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Continued fractions on the Heisenberg group

by

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1. Introduction. Real continued fractions and their many variations have played an important part in Diophantine approximation, hyperbolic geometry, and the study of quadratic irrationals. Many higher-dimensional generalizations of continued fractions have been developed to extend this powerful theory, with varying success. In this paper, we develop a notion of continued fractions in the *non-commutative* setting of the Heisenberg group, whose structure is directly analogous to that of real continued fractions and yields a strikingly similar theory.

We will work with the Heisenberg group primarily in its Siegel model (see also §2.1), namely the set of points

$$\mathcal{S} = \{(u,v) \in \mathbb{C}^2 : u\overline{u} - (v+\overline{v}) = 0\}$$

with the group law given by $(u_1,v_1) * (u_2,v_2) = (u_1+u_2, v_1+v_2+\overline{u_1}u_2)$.

The integer points $\mathcal{S}(\mathbb{Z}) := \mathcal{S} \cap \mathbb{Z}[\mathbb{i}]^2$ form a co-compact subgroup of $\mathcal{S}$, analogously to $\mathbb{Z} \subset \mathbb{R}$. Fix a fundamental domain $K \subset \mathcal{S}$ for $\mathcal{S}(\mathbb{Z})$. We define a *complex* two-dimensional continued fraction via a Gauss map $T : K \to K$ given by

$$T(u,v) = \begin{cases} (0,0) & \text{if } (u,v) = (0,0), \\ (\alpha,\beta)^{-1} * \left(-\dfrac{u}{v}, \dfrac{1}{v}\right) & \text{if } (u,v) \neq (0,0), \end{cases}$$

for some appropriately chosen integer point $(\alpha,\beta) \in \mathcal{S} \cap \mathbb{Z}[\mathbb{i}]^2$ depending on (u,v). The term in the second case above is given more directly by

$$\left(-\alpha - \frac{u}{v}, \overline{\beta} + \overline{\alpha}\frac{u}{v} + \frac{1}{v}\right),$$

which bears strong resemblence to multi-dimensional continued fractions like the Jacobi–Perron algorithm.

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Surprisingly, we recover not only standard results of convergence (see Theorem 1.3), but also several simple, direct analogs of classical formulas for regular continued fractions—formulas which lack simple analogs for any other known multi-dimensional continued fraction. This suggests that continued fractions are a reasonable and natural object of study on the Heisenberg group.

There are a variety of related constructions which the reader may find interesting, such as one-dimensional complex continued fractions [5] and multidimensional real continued fractions [12]. See also [9] for many open questions about one-dimensional real continued fraction variants.

The present paper opens up the way for many new questions. In a forthcoming paper [15], the second author connects our study to that of Diophantine approximation on the Heisenberg group, showing that the continued fractions studied here satisfy an analog of Khinchin's theorem and a weak form of best approximation (see also [6]). We also have some results characterizing periodic continued fraction expansions [16], and hope in the future to extend the current theorems to other choices of lattices in the Heisenberg group and other boundaries of hyperbolic rank-one symmetric spaces (see Remark 2.5 and [14]). The dynamical properties of the Gauss map, including ergodicity, mixing, and any connection to geodesics in complex hyperbolic space are unknown.

1.1. Main results. We now phrase our results for the *nearest-integer* continued fractions on $\mathcal{S}$. Here, distances are measured using the *Korányi norm* $\|(u,v)\| = |v|^{1/2}$, whose topology agrees with that induced by the embedding $\mathcal{S} \hookrightarrow \mathbb{C}^2$.

The Siegel model has two notions of inversion. The inverse of a point with respect to the group action is given by $(u,v)^{-1} = (-u, \overline{v})$; it corresponds to the map $x \mapsto -x$ for $\mathbb{R}$. The Korányi inversion in the unit sphere is given by $\iota(u,v) := (-u/v, 1/v)$; it corresponds to the map $x \mapsto 1/x$ in $\mathbb{R}$.

We will denote the integer points and rational points of $\mathcal{S}$ by $\mathcal{S}(\mathbb{Z}) = \mathcal{S} \cap \mathbb{Z}[\mathbb{i}]^2$ and $\mathcal{S}(\mathbb{Q}) = \mathcal{S} \cap \mathbb{Q}[\mathbb{i}]^2$, respectively. We will usually denote a point of the Siegel model by $h = (u,v)$ and an integer point by $\gamma = (\alpha, \beta)$. Given a generic point $h \in \mathcal{S}$, there exists a unique integer point $[h] \in \mathcal{S}(\mathbb{Z})$ that minimizes $\|[h]^{-1} * h\|$. We think of $[h]^{-1} * h$ as the fractional part of h.

Fix the fundamental domain $K_D = \{h : [h] = (0,0)\}$ for $\mathcal{S}(\mathbb{Z}) \subset \mathcal{S}$ (see Figure 3).

Definition 1.1. The *continued fraction digits* $CF(h) = \{\gamma_i\}$ and *forward iterates* $\{h_i\}$ of a point $h \in \mathcal{S}$, with respect to K_D, are defined inductively by

$$\gamma_0 = [h], \quad h_0 = \gamma_0^{-1} * h, \qquad \gamma_{i+1} = [\iota(h_i)], \quad h_{i+1} = \gamma_{i+1}^{-1} * \iota(h_i).$$

Note that $\iota(0)$ is undefined. Thus, the process may terminate after finitely many steps. In Theorem 3.10 we will characterize points for which this happens, and then focus on points with infinitely many digits. We will also generally assume that $\gamma_0 = 0$ unless otherwise specified.

The Gauss map $T : K_D \to K_D$ is given explicitly by

$$Th = \begin{cases} (0,0) & \text{if } h = (0,0), \\ [\iota h]^{-1} * \iota h & \text{if } h \neq (0,0). \end{cases}$$

By construction, $CF(Th)$ is a forward shift of the sequence of digits given by $CF(h)$.

Definition 1.2. Let $\{\gamma_i\}$ be a sequence of elements of $\mathcal{S}(\mathbb{Z})$. For a finite sequence, define the associated continued fraction

$$\mathbb{K}\{\gamma_i\} = \mathbb{K}\{\gamma_i\}_{i=0}^{n} := \gamma_0 \iota \gamma_1 \iota \cdots \iota \gamma_n,$$

suppressing product notation and parentheses. It is clear that if $CF(h)$ is finite, then $\mathbb{K}CF(h) = h$. For an infinite sequence, we write

$$\mathbb{K}\{\gamma_i\} = \mathbb{K}\{\gamma_i\}_{i=0}^{\infty} := \lim_{n\to\infty} \mathbb{K}\{\gamma_i\}_{i=0}^{n},$$

provided the limit exists.

Our main result is to show that $\mathbb{K}$ and CF define a valid notion of a continued fraction expansion for a point in $\mathcal{S}$.

Theorem 1.3.

(1) *Let $\{\gamma_i\}$ be a sequence of elements of $\mathcal{S}(\mathbb{Z})$ satisfying $\|\gamma_i\| > 2+\epsilon$ for some $\epsilon > 0$ and each i. Then $\mathbb{K}\{\gamma_i\}$ exists and is unique regardless of whether $\{\gamma_i\}$ is finite or infinite* (*Theorem* 3.7).
(2) *A point $h \in \mathcal{S}$ satisfies $h = \mathbb{K}\{\gamma_i\}_{i=0}^{n}$ for a finite sequence $\{\gamma_i\}$ of elements of $\mathcal{S}(\mathbb{Z})$ if and only if $h \in \mathcal{S}(\mathbb{Q})$* (*Theorem* 3.10).
(3) *Every point in $\mathcal{S}$ has a continued fraction expansion. In fact, for all $h \in \mathcal{S}$, the limit $\mathbb{K}CF(h)$ is unique and equal to h* (*Theorem* 3.21).

Throughout §3, we obtain variants of classical continued fraction results. We show a relationship between the denominator of a rational point and the length of its continued fraction expansion (Theorem 3.11). We find a recursive formula for the approximants $\mathbb{K}\{\gamma_i\}_{i=1}^{n}$ (Theorem 3.18), and show that the distance between $h \in \mathbb{H}$ and its approximants $\mathbb{K}\{\gamma_i\}_{i=1}^{n}$ satisfies a variant of a classical relation (Theorem 3.23). We prove that the convergence of $\mathbb{K}CF(h)$ is uniform on a full-measure set in Theorem 3.25.

We can say very little about which strings of digits are admissible (i.e., which strings can appear in $CF(h)$ for some h). Theorem 3.7 implies that all strings with sufficiently large digits are admissible for nearest-integer continued fractions, but what strings having small digits are admissible is currently unknown.

2. The Heisenberg group. We will think of the Heisenberg group in three different ways. For geometric purposes, including illustration and discussion of measures, we will identify it with $\mathbb{R}^3$ (with the appropriate group structure and geometry). For the majority of the paper, however, we will be concerned with the representation of the Heisenberg group as a subgroup of the unitary matrices $U(2,1)$ or as a subset $\mathcal{S}$ of $\mathbb{C}^2$ as in the introduction. This is in direct analogy with thinking of the real numbers as elements of $SL(2,\mathbb{R})$ or as the real axis within $\mathbb{C}^1$. We now discuss these models, and then record some information on discrete subgroups of the Heisenberg group and their fundamental domains; see also [3, 4, 8].

We emphasize that the topological and measure-theoretic notions we consider do not (qualitatively) depend on the model we choose, nor on the metric. In particular, convergence can be shown using the intrinsic gauge metric, or using metrics intrinsic to the model, such as the Euclidean metrics on $\mathbb{R}^3$ or $\mathbb{C}^2$.

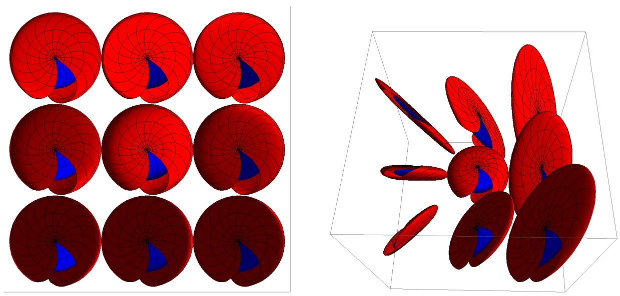

Fig. 1. Two views of nested spheres in $\mathbb{H}$, centered at $(i,j,0)$ with $i,j \in \{-1,0,1\}$, related to each other by left translation by elements of $\mathbb{H}(\mathbb{Z})$. In the top view (left), the spheres look identical. A side view (right) shows an additional shear in the t coordinate.

2.1. Geometric model. We define $\mathbb{H}$ as the space $\mathbb{R}^3$ with group law

$$(x,y,t) * (x',y',t') = (x+x', y+y', t+t'+2(xy'-yx')).$$

Combining coordinates by taking $z = x + \mathbb{i}y$, $\mathbb{H}$ becomes $\mathbb{C} \times \mathbb{R}$ with group law

$$(z,t) * (z',t') = (z+z', t+t'+2\,\mathrm{Im}(\overline{z}z')).$$

We will think of these as the same model, and use it primarily when geometry or visualization are concerned. There are several standard (topologically equivalent) metrics on $\mathbb{H}$; we will work with the *gauge metric*. The gauge $\|\cdot\|$ and distance d are defined by:

$$\|(z,t)\| = \sqrt[4]{|z|^4+t^2}, \qquad d(h,k) = \|h^{-1} * k\|,\ h,k \in \mathbb{H}.$$

There are three basic types of transformations we are interested in:

(1) left translations $h \mapsto k * h$ for $k \in \mathbb{H}$,
(2) rotations $(z,t) \mapsto (e^{\mathbb{i}\theta} z, t)$ for $\theta \in \mathbb{R}$,
(3) the Korányi inversion $\iota : \mathbb{H} \setminus \{0\} \to \mathbb{H} \setminus \{0\}$ given by

$$\iota(z,t) = \left(\frac{-z}{|z|^2 + \mathbb{i}t}, \frac{-t}{|z|^4 + t^2} \right).$$

Translations and rotations do not distort distances or volume (that is, the Lebesgue measure λ on $\mathbb{R}^3$). The Korányi inversion is a conformal map (with respect to the gauge metric, see [7]) with the following important property.

LEMMA 2.1 (see [2, p. 19]). *Let $h, k \in \mathbb{H} \setminus \{0\}$. Then*

$$d(\iota h, \iota k) = \frac{d(h,k)}{\|h\| \, \|k\|}.$$

In particular, one has $\|\iota h\| = \|h\|^{-1}$, so that the inside and outside of the unit ball are interchanged. Note that individual points on the unit sphere are not fixed.

REMARK 2.2. We will show in Lemma 2.9 that ι has a particularly simple form in the unitary model.

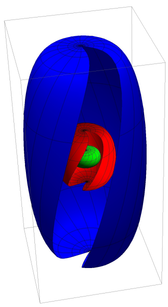

Fig. 2. Spheres in $\mathbb{H}$ centered at the origin, with radius $2, 1, 1/2$, with sectors removed to display nested spheres. The spheres are parametrized by applying ι to a plane; the radial lines of the plane provide the characteristic foliation on the spheres.

2.2. Real nilpotent model. It is common to describe $\mathbb{H}$ as the group of nilpotent upper-triangular 3-by-3 real matrices. Our definition is related to this *real nilpotent model* via the Lie group isomorphism

$$(x,y,t) \mapsto \begin{pmatrix} 1 & x & t/4 + xy/2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

We will not use the real nilpotent model, although our results can be rephrased for it. Note that under (2.1), $\mathbb{H}(\mathbb{Z})$ is *not* identified with matrices with integer entries.

2.3. Unitary representation. For calculation purposes, we will use the (*Siegel*) *unitary representation* of $\mathbb{H}$. Namely, we will embed $\mathbb{H}$ in $GL(3,\mathbb{C})$ via the homomorphism

$$\mathbb{U} : (z,t) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ z(1+\mathbb{i}) & 1 & 0 \\ |z|^2 + t\mathbb{i} & \overline{z}(1-\mathbb{i}) & 1 \end{pmatrix}. \tag{2.2}$$

Remark 2.3. In literature, one sees a factor of $\sqrt{2}$ rather than $1+\mathbb{i}$ or $1-\mathbb{i}$ in the embedding. The latter is more convenient for our purposes.

Let $\mathbb{J}$ be the Hermitian inner product given by

$$\mathbb{J}((z_0,z_1,z_2),(w_0,w_1,w_2)) = (\overline{z_0}\ \overline{z_1}\ \overline{z_2}) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}.$$

In particular, we record

$$|(z_0,z_1,z_2)|_{\mathbb{J}}^2 = \mathbb{J}((z_0,z_1,z_2),(z_0,z_1,z_2)) = |z_1|^2 - 2\operatorname{Re}(\overline{z_0}z_2). \tag{2.3}$$

We will refer to a vector of norm 0 as a *null vector.*

Abusing notation, we will also use $\mathbb{J}$ to denote the skew-diagonal matrix above. Note that $\mathbb{J}$ has signature $(2,1)$: it has two positive and one negative eigenvalue.

The unitary group $U(2,1) \subset GL(3,\mathbb{C})$ is the set of matrices $M \in GL(3,\mathbb{C})$ satisfying $\mathbb{J}(M\vec{z}, M\vec{w}) = \mathbb{J}(\vec{z},\vec{w})$ for all $\vec{z},\vec{w} \in \mathbb{C}^3$. Equivalently, M satisfies $M^\dagger \mathbb{J} M = \mathbb{J}$, where $\dagger$ denotes the conjugate transpose. We will additionally distinguish the subgroups $SU(2,1)$ and $S^{\pm}U(2,1)$ consisting of matrices M in $U(2,1)$ satisfying, respectively, $\det M = 1$ or $\det M = \pm 1$. We note that $\mathbb{U}(\mathbb{H}) \subset SU(2,1)$.

2.4. Siegel model. The Siegel model provides a geometric view of the unitary representation and a simpler formula for the Korányi inversion. We will in fact define two closely related models, the *planar Siegel model* that views a point $h \in \mathbb{H}$ as a vector $(u,v) \in \mathbb{C}^2$, and the *projective Siegel model* that views h as a point in complex projective space with homogeneous coordinates $(1:u:v)$. We will denote both models by $\mathcal{S}$.

We first identify a point $h \in \mathbb{H}$ with geometric coordinates (z,t) with the vector

$$(1, z(1+\mathbb{i}), |z|^2 + \mathbb{i}t) \in \mathbb{C}^3. \tag{2.4}$$

Note that this is exactly the image of the vector $(1,0,0)$ under the unitary transformation $\mathbb{U}(z,t)$. We will say that h has *planar Siegel coordinates*

$$(z(1+\mathbb{i}), |z|^2 + \mathbb{i}t) \in \mathbb{C}^2. \tag{2.5}$$

The *planar Siegel model* of $\mathbb{H}$ is the set of points in $\mathbb{C}^2$ of the form (2.5).

Sometimes, a unitary transformation will take $(1, z(1+\mathbb{i}), |z|^2 + \mathbb{i}t)$ to a point that is not of the same form, but can be rescaled to be such. It will therefore be useful to think of vectors up to rescaling, that is, as elements of the complex projective space $\mathbb{CP}^2$.

Recall that the complex projective plane $\mathbb{CP}^2$ is the projectivization of $\mathbb{C}^3$, i.e., the set of non-zero vectors up to rescaling by a non-zero complex number. A point in $\mathbb{CP}^2$ has *homogeneous coordinates* $(z_0 : z_1 : z_2)$, well-defined up to rescaling.

We can now define the *projective Siegel model* of $\mathbb{H}$ as the set of points in $\mathbb{CP}^2$ with homogeneous coordinates $(1 : z(1+\mathbb{i}) : |z|^2 + \mathbb{i}t)$.

Abusing notation, we will denote both Siegel models by $\mathcal{S}$, with the identification $(u, v) \leftrightarrow (1 : u : v)$. We have the following simple characterization of points in $\mathcal{S}$.

Lemma 2.4. *Let $(z_0 : z_1 : z_2) \in \mathbb{CP}^2$ be a null point, that is, $\|(z_0, z_1, z_2)\|_{\mathbb{J}}^2 = 0$. Then either $(z_0 : z_1 : z_2) \in \mathcal{S}$ or $(z_0 : z_1 : z_2) \cong (0 : 0 : 1)$.*

We denote the closure of $\mathcal{S}$ in $\mathbb{CP}^2$ by $\overline{\mathcal{S}} = \mathcal{S} \cup \{(0 : 0 : 1)\}$.

Remark 2.5. The region $\{(z_0 : z_1 : z_2) \in \mathbb{CP}^2 : \|(z_0, z_1, z_2)\|_{\mathbb{J}}^2 < 0\}$ bounded by $\overline{\mathcal{S}}$ is the *Siegel domain*. Complex hyperbolic space is defined on this region and has strong connections with the Heisenberg group (see e.g. [2, 4, 7, 8]). In particular, we hope to discuss the relation of Heisenberg continued fractions to geodesic coding in complex hyperbolic space in an upcoming paper, following [14].

Note that the gauge norm is easy to write in the Siegel model:

Lemma 2.6. *Let $(u, v) \in \mathcal{S}$. Then the gauge norm of (u, v) is $\|(u, v)\| = |v|^{1/2}$.*

Proof. An element of $\mathcal{S}$ has the form $(u, v) = (z(1+\mathbb{i}), |z|^2 + t\mathbb{i})$ for some $(z, t) \in \mathbb{H}$. The gauge norm of (z, t) is given by $\|(z, t)\| = \sqrt[4]{|z|^4 + t^2} = |v|^{1/2}$. ∎

The gauge distance is defined as $d(h, k) = \|h^{-1}k\|$. With this in mind, we show:

Lemma 2.7. *In the planar Siegel model, we have*

$$(u_1, v_1)^{-1} * (u_2, v_2) = (u_2 - u_1, \overline{v_1} - \overline{u_1}u_2 + v_2).$$

Proof. We have associated to $(u_1, v_1)^{-1}$ the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -u_1 & 1 & 0 \\ \overline{v_1} & -\overline{u_1} & 1 \end{pmatrix}.$$

Applying this matrix to the point $(1, u_2, v_2)$, we get the vector

$$(1, u_2 - u_1, \overline{v_1} - \overline{u_1}u_2 + v_2).$$

Taking the last two coordinates yields the desired formula. ■

We now study the action of $S^{\pm}U(2,1)$ matrices on the Heisenberg group in the Siegel models. General linear matrices act on $\mathbb{CP}^2$ by acting on the homogeneous coordinates. Since we have $\mathbb{C}^2 \hookrightarrow \mathbb{CP}^2$ by taking $(u,v) \mapsto (1 : u : v)$, we also obtain an action on $\mathbb{C}^2$.

Lemma 2.8. *Let $M = (a_{i,j}) \in GL(3,\mathbb{C})$ and $(u,v) \in \mathbb{C}^2 \hookrightarrow \mathbb{CP}^2$. Then M acts on (u,v) as*

$$M(u,v) = \left(\frac{a_{2,1} + a_{2,2}u + a_{2,3}v}{a_{1,1} + a_{1,2}u + a_{1,3}v}, \frac{a_{3,1} + a_{3,2}u + a_{3,3}v}{a_{1,1} + a_{1,2}u + a_{1,3}v}\right).$$

Proof. The point (u,v) corresponds to a point in $\mathbb{CP}^2$ with homogeneous coordinates $(1 : u : v)$. We then have

$$M\begin{pmatrix} 1 \\ u \\ v \end{pmatrix} = \begin{pmatrix} a_{1,1} + a_{1,2}u + a_{1,3}v \\ a_{2,1} + a_{2,2}u + a_{2,3}v \\ a_{3,1} + a_{3,2}u + a_{3,3}v \end{pmatrix}.$$

To view $M(1 : u : v)$ as a point in $\mathbb{C}^2$, we renormalize so that the first coordinate is 1, and take the remaining two coordinates. ■

Elements of $GL(3,\mathbb{C})$ do not necessarily preserve the set $\overline{\mathcal{S}}$, but the unitary matrices $U(2,1)$ preserve $\mathbb{J}$ and therefore $\overline{\mathcal{S}}$. In particular, elements of $\mathbb{U}(\mathbb{H})$ act transitively on $\mathcal{S}$ while fixing the point $(0 : 0 : 1)$. We also use the symbol $\mathbb{U}(\iota)$ to denote the matrix

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Lemma 2.9. *$\mathbb{U}(\iota)$ acts on $\mathbb{H}$ by the Korányi inversion ι.*

Proof. We compute, for a point in $\mathbb{H}$ with geometric coordinates (z,t) and projective Siegel coordinates $(1 : z(1+\mathbb{i}) : |z|^2 + t\mathbb{i})$:

$$\begin{aligned}
\mathbb{U}(\iota)(1 : z(1+\mathbb{i}) : |z|^2 + t\mathbb{i}) &= (|z|^2 + t\mathbb{i} : -z(1+\mathbb{i}) : 1) \\
&= \left(1 : \frac{-z}{|z|^2 + t\mathbb{i}}(1+\mathbb{i}) : \frac{1}{|z|^2 + t\mathbb{i}}\right) = \left(1 : \frac{-z}{|z|^2 + t\mathbb{i}}(1+\mathbb{i}) : \frac{|z|^2 - t\mathbb{i}}{|z|^4 + t^2}\right) \\
&= \left(1 : \frac{-z}{|z|^2 + t\mathbb{i}}(1+\mathbb{i}) : \left|\frac{-z}{|z|^2 + t\mathbb{i}}\right|^2 + \frac{-t}{|z|^4 + t^2}\mathbb{i}\right).
\end{aligned}$$

We thus see that under $\mathbb{U}(\iota)$, the geometric coordinates (z,t) are mapped to $\left(\frac{-z}{|z|^2 + t\mathbb{i}}, \frac{-t}{\|(z,t)\|^4}\right)$, as desired. ■

2.5. Lattices and fundamental domains. Let $\mathbb{H}(\mathbb{Z})$ and $\mathcal{S}(\mathbb{Z})$ be the set of Heisenberg points with integer coordinates in the appropriate model. Likewise, we will denote by $\mathbb{H}(\mathbb{Q})$ and $\mathcal{S}(\mathbb{Q})$ the set of points in $\mathcal{S}$ with rational coordinates. In the geometric model $\mathbb{H} = \mathbb{C} \times \mathbb{R}$, we have $\mathbb{H}(\mathbb{Z}) = \mathbb{Z}[\mathbb{i}] \times \mathbb{Z}$. In the Siegel model, $\mathcal{S}(\mathbb{Z})$ is the set of points $(u, v) \in \mathcal{S}$ such that $u \in (1+\mathbb{i})\mathbb{Z}[\mathbb{i}]$, $v \in \mathbb{Z}[\mathbb{i}]$. In the unitary model, we have $\mathbb{U}(\mathbb{H}(\mathbb{Z})) \subset SU(2,1;\mathbb{Z}[\mathbb{i}])$, where the latter denotes the subset of $SU(2,1)$ with Gaussian integer coefficients, and is known as the *Picard modular group* (or the *Gauss–Picard modular group*).

We are now interested in the structure and geometry of $\mathbb{H}(\mathbb{Z})$. We record its generators in the geometric model:

Lemma 2.10. *The group* $\mathbb{H}(\mathbb{Z})$ *is generated by the elements* $(1, 0)$*,* $(\mathbb{i}, 0)$*, and* $(0, 1)$*.*

The groups $\mathbb{H}[\mathbb{Z}]$ and $SU(2,1;\mathbb{Z}[\mathbb{i}])$ are closely linked:

Theorem 2.11 (Falbel–Francics–Lax–Parker [3], see also [17]). *The group* $SU(2,1;\mathbb{Z}[\mathbb{i}])$ *is generated by the matrices* $\mathbb{U}(1, 0)$*,* $\mathbb{U}(0, 1)$*,* $-\mathbb{U}(\iota)$*, and the matrix*

$$\begin{pmatrix} \mathbb{i} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mathbb{i} \end{pmatrix}$$

corresponding to the mapping $(z, t) \mapsto (-\mathbb{i}z, t)$*.*

We now discuss fundamental domains for $\mathbb{H}(\mathbb{Z})$. Our definition will differ slightly from the standard one. We require K, our fundamental domain, to consist of an open subset of $\mathbb{H}$ and some measurable subset of its boundary (which is not necessarily piecewise smooth) such that $\bigcup\{\gamma * K : \gamma \in \mathbb{H}(\mathbb{Z})\} = \mathbb{H}$ and $K \cap (\gamma * K) \neq \emptyset$ implies $\gamma = 0$. We then have:

Lemma 2.12. *Let* K *be a fundamental domain for* $\mathbb{H}(\mathbb{Z})$*. Then the map* $[p]_K : \mathcal{S} \to \mathbb{H}(\mathbb{Z})$ *mapping all points of* γK *to* γ *is well-defined.*

The next lemma is an immediate consequence of the definitions:

Lemma 2.13. *The following regions are fundamental domains for* $\mathbb{H}(\mathbb{Z})$*:*

- *The unit cube* $K_C = [-1/2, 1/2) \times [-1/2, 1/2) \times [-1/2, 1/2)$*.*
- *The Dirichlet domain* $K_D = \{h \in \mathbb{H} : d(0, h) \leq d(\gamma, h) \text{ for all } \gamma \in \mathcal{S}(\mathbb{Z})\}$*, with a choice of excluded boundary points* (*see Figure* 3)*.*

Denote the unit sphere in $\mathbb{H}$ by S. For a subset $A \subset \mathbb{H}$, let $\mathrm{rad}(A)$ denote the supremum of the norms of the points of A, and let $\lambda(A)$ denote its Lebesgue measure (in the geometric model).

Lemma 2.14. *Every fundamental domain* K *for* $\mathbb{H}(\mathbb{Z})$ *satisfies* $\lambda(K) = 1$*. Furthermore, the domains* K_C *and* K_D *satisfy* $\mathrm{rad}(K_C) = \mathrm{rad}(K_D) = \sqrt[4]{1/2}$*.*

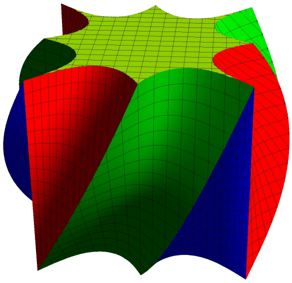

Fig. 3. The Dirichlet domain for $\mathbb{H}(\mathbb{Z})$ centered at the origin

Proof. The radius of K_C is easy to compute because $\|\cdot\|$ behaves similarly to the Euclidean norm. As in the Euclidean case, the norm is maximized by each corner of the cube. We have $\|(1/2 + \mathbb{i}1/2, 1/2)\| = \sqrt[4]{1/2}$.

The radius of K_D seems difficult to compute directly, as the boundary of K_D is more complicated (see Figure 3). We will therefore argue indirectly by means of K_C. Let $h \in K_D$, and choose $g \in \mathbb{H}(\mathbb{Z})$ so that $g * h \in K_C$. We then have $\|g * h\| \le \mathrm{rad}(K_C) = \sqrt[4]{1/2}$. This implies that $d(g^{-1}, h) \le \sqrt[4]{1/2}$. Now, by definition of K_D, $d(0, h) \le d(g^{-1}, h) \le \sqrt[4]{1/2}$, so we must also have $\|h\| \le \sqrt[4]{1/2}$, hence $\mathrm{rad}(K_D) \le \sqrt[4]{1/2}$. To prove equality, one shows directly that the point $(1/2 + \mathbb{i}1/2, 1/2)$ is on the boundary of K_D.

For the volume computation, it is clear that $\lambda(K_C) = 1$. To compute $\lambda(K)$ for an arbitrary fundamental domain K, note that Lebesgue measure is preserved by left translation in the Heisenberg group (which acts by shears). Since K_C can be constructed by rearranging measurable pieces of K, the two fundamental domains must have the same volume. ∎

Remark 2.15. Note that we defined $U(2, 1; \mathbb{Z}[\mathbb{i}])$ with a particular Hermitian form $\mathbb{J}$ in mind. Different Hermitian forms $\mathbb{J}$ provide isomorphic Lie groups $U(2, 1)$, but the lattice $U(2, 1; \mathbb{Z}[\mathbb{i}])$ depends on the choice of the Hermitian form. If two forms are related by an integer change of coordinates, then the associated lattices are equivalent. If the change of coordinates is not integral, the lattices are not isomorphic as groups (even up to finite index); see [10, 11]. Nonetheless, in the literature one mostly sees mention of the Picard modular group defined by a Hermitian form equivalent to our $\mathbb{J}$.

3. Heisenberg continued fractions. Fix a fundamental domain K for the group $\mathbb{H}(\mathbb{Z})$ such that $\mathrm{rad}(K) < 1$ (e.g., K_C or K_D in Lemma 2.13). We shall be a bit loose with notation and consider K as being in the Siegel model $\mathcal{S}$ from here on. We begin by establishing some notation.

Definition 3.1. Given an arbitrary sequence $\{\gamma_i\}_{i=1}^n$ of non-zero digits in $\mathcal{S}(\mathbb{Z})$, we write the associated continued fraction as

$$\mathbb{K}\{\gamma_i\} = \mathbb{K}\{\gamma_i\}_{i=1}^{n} = \iota\gamma_1\iota\gamma_2\cdots\iota\gamma_n. \tag{3.1}$$

For an infinite sequence $\{\gamma_i\}_{i=1}^{\infty}$, we define

$$\mathbb{K}\{\gamma_i\} = \mathbb{K}\{\gamma_i\}_{i=1}^{\infty} := \lim_{n\to\infty} \mathbb{K}\{\gamma_i\}_{i=1}^{n},$$

if this limit exists.

The goal of this section is to show that the limit does exist in several important cases, and that the computation of $\mathbb{K}\{\gamma_i\}$ may be simplified by using a recursive algorithm.

Definition 3.2. We associate with K:

(1) A "nearest-integer" map $[\cdot] : \mathcal{S} \to \mathcal{S}(\mathbb{Z})$, characterized by

$$[h] = \gamma \quad \text{for each } \gamma \in \mathcal{S}(\mathbb{Z}) \text{ and } h \in \gamma K.$$

Note that $[\cdot]$ selects the nearest integer in the gauge metric exactly if K is the Dirichlet domain K_D.

(2) The *Gauss map* $T : K \setminus \{(0,0)\} \to K$ given by

$$Th = [\iota h]^{-1}\iota h.$$

Remark 3.3. Working with the geometric model, one sees that the intersection of K with any of the axes (t, x, or y) is preserved by the Gauss map T. If the intersection corresponds to the interval $[-1/2, 1/2)$ along the axis, then the action of T restricted to that intersection is essentially isomorphic to the nearest-integer Gauss map. The theory of continued fractions we develop likewise restricts to the classical nearest-integer continued fraction theory on this axis.

Furthermore, an intersection of $[-\alpha, 1-\alpha)$ provides a system essentially isomorphic to Nakada's α-continued fractions. However, the system does *not* restrict to complex continued fractions along the complex coordinate, because of the shearing component of Heisenberg translations.

Definition 3.4. Given a point $h \in K$, we have:

(1) the *forward iterates* $h_i := T^i h \in K$ for each i;
(2) the *continued fraction digits* $\gamma_i := [\iota h_{i-1}] \in \mathcal{S}(\mathbb{Z})$ for each i;
(3) the *rational approximants* $\mathbb{K}\{\gamma_i\}_{i=1}^{n} \in \mathcal{S}(\mathbb{Q})$ for each n.

Because T is defined on $K \setminus \{0\}$, the process of defining forward iterates, continued fraction digits, and rational approximants terminates if for some i we have $h_i = 0$. In Theorem 3.10 we will characterize the points h for which this happens.

More generally, for a point $h \in \mathcal{S}$ we can take $\gamma_0 = [h]$, $h_0 = \gamma_0^{-1}h$ and obtain the remaining digits $\{\gamma_i\}_{i=1}^{\infty}$ of $CF(h)$ from $h_0 \in K$ as before. However, our focus will be on points in K.

It is easy to see that, on finite sequences, $\mathbb{K}$ is the inverse operation to CF:

LEMMA 3.5. *For $h \in K$ with $CF(h)$ a finite sequence, we have $\mathbb{K}CF(h) = h$.*

REMARK 3.6. The operation $\mathbb{K}$ is defined without reference to a specific fundamental domain K. Thus, while we will show that $\mathbb{K}CF(h) = h$, we do not in general have $CF(\mathbb{K}\{\gamma_i\}) = \{\gamma_i\}$. Indeed, problems arise when the γ_i get too close to the unit sphere.

For example, let $K = K_C$, the unit cube, and let $\{\gamma_i\} = \{(a_1, b_1) = (1+\mathbb{i}, 1)\}$. We have

$$\mathbb{K}\{\gamma_i\} = \iota(1+\mathbb{i}, 1) = (-(1+\mathbb{i}), 1).$$

Attempting to reverse the process, we find that $(a_0, b_0) = [(-(1+\mathbb{i}), 1)] = (-(1+\mathbb{i}), 1)$, and $(-(1+\mathbb{i}), 1)^{-1} * (-(1+\mathbb{i}), 1) = (0, 0)$, therefore

$$CF(-(1+\mathbb{i}), 1) = \{(a_0, b_0) = (-(1+\mathbb{i}), 1)\}.$$

This non-uniqueness of continued fraction expansions is analogous to how in regular continued fractions we have, for example,

$$\frac{1}{5+\frac{1}{1}} = \frac{1}{6}.$$

3.1. Pringsheim-type theorem. The Pringsheim theorem for regular continued fractions guarantees convergence of a continued fraction whose digits are sufficiently large. A variant holds for the Heisenberg group:

THEOREM 3.7 (Pringsheim-type theorem). *Let $\{\gamma_i\}_{i=1}^{\infty}$ be a sequence of points in $\mathcal{S}$ such that for each i we have $\|\gamma_i\| > 2 + \epsilon$ for some $\epsilon > 0$. Then the limit $\mathbb{K}\{\gamma_i\}$ exists. Furthermore, if $\|\gamma_i\| > 2 + \sqrt[4]{1/2}$ for all i, then $CF(\mathbb{K}\{\gamma_i\}) = \{\gamma_i\}$ where the continued fraction expansion is taken with respect to the Dirichlet region K_D.*

Proof. Recall that left multiplication by any $\gamma \in \mathcal{S}$ is an isometry, and that ι satisfies the relation $d(\iota h, \iota k) = \frac{d(h,k)}{\|h\|\,\|k\|}$ for all $h, k \in \mathcal{S}$ (Lemma 2.1).

Let B be the unit ball in the Heisenberg group, including the boundary. Suppose that $\gamma \in \mathcal{S}$ with $\|\gamma\| > 2+\epsilon$. We claim that $\iota\gamma B \subset B$. Indeed, every point $h \in \gamma B$ satisfies $\|\gamma h\| > 2+\epsilon-\|h\| \geq 1+\epsilon$, so that $\|\iota\gamma h\| \leq (1+\epsilon)^{-1}$, and we conclude $\iota\gamma B \subset B$.

Now, for each n, we have (the identity element 0 being contained in B)

$$\mathbb{K}\{\gamma_i\}_{i=1}^{n} = \iota\gamma_1\iota\gamma_2\cdots\iota\gamma_n = \iota\gamma_1\iota\gamma_2\cdots\iota\gamma_n 0 \in \iota\gamma_1\iota\gamma_2\cdots\iota\gamma_n B.$$

These sets form a nested sequence:

$$\iota\gamma_1\iota\gamma_2\cdots\iota\gamma_n B = \iota\gamma_1\iota\gamma_2\cdots\iota\gamma_{n-1}(\iota\gamma_n B) \subset \iota\gamma_1\iota\gamma_2\cdots\iota\gamma_{n-1}B.$$

By the above calculation, the diameter of the cylinder set $\iota\gamma_1\iota\gamma_2\cdots\iota\gamma_n B$ is bounded above by $(1+\epsilon)^{-2n}$. Therefore the sequence of fractions $\mathbb{K}\{\gamma_i\}_{i=1}^n$ (as n varies) is a Cauchy sequence, and hence converges to some $\mathbb{K}\{\gamma_i\}$.

We thus see that $\mathbb{K}\{\gamma_i\}$ exists.

Now suppose $\|\gamma_i\| > 2 + \sqrt[4]{1/2}$. If we run through the proof with K_D, the Dirichlet region, in place of B, then by the triangle inequality, we can show that the sets $\iota\gamma_1\iota\gamma_2\cdots\iota\gamma_n K_D$ are nested. (The sets are in fact properly nested, so that $\mathbb{K}\{\gamma_i\}$ cannot escape to a set's boundary.) Thus the point $\mathbb{K}\{\gamma_i\}$ is contained in these nested sets. This is equivalent to the second assertion of the theorem. ∎

3.2. Rational points. We will now show that a point in $\mathcal{S}$ has rational coordinates if and only if it has a finite continued fraction expansion. Our proof is motivated by the work of Falbel–Francsics–Lax–Parker [3].

Recall that for a point $h \in K$ that is of interest to us, we write $h = (u, v) \in \mathbb{C}^2$ in the planar Siegel model. We also think of (u, v) as the element of $\mathbb{CP}^2$ with homogeneous coordinates $(1 : u : v)$. In other words, it is the vector $(1, u, v)$ considered up to multiplication by a non-zero complex number.

Definition 3.8. Given an element $\gamma \in \mathcal{S}(\mathbb{Z})$ with planar Siegel coordinates $(\alpha, \beta) \in (\mathbb{Z}[\mathbb{i}] \times \mathbb{Z}[\mathbb{i}]) \cap \mathcal{S}$, define

$$A_\gamma := \mathbb{U}(\iota)\mathbb{U}(\gamma) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \overline{\alpha} & 1 \end{pmatrix} = \begin{pmatrix} -\beta & -\overline{\alpha} & -1 \\ \alpha & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Lemma 3.9. *In the Siegel projective model, we have*

$$\mathbb{K}\{\gamma_i\}_{i=1}^n = A_{\gamma_1}\cdots A_{\gamma_n}(1:0:0).$$

Proof. Abstractly, we have the definition $\mathbb{K}\{\gamma_i\}_{i=1}^n = \iota\gamma_1\iota\cdots\iota\gamma_n$. Using the identity element $0 \in \mathcal{S}$, we may also write $\mathbb{K}\{\gamma_i\}_{i=1}^n = \iota\gamma_1\iota\cdots\iota\gamma_n 0$. In the projective Siegel model, 0 is interpreted as the point $(1:0:0) \in \mathbb{CP}^2$. The inversion ι and left multiplication by γ_i are, respectively, interpreted as the unitary matrices $\mathbb{U}(\iota)$ and $\mathbb{U}(\gamma_i)$. Thus, $\mathbb{K}\{\gamma_i\}_{i=1}^n = A_{\gamma_1}\cdots A_{\gamma_n}(1:0:0)$, as desired. ∎

We are now in a position to characterize rational Heisenberg points in terms of their continued fraction expansion.

Theorem 3.10. *Let $h \in \mathcal{S}$. Then $h \in \mathcal{S}(\mathbb{Q})$ if and only if $h = \mathbb{K}\{\gamma_i\}_{i=0}^n$ for some finite sequence $\{\gamma_i\}_{i=0}^n$.*

Proof. Suppose $h = \mathbb{K}\{\gamma_i\}_{i=0}^n$. Then from the definition of $\mathbb{K}$ and the fact that $\gamma_i \in \mathcal{S}(\mathbb{Z})$ it is clear that $h \in \mathcal{S}(\mathbb{Q})$.

Conversely, fix $K = K_D$ and assume by way of contradiction that there exists an element $h \in \mathcal{S}(\mathbb{Q})$ with an infinite continued fraction sequence $CF(h) = \{\gamma_i\}_{i=1}^{\infty}$. Without loss of generality, we may assume $h \in K$ (this corresponds to discarding the digit γ_0 of h).

The idea of the proof is to show that the forward iterates h_i of h can be written as fractions whose denominators decrease with i. Write, in planar Siegel coordinates,

$$h = \left(\frac{r}{q}, \frac{p}{q}\right),$$

with $q, r, p \in \mathbb{Z}[\mathbb{i}]$. Because $h \in K$, we deduce from Lemma 2.6 that $|p/q| \leq \operatorname{rad}(K)^2 < 1$.

Consider the first forward iterate $h_1 = Th = \gamma_1^{-1}\iota h$ as a vector in $\mathbb{C}^3$:

$$\begin{aligned}\begin{pmatrix} q^{(1)} \\ r^{(1)} \\ p^{(1)} \end{pmatrix} := A_{\gamma_1}^{-1}\begin{pmatrix} q \\ r \\ p \end{pmatrix} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & \alpha_1 \\ -1 & -\overline{\alpha_1} & -\overline{\beta_1} \end{pmatrix}\begin{pmatrix} q \\ r \\ p \end{pmatrix} \\ &= \begin{pmatrix} -p \\ r + \alpha_1 p \\ -q - \overline{\alpha_1} r - \overline{\beta_1} p \end{pmatrix}.\end{aligned}$$

Thus, h_1 is a rational point with planar Siegel coordinates $h_1 = (r^{(1)}/q^{(1)}, p^{(1)}/q^{(1)})$. Furthermore, we have $q^{(1)} = -p$, hence

$$\left|\frac{q^{(1)}}{q}\right| = \left|\frac{p}{q}\right| = \|h\|^2 < \operatorname{rad}(K)^2 < 1. \tag{3.2}$$

Repeating this procedure recursively, we get rational coordinates $h_i = (r^{(i)}/q^{(i)}, p^{(i)}/q^{(i)})$ for each forward iterate h_i, satisfying $|q^{(i)}| = |p^{(i-1)}|$. Since $h_i \in K$ for all i, we obtain, for each n,

$$|q^{(n)}| \leq |q|(\operatorname{rad}(K))^{2n} \tag{3.3}$$

For sufficiently large n, we conclude $|q^{(n)}| < 1$, which implies that $q^{(n)} = 0$; but this is only possible if $h_{n-1} = 0$ and $CF(h)$ is, in fact, finite. ∎

As a corollary to the proof of Theorem 3.10, we obtain

THEOREM 3.11 (Denominator growth theorem). *Fix a fundamental domain K for $\mathcal{S}(\mathbb{Z})$ with $\operatorname{rad}(K) < 1$. Let $h \in \mathcal{S}(\mathbb{Q})$, with $CF(h) = \{\gamma_i\}_{i=0}^{n}$ the continued fraction expansion associated to K. Suppose one can write h as a fraction with denominator $q \in \mathbb{Z}[\mathbb{i}]$. Then*

$$|q| \geq (\operatorname{rad}(K))^{-2n}.$$

Proof. The result follows directly from (3.3). ∎

REMARK 3.12. One may hope for a stronger statement, that for a sequence $\{\gamma_i\}_{i=1}^{\infty}$ of elements of $\mathcal{S}(\mathbb{Z})$, the norms of the denominators q_n of the partial fractions $\mathbb{K}\{\gamma_i\}_{i=1}^{n}$ are an increasing sequence. However, we are unable to prove this without assuming that $\|\gamma_i\| \geq 2$ for all i. Indeed, the corresponding statement is false for some variants of continued fractions (see [9]).

3.3. Recursive formula. We will now find a simple recursive formula for $\mathbb{K}\{\gamma_i\}_{i=1}^{\infty}$.

DEFINITION 3.13. Let $\{\gamma_i\}$ be a sequence of elements of $\mathcal{S}[\mathbb{Z}]$. Define

$$Q_n := A_{\gamma_1} \cdots A_{\gamma_n}, \qquad (q_n, r_n, p_n) := Q_n(1, 0, 0).$$

We note the following consequence of Lemma 3.9.

LEMMA 3.14. *In the above notation,* $\mathbb{K}\{\gamma_i\}_{i=1}^{n} = (r_n/q_n, p_n/q_n)$ *in the planar Siegel model.*

REMARK 3.15. It should be noted that Theorem 3.11 does not imply that $q_n \geq 2^{n/2}$. Recall from Remark 3.6 that if $CF(h) = \{\gamma_i\}_{i=0}^{\infty}$, we do not necessarily have $CF(\mathbb{K}\{\gamma_i\}_{i=0}^{n}) = \{\gamma_i\}_{i=0}^{n}$.

Lemma 3.14 states that the partial fraction $\mathbb{K}\{\gamma_i\}_{i=1}^{n}$ is encoded in the matrix Q_n. As in the case of regular continued fractions, Q_n stores additional information:

LEMMA 3.16. *In the above notation, the matrices* Q_n *have the form*

$$Q_n = \begin{pmatrix} q_n & \mathfrak{q}_n & -q_{n-1} \\ r_n & \mathfrak{r}_n & -r_{n-1} \\ p_n & \mathfrak{p}_n & -p_{n-1} \end{pmatrix},$$

where the elements $\mathfrak{q}_n, \mathfrak{r}_n, \mathfrak{p}_n$ *are given by:*

$$\begin{aligned} \mathfrak{q}_n &= (-1)^n\, \overline{r_n q_{n-1} - q_n r_{n-1}}, \\ \mathfrak{r}_n &= (-1)^n\, \overline{p_n q_{n-1} - q_n p_{n-1}}, \\ \mathfrak{p}_n &= (-1)^n\, \overline{p_n r_{n-1} - r_n p_{n-1}}. \end{aligned}$$

Moreover, the matrix Q_n *has determinant* $(-1)^n$.

Proof. The first column of Q_n is as stated by the definition of the vector (q_n, r_n, p_n). The third column follows from the identity $Q_n = Q_{n-1}A_{\gamma_n}$. The value of the determinant follows from the fact that each A_{γ_i} has determinant -1. Finally, the second column follows from comparing the middle rows of the relation $Q_n^{\dagger}\mathbb{J} = \mathbb{J}Q_n^{-1}$. ∎

We record the following for later use:

LEMMA 3.17. *The identity $Q_n^\dagger \mathbb{J} = \mathbb{J} Q_n^{-1}$ is equivalent to*

$$\overline{\begin{pmatrix} -p_n & r_n & -q_n \\ -\mathfrak{p}_n & \mathfrak{r}_n & -\mathfrak{q}_n \\ p_{n-1} & -r_{n-1} & q_{n-1} \end{pmatrix}}$$

$$= (-1)^n \begin{pmatrix} p_n\mathfrak{r}_n - \mathfrak{p}_n r_n & \mathfrak{p}_n q_n - \mathfrak{q}_n p_n & r_n \mathfrak{q}_n - \mathfrak{r}_n q_n \\ p_{n-1} r_n - p_n r_{n-1} & p_n q_{n-1} - p_{n-1} q_n & r_{n-1} q_n - r_n q_{n-1} \\ p_{n-1}\mathfrak{r}_n - \mathfrak{p}_n r_{n-1} & \mathfrak{p}_n q_{n-1} - p_{n-1}\mathfrak{q}_n & r_{n-1}\mathfrak{q}_n - \mathfrak{r}_n q_{n-1} \end{pmatrix}.$$

We can now obtain a recursive form for the partial fractions $\mathbb{K}\{\gamma_i\}_{i=1}^n$.

THEOREM 3.18. *Let $\{\gamma_i\}_{i=1}^\infty$ be a sequence of elements of $\mathcal{S}(\mathbb{Z})$ represented in the planar Siegel model by the vectors $\{(\alpha_i, \beta_i)\}_{i=1}^\infty$. Let $(q_{-1}, p_{-1}, r_{-1}) = (0, 0, 1)$ and $(q_0, p_0, r_0) = (1, 0, 0)$. Define, recursively, for $n \geq 0$,*

$$\begin{pmatrix} q_{n+1} \\ r_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} q_n & (-1)^n \overline{r_n q_{n-1} - q_n r_{n-1}} & -q_{n-1} \\ r_n & (-1)^n \overline{p_n q_{n-1} - q_n p_{n-1}} & -r_{n-1} \\ p_n & (-1)^n \overline{p_n r_{n-1} - r_n p_{n-1}} & -p_{n-1} \end{pmatrix} \begin{pmatrix} -\beta_{n+1} \\ \alpha_{n+1} \\ -1 \end{pmatrix}.$$

Then for each n we have, in the planar Siegel model,

$$\mathbb{K}\{\gamma_i\}_{i=1}^n = \left(\frac{r_n}{q_n}, \frac{p_n}{q_n}\right).$$

Proof. Earlier in the section, we defined matrices A_{γ_i} (which append the digit γ_i to a continued fraction) and $Q_n = A_{\gamma_1} \cdots A_{\gamma_n}$. We set $(q_n, r_n, p_n) = Q_n(1, 0, 0)$. Now we claim that this agrees with the definition in the statement of the theorem. Lemma 3.14 will then show $\mathbb{K}\{\gamma_i\}_{i=1}^n = (r_n/q_n, p_n/q_n)$.

If we take Q_0 to be the identity matrix, the following computation provides the equivalence (see the definition of $A_{\gamma_{n+1}}$ and Lemma 3.16 for the form of the two matrices):

$$\begin{pmatrix} q_{n+1} \\ r_{n+1} \\ p_{n+1} \end{pmatrix} = Q_{n+1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = Q_n A_{\gamma_{n+1}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} q_n & \mathfrak{q}_n & -q_{n-1} \\ r_n & \mathfrak{p}_n & -r_{n-1} \\ p_n & \mathfrak{r}_n & -p_{n-1} \end{pmatrix} \begin{pmatrix} -\beta_{n+1} & -\overline{\alpha}_{n+1} & -1 \\ \alpha_{n+1} & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} q_n & \mathfrak{q}_n & -q_{n-1} \\ r_n & \mathfrak{p}_n & -r_{n-1} \\ p_n & \mathfrak{r}_n & -p_{n-1} \end{pmatrix} \begin{pmatrix} -\beta_{n+1} \\ \alpha_{n+1} \\ -1 \end{pmatrix}.$$

Rewriting $\mathfrak{q}_n, \mathfrak{r}_n, \mathfrak{p}_n$ in terms of the other entries in Q_n completes the proof. ∎

3.4. Continued fraction representation theorem. We are now ready to prove the convergence of continued fraction expansions. In fact, we obtain a variation on the strong convergence property, which for regular continued fractions says that not only do the convergents p_n/q_n converge to the original point x, but also $q_n x - p_n$ converges to 0. While for our new continued fractions we do not obtain strong convergence in the sense of Schweiger [12], our convergence estimate is obtained via a similar method to strong convergence for regular continued fractions.

We also note that we obtain such an explicit convergence estimate by exploiting a special form for Q_n^{-1} that follows from the identity $M^\dagger \mathbb{J} M = \mathbb{J}$ that defines $U(2,1)$. Other continued fraction theories are complicated by the lack of a simple form for Q_n^{-1}.

Before we can prove convergence, we need to show that q_n will never equal 0. We do so in two steps.

LEMMA 3.19. *We have*

$$\begin{pmatrix} q_n + \mathfrak{q}_n u_n - q_{n-1} v_n \\ r_n + \mathfrak{r}_n u_n - r_{n-1} v_n \\ p_n + \mathfrak{p}_n u_n - p_{n-1} v_n \end{pmatrix} = (-1)^n \begin{pmatrix} \dfrac{1}{v v_1 \cdots v_{n-1}} \\ \dfrac{u}{v v_1 \cdots v_{n-1}} \\ \dfrac{1}{v_1 \cdots v_{n-2}} \end{pmatrix}. \tag{3.4}$$

Proof. By Lemma 3.16, the vector on the left-hand side of (3.4) equals

$$Q_n(1, u_n, v_n) = A_{\gamma_1} \cdots A_{\gamma_n}(1, u_n, v_n).$$

Recall that the forward iterates of h are given by

$$h_i = T^i h = A_{\gamma_i}^{-1} \cdots A_{\gamma_1}^{-1} h,$$

and have planar Siegel coordinates (u_i, v_i), corresponding to the points $(1 : u_i : v_i) \in \mathbb{CP}^2$.

More generally, we have $A_{\gamma_i} \cdots A_{\gamma_n}(1 : u_n : v_n) = h_i$. Write $A_{\gamma_n}(1, u_n, v_n) =: (a, b, c)$. Since A_{γ_n} has the form (see Definition 3.8)

$$\begin{pmatrix} -\beta & -\overline{\alpha} & -1 \\ \alpha & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

we find that $c = -1$. Since $(b/a, c/a) = (u_{n-1}, v_{n-1})$, we conclude

$$A_{\gamma_n}(1, u_n, v_n) = \left(-\frac{1}{v_{n-1}}, -\frac{u_{n-1}}{v_{n-1}}, -1 \right).$$

Continuing in the same fashion we deduce that

$$A_{\gamma_{n-1}}A_{\gamma_n}(1,u_n,v_n) = A_{\gamma_{n-1}}\left(-\frac{1}{v_{n-1}}, -\frac{u_{n-1}}{v_{n-1}}, -1\right)$$
$$= \left(\frac{1}{v_{n-1}v_{n-2}}, \frac{u_{n-2}}{v_{n-1}v_{n-2}}, \frac{1}{v_{n-2}}\right).$$

After n iterations, the process yields the desired formula. ∎

Lemma 3.20. *For $n \geq 0$, the number q_n never equals* 0.

Proof. Assume, by way of contradiction, that $q_n = 0$. Then by Lemmas 3.16 and 3.17, we have $\mathfrak{q}_n = 0$ as well (r_n also must equal 0, but we will not use this fact). Since the matrix Q_n has determinant $(-1)^n$ and each entry is a Gaussian integer, q_{n-1} must have norm 1.

Therefore,

$$|q_n + \mathfrak{q}_n u_n - q_{n-1}v_n| = |v_n| < 1. \tag{3.5}$$

However, Lemma 3.19 implies

$$|q_n + \mathfrak{q}_n u_n - q_{n-1}v_n| = |vv_1v_2\ldots v_{n-1}|^{-1} > 1, \tag{3.6}$$

which is a contradiction. Therefore our assumption that $q_n = 0$ must be false. ∎

Now we can continue with the proof of convergence.

Theorem 3.21. *Let $h \in \mathcal{S}$ and let K be a fundamental domain for $\mathcal{S}(\mathbb{Z})$ with* $\mathrm{rad}(K) < 1$. *Then*

$$\mathbb{K}CF(h) = h.$$

Furthermore, if $CF(h) = \{\gamma_i\}$ is a sequence with at least n terms and q_n is the denominator of the nth rational approximant, then the rational approximants satisfy

$$d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) \leq \frac{\mathrm{rad}(K)^{n+1}}{|q_n|^{1/2}}$$

for both rational and irrational points in $\mathcal{S}$.

Proof. Recall from Lemma 3.14 that the associated rational approximants $\mathbb{K}\{\gamma_i\}_{i=1}^n$ have planar Siegel coordinates $(r_n/q_n, p_n/q_n)$, associated to the vector $(q_n, r_n, p_n) \in \mathbb{C}^3$. Recall also that the forward iterates $T^n h$ have planar Siegel coordinates (u_n, v_n), and we know $|v_n|^{1/2} \leq \mathrm{rad}(K) < 1$ from Lemma 2.6.

To prove the theorem, it suffices to show that

$$d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) = \frac{\prod_{i=0}^n |v_i|^{1/2}}{|q_n|^{1/2}}.$$

Indeed, by Lemmas 2.7 and 2.6, we have

$$\begin{aligned} d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) &= d\bigg(\Big(\frac{r_n}{q_n}, \frac{p_n}{q_n}\Big), h\bigg) \\ &= \bigg\| \bigg(u - \frac{r_n}{q_n}, v - \overline{\Big(\frac{r_n}{q_n}\Big)}u + \overline{\Big(\frac{p_n}{q_n}\Big)}\bigg) \bigg\| \\ &= \bigg| v - \overline{\Big(\frac{r_n}{q_n}\Big)}u + \overline{\Big(\frac{p_n}{q_n}\Big)} \bigg|^{1/2} = \frac{|\overline{q_n}v - \overline{r_n}u + \overline{p_n}|^{1/2}}{|q_n|^{1/2}}. \end{aligned}$$

We now view h as the vector $(1, u, v)$ and represent the operation T^n by the unitary matrix Q_n^{-1}. The vector

$$Q_n^{-1}\begin{pmatrix} 1 \\ u \\ v \end{pmatrix} = \overline{\begin{pmatrix} -p_{n-1} & r_{n-1} & -q_{n-1} \\ -\mathfrak{p}_n & \mathfrak{r}_n & -\mathfrak{q}_n \\ p_n & -r_n & q_n \end{pmatrix}}\begin{pmatrix} 1 \\ u \\ v \end{pmatrix}$$

is then a scalar multiple of $(1, u_n, v_n)$. In particular,

$$v_n = -\frac{\overline{p_n} - \overline{r_n}u + \overline{q_n}v}{\overline{p_{n-1}} - \overline{r_{n-1}}u + \overline{q_{n-1}}v}.$$

By multiplying these formulas for various indices we obtain

$$\begin{aligned} \prod_{i=1}^n v_i &= (-1)^n \prod_{i=1}^n \frac{\overline{p_i} - \overline{r_i}u + \overline{q_i}v}{\overline{p_{i-1}} - \overline{r_{i-1}}u + \overline{q_{i-1}}v} = (-1)^n \frac{\overline{p_n} - \overline{r_n}u + \overline{q_n}v}{\overline{p_0} - \overline{r_0}u + \overline{q_0}v} \\ &= (-1)^n \frac{\overline{p_n} - \overline{r_n}u + \overline{q_n}v}{v} \end{aligned}$$

This yields the interesting formula

$$\overline{p_n} - \overline{r_n}u + \overline{q_n}v = (-1)^n \prod_{i=0}^n v_i. \tag{3.7}$$

We then have

$$d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) = \frac{|\overline{q_n}v - \overline{r_n}u + \overline{p_n}|^{1/2}}{|q_n|^{1/2}} = \frac{|\prod_{i=0}^n v_i|^{1/2}}{|q_n|^{1/2}}.$$

Noting that $q_n \in \mathbb{Z}[i]$ and that $q_n \neq 0$ by Lemma 3.20 completes the proof. ∎

COROLLARY 3.22. *If $h \in K \setminus \mathcal{S}(\mathbb{Q})$, then $|q_n|$ tends to ∞.*

Proof. This follows almost immediately from the fact that there are only finitely many rational points $(r/q, p/q) \in \mathcal{S}$ that are written in lowest terms, are inside the unit sphere, and have $|q|$ bounded. Since the volume of ϵ-radius balls centered at these points shrinks to zero as ϵ shrinks to zero, no irrational point h can be arbitrarily well approximated by such points. ∎

As a corollary to the proof of Theorem 3.21 we obtain a new analog of the classical formula for regular continued fractions:

$$\left|x - \frac{p_n}{q_n}\right| = \frac{1}{q_n(q_{n+1} + q_n \cdot T^{n+1}x)}.$$

The left-hand side of this formula may be considered to be the distance between x and the point p_n/q_n. Recall that in Theorem 3.21 we showed that

$$d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) = \left|v - \overline{\left(\frac{r_n}{q_n}\right)}u + \overline{\left(\frac{p_n}{q_n}\right)}\right|^{1/2}. \tag{3.8}$$

THEOREM 3.23. *Let $h \in \mathcal{S}$ with continued fraction digits $CF(h) = \{\gamma_i\}$, associated to a fundamental domain K with* $\mathrm{rad}(K) < 1$, *and rational approximants $\mathbb{K}\{\gamma_i\}_{i=1}^n = (r_n/q_n, p_n/q_n)$. Then, in the notation of Lemma* 3.16,

$$v - \overline{\left(\frac{r_n}{q_n}\right)}u + \overline{\left(\frac{p_n}{q_n}\right)} = \frac{1}{\overline{q_n}(q_{n+1} + \mathfrak{q}_{n+1}u_{n+1} - q_n v_{n+1})}.$$

Proof. This follows immediately from (3.7) and Lemma 3.19. ∎

REMARK 3.24. There are two interesting approximation formulas we have given in this section:

$$v - \overline{\left(\frac{r_n}{q_n}\right)}u + \left(\frac{p_n}{\overline{q_n}}\right) = \frac{\prod_{i=0}^n v_i}{q_n} = \frac{1}{\overline{q_n}(q_{n+1} + \mathfrak{q}_{n+1}u_{n+1} - q_n v_{n+1})}.$$

Their analogs exist in other multi-dimensional continued fraction algorithms and look in some ways similar and in some ways very different. We will illustrate them with examples from the two-dimensional Jacobi–Perron formula, and for expediency, we only mention that here T is the Jacobi–Perron map, $x = (x_1, x_2)$ is a given point, $y = (y_1, y_2)$ is a point such that $x = T^s y$, and $(A_1^{(s)}/A_0^{(s)}, A_2^{(s)}/A_0^{(s)})$ is the nth convergent.

The first formula takes the form

$$\begin{aligned}&\left|x_j - \frac{A_j^{(s+3)}}{A_0^{(s+3)}}\right| \\ &\quad = \left|\frac{y_1(A_j^{(s+1)}A_0^{(s+3)} - A_0^{(s+1)}A_j^{(s+3)}) + y_2(A_j^{(s+2)}A_0^{(s+3)} - A_0^{(s+2)}A_j^{(s+3)})}{A_0^{(s+3)}(A_0^{(s+3)} + y_1 A_0^{(s+1)} + y_2 A_0^{(s+2)})}\right|.\end{aligned}$$

Similar formulas for other multi-dimensional continued fractions can be derived from Perron's identity (see [12, Section 15.2]). Unlike the formula we have given, however, Perron's identity approximates one coordinate of the point x at a time and the coefficients of the convergents appear in the numerator on the right-hand side.

On the other hand, if one considers the simplex formed by two successive convergents, it has area

$$V(x;s) = \frac{1}{2!A_0^{(s+1)}A_0^{(s+2)}(A_0^{(s+3)} + y_1A_0^{(s+1)} + y_2A_0^{(s+2)})},$$

which looks very similar to the formula we give above. The volume of similar simplices has been used as a way to analyze the approximation of the Jacobi–Perron algorithm [1, 13].

3.5. Uniform convergence. We continue with the assumptions of Theorem 3.21 and the notation of Lemma 3.16. The purpose of this section is to study the points $(\mathfrak{r}_n/\mathfrak{q}_n, \mathfrak{p}_n/\mathfrak{q}_n)$, and to understand when they converge (in the appropriate sense) to h. When this happens, we say that the continued fraction converges uniformly.

We will say a point $h = (u, v)$ is *degenerate* if $u_n = 0$ for some n, and *non-degenerate* otherwise. Degenerate points are named such since their dynamical properties eventually simplify to those of one-dimensional real continued fractions. We will prove the following theorem:

THEOREM 3.25. *Let $h \in K$. If h is non-degenerate, then $(\mathfrak{r}_n/\mathfrak{q}_n, \mathfrak{p}_n/\mathfrak{q}_n)$ converges to h (in the Euclidean sense as elements of $\mathbb{C}^2$) as $n \to \infty$. If h is degenerate, then the points $(\mathfrak{r}_n/\mathfrak{q}_n, \mathfrak{p}_n/\mathfrak{q}_n)$ are eventually constant.*

It should be emphasized that that *none* of the points $(\mathfrak{r}_n/\mathfrak{q}_n, \mathfrak{p}_n/\mathfrak{q}_n)$ are actually in $\mathcal{S}$, due to the following lemma.

LEMMA 3.26. *We have*

$$|\mathfrak{r}_n|^2 - 2\,\mathrm{Re}(\overline{\mathfrak{q}_n}\mathfrak{p}_n) = 1. \tag{3.9}$$

Proof. This can be easily found by using the fact from Lemma 3.16 that $\det Q_n = (-1)^n$. If we write down this determinant in terms of the matrix coefficients and then simplify, this gives the left-hand side of (3.9) times a factor of $(-1)^n$. ∎

The importance of non-degeneracy comes from the following lemma.

LEMMA 3.27. *If $u_n = 0$, then $u_{n+1} = 0$ and $\mathfrak{q}_{n+1} = \mathfrak{q}_n$.*
If h is non-degenerate, then $|\mathfrak{q}_n|$ tends to infinity as n grows.

Note that it is possible for $\mathfrak{q}_n$ to equal 0, but if h is non-degenerate then this can only occur finitely many times.

Proof of Lemma 3.27. If $u_n = 0$, then the corresponding point $T^n h$ has z-coordinate (in the geometric model) equal to 0. A quick calculation shows that $\gamma_{n+1} = [\iota T^n h]$ must have z-coordinate equal to 0, and therefore, so must $T^{n+1}h$. Converting this back to Siegel model coordinates shows that

$u_{n+1} = 0$, and since the matrix $A_{\gamma_{n+1}}$ takes the form

$$\begin{pmatrix} * & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

we have $\mathfrak{q}_{n+1} = \mathfrak{q}_n$.

Now suppose h is non-degenerate. In particular assume that if $n > N$, then $u_n \neq 0$. By modifying the argument that yielded (3.7), we can easily obtain

$$\overline{\mathfrak{p}_n} - \overline{\mathfrak{r}_n}u + \overline{\mathfrak{q}_n}v = (-1)^{n-1}\prod_{i=0}^{n-1} v_i \cdot u_n. \tag{3.10}$$

Since u_n is bounded and non-zero and each v_i has norm strictly between 0 and $\operatorname{rad}(K)$, we see that the right-hand side of (3.10) comes arbitrarily close to, but never equals, 0 as n increases.

Suppose, by way of contradiction, that there exist infinitely many $\{n_m\}_{m=1}^{\infty}$ such that $|\mathfrak{q}_{n_m}| < M$ for some M. Lemma 3.17 implies that

$$\overline{q_n} = (-1)^{n+1}(r_n\mathfrak{q}_n - \mathfrak{r}_n q_n), \qquad \overline{r_n} = (-1)^n(\mathfrak{p}_n q_n - \mathfrak{q}_n p_n)$$

and therefore

$$\mathfrak{r}_n = \frac{r_n}{q_n}\mathfrak{q}_n + (-1)^n\frac{\overline{q_n}}{q_n}, \tag{3.11}$$

$$\mathfrak{p}_n = \frac{p_n}{q_n}\mathfrak{q}_n + (-1)^n\frac{\overline{r_n}}{q_n}. \tag{3.12}$$

Since there are only finitely many values that $\mathfrak{q}_{n_m}$ can take, these equations imply that there are also only finitely many values that the tuple $(\mathfrak{q}_{n_m}, \mathfrak{r}_{n_m}, \mathfrak{p}_{n_m})$ can take; and hence only finitely many values for $\overline{\mathfrak{p}_{n_m}} - \overline{\mathfrak{r}_{n_m}}u + \overline{\mathfrak{q}_{n_m}}v$. This contradicts the fact that the left-hand side of (3.10) gets arbitrarily close to 0 without equaling it.

Hence $|q_n|$ must tend to infinity as n grows. ∎

Note that (3.10) provides a necessary condition for non-degeneracy: if there do not exist $a, b, c \in \mathbb{Z}[\mathbb{i}]$ with $a + bu + cv = 0$ and $|b|^2 - 2\operatorname{Re}(\bar{c}a) = 1$, then h is non-degenerate. It is not clear whether this is a sufficient condition as well.

Proof of Theorem 3.25. Assume that h is non-degenerate. From (3.11) and (3.12), we have

$$\frac{\mathfrak{r}_n}{\mathfrak{q}_n} = \frac{r_n}{q_n} + (-1)^n\frac{\overline{q_n}}{q_n}\cdot\frac{1}{\mathfrak{q}_n}, \qquad \frac{\mathfrak{p}_n}{\mathfrak{q}_n} = \frac{p_n}{q_n} + (-1)^n\frac{\overline{r_n}}{q_n}\cdot\frac{1}{\mathfrak{q}_n},$$

provided n is large enough so that $\mathfrak{q}_n$ is non-zero. We observe that r_n/q_n and p_n/q_n converge to u and v respectively. Since both $\overline{q_n}/q_n$ and $\overline{r_n}/q_n$ are bounded, and since $|\mathfrak{q}_n|$ goes to infinity, this proves that $\mathfrak{r}_n/\mathfrak{q}_n$ and $\mathfrak{p}_n/\mathfrak{q}_n$ converge to u and v respectively. ∎

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