(k)M(D/k)+R_2^*(K)M(D/K). \tag{7.2}$$

Lemma 7.1. *When $K=462\,848$, we have*

$$\sum_{k\le K}\frac{|\Lambda\star\Lambda-\Lambda\log+2\gamma|(k)}{k}+\frac{|R_2^*(K)|}{K}\le 0.03739\times 4345.$$

We can use the simple bound (6.3) to get, for $D/K\ge 2\,160\,535$,

$$\Big|\sum_{d\le D}\mu(d)\log^2 d\Big|/D \le \frac{2\gamma}{D}+0.0213\left(\frac{6}{\pi^2}\log\frac{D}{K}+1.166\right)+0.03739 \le 0.0130\log D-0.144$$

with $K=462\,848$. Note that this lower bound of K has been chosen to satisfy

$$462\,848\times 2\,160\,535\le 10^{12}.$$

Concerning the smaller values, we use summation by parts:

$$\sum_{d\le D}\mu(d)\log^2 d=\sum_{d\le D}\mu(d)\log^2 D-2\int_1^D\sum_{d\le t}\mu(d)\frac{\log t\,dt}{t},$$

which gives, when $33\le D\le 10^{12}$,

$$\begin{aligned}\Big|\sum_{d\le D}\mu(d)\log^2 d\Big| &\le 0.571\sqrt{D}\log^2 D+2\Big|\int_1^{33}\sum_{d\le t}\mu(d)\frac{\log t\,dt}{t}\Big|\\ &\quad+2\cdot 0.571\int_{33}^{D}\frac{\log t\,dt}{\sqrt{t}}\\ &\le 0.571\sqrt{D}\log^2 D+2.284\sqrt{D}\log D+4.568\sqrt{D}-43,\end{aligned}$$

and this is $\le 0.0130\log D-0.144$ when $D\ge 8\,613\,000$. We extend this bound to $D\ge 2\,161\,205$ by direct computations using Pari/Gp.

Let us state formally:

Lemma 7.2. *For $D\ge 2\,161\,205$, we have*

$$\Big|\sum_{d\le D}\mu(d)\log^2 d\Big|/D\le 0.0130\log D-0.144.$$

8. A general formula and proof of Theorem 1.1. Let $(f(n))$ be a sequence of complex numbers. We consider, for integer $k\ge 0$, the weighted summatory function

$$M_k(f,D)=\sum_{n\le D}f(n)\log^k n. \tag{8.1}$$

We want to derive information on $M_0(f,D)$ from information on $M_k(f,D)$. The traditional way to do that is in essence due to [14] and goes via a differential equation. It turns out that it is clearer and somewhat more precise to use the identity that follows.

Lemma 8.1. *For $k\ge 0$ and $D\ge D_0$ we have*

$$M_0(f,D)=\frac{M_k(f,D)}{\log^k D}+M_0(f,D_0)-\frac{M_k(f,D_0)}{\log^k D_0}-k\int_{D_0}^{D}\frac{M_k(f,t)}{t\log^{k+1}t}\,dt.$$

This formula in a special case is also used in [21] and [10].

Proof. Indeed, we have

$$k\int_{D_0}^{D}\frac{M_k(f,t)}{t\log^{k+1}t}\,dt=-\frac{M_k(f,D_0)}{\log^k D_0}+\sum_{n\le D}f(n)\frac{\log^k n}{\log^k D}-\sum_{D_0<n\le D}f(n).\ \blacksquare$$

Proof of Theorem 1.1. In the notation of Lemma 8.1, we have $M(D) = M_0(\mu, D)$. By Lemma 7.2 with $D_0 = 2\,161\,205$ we have

$$\begin{aligned} |M(D)| &\le \frac{0.0130\log D - 0.144}{\log^2 D} D + M(D_0) - \frac{M_2(\mu, D_0)}{\log^2 D_0} \\ &\qquad + 2\int_{D_0}^{D} \frac{0.0130\log t - 0.144}{\log^3 t}\,dt \\ &\le \frac{0.0130\log D - 0.144}{\log^2 D} D - 3.48 + 2\int_{D_0}^{D} \frac{0.0130\log t - 0.144}{\log^3 t}\,dt \\ &\le \frac{0.0130\log D - 0.118}{\log^2 D} D - 3.48 \\ &\qquad - 0.0260\frac{D_0}{\log^2 D_0} - \int_{D_0}^{D} \frac{0.236}{t\log^3 t}\,dt. \end{aligned}$$

(We used Pari/Gp to compute the quantity $M(D_0) - M_2(\mu, D_0)/\log^2 D_0$). We conclude by direct verification, again relying on Pari/Gp. ■

9. From M to m. We take the following lemma from [11, (1.1)].

LEMMA 9.1 (El Marraki). *We have*

$$|m(D)| \le \frac{|M(D)|}{D} + \frac{1}{D}\int_1^D \frac{|M(t)|\,dt}{t} + \frac{\log D}{D}.$$

This lemma may look trivial enough, but its teeth are hidden. Indeed, the usual summation by parts would bound $|m(D)|$ by an expression containing the integral of $|M(t)|/t^2$. An upper bound for $|M(t)|$ of the shape $ct/\log t$ would then result in the useless bound $m(D) \ll \log\log D$.

Proof of Lemma 9.1. We reproduce the proof, as it is short and the preprint we refer to is difficult to find. We have two equations, namely

$$m(D) = \frac{M(D)}{D} + \int_1^D \frac{M(t)\,dt}{t^2} \tag{9.1}$$

and

$$\int_1^D \left[\frac{D}{t}\right]\frac{M(t)\,dt}{t} = \log D. \tag{9.2}$$

We deduce from the above that

$$m(D) = \frac{M(D)}{D} + \frac{1}{D}\int_1^D \left(\frac{D}{t} - \left[\frac{D}{t}\right]\right)\frac{M(t)\,dt}{t} + \frac{\log D}{D}.$$

The lemma follows readily. ■

Proof of Corollary 1.2. We have, when $D \geq D_0 = 1\,078\,853$,

$$
\begin{aligned}
|m(D)| \leq \frac{0.0130 \log D - 0.118}{(\log D)^2} &+ \frac{1}{D}\int_{D_0}^{D} \frac{0.0130 \log t - 0.118}{(\log t)^2}\,dt \\
&+ \frac{1}{D}\int_{1}^{D_0} \frac{|M(t)|\,dt}{t} + \frac{\log D}{D}, \\
\leq \frac{0.0130 \log D - 0.118}{(\log D)^2} &+ \frac{1}{D}\int_{D_0}^{D} \frac{0.0130\,dt}{\log t} \\
&- \frac{1}{D}\int_{D_0}^{D} \frac{0.118\,dt}{(\log t)^2} + \frac{301 + \log D}{D}.
\end{aligned}
$$

We continue by an integration by parts and some numerical computations:

$$
\begin{aligned}
|m(D)| &\leq \frac{0.0260 \log D - 0.118}{(\log D)^2} - \frac{0.105}{D}\int_{D_0}^{D} \frac{dt}{(\log t)^2} + \frac{-9795 + \log D}{D}, \\
&\leq \frac{0.0260 \log D - 0.118}{(\log D)^2} - \frac{1}{D}\int_{D_0}^{D} \frac{dt}{t} + \frac{-9795 + \log D}{D}.
\end{aligned}
$$

This proves that $|m(D)|(\log D)^2 \leq 0.0260 \log D - 0.118$ as soon as $D \geq 1\,078\,853$. We extend this bound by direct inspection. ■

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