ACTA ARITHMETICA
170.3 (2015)

A density theorem and extreme values of automorphic L-functions at one

by

Yingnan Wang (Shenzhen)

1. Introduction. The distribution of the values of Dirichlet L-functions at $s = 1$ was first studied by Littlewood [15] in 1928 and has attracted many mathematicians' interests. These values are related to the class numbers of quadratic fields via Dirichlet's class number formula. For a survey, the readers may refer to [4]. One may generalize this problem to symmetric power L-functions which are a generalization of Dirichlet L-functions. In the framework of automorphic representation theory, the Dirichlet L-functions constitute the family of L-functions attached to the automorphic cuspidal representations of $\mathrm{GL}_1(\mathbb{Q})$, and L-functions attached to primitive holomorphic cusp forms or primitive Maass forms are the case of $\mathrm{GL}_2(\mathbb{Q})$. According to the Langlands functoriality conjecture, a symmetric mth power L-function attached to a primitive holomorphic cusp form or a primitive Maass form is an L-function attached to some automorphic cuspidal representation of $\mathrm{GL}_{m+1}(\mathbb{Q})$ (the cases $m \le 4$ have been proved).

Symmetric power L-functions have important analytic properties carrying a lot of information on the initial automorphic representation. For example, Shimura [20] proved that for any primitive holomorphic cusp form or primitive Maass form, its symmetric square L-function is entire and satisfies a functional equation; moreover, its Rankin–Selberg L-function can be factorized into a product of its symmetric square L-function and a factor involving Riemann zeta-function, and the value of its symmetric square L-function at $s = 1$ is a scalar multiple of its L^2 norm. For symmetric power L-functions, one main difference from the Dirichlet L-functions is that the combinatorial analysis is much more complicated due to the fact that the degrees of their Euler products are higher than 1 and so the Dirichlet coefficients are not completely multiplicative.

2010 *Mathematics Subject Classification*: Primary 11F67.
Key words and phrases: density theorem, Maass forms, automorphic L-functions, extreme values.

DOI: 10.4064/aa170-3-1

Just as for the Riemann zeta-function, all the nontrivial zeros of symmetric power L-functions attached to primitive Maass forms lie in the critical strip $0 \le \Re s \le 1$ and it is also conjectured that they all lie on the vertical line $\Re s = 1/2$, which is the so-called *generalized Riemann hypothesis* (GRH). Unfortunately, GRH is out of reach currently. To get around this difficulty in many applications, one makes use of the classical tools called density theorems which are the estimates on the number of "exceptional" nontrivial zeros in the critical strip, especially in the regions close to the boundary of the critical strip. For the classical case, one famous example is that the nonvanishing of Riemann zeta-function in a region close to the line $\Re s = 1$ implies the prime number theorem.

In 1999, Luo [16] first proved a density theorem on symmetric square L-functions attached to primitive Maass forms with large eigenvalues and then applied it to study the distribution of the values of symmetric square L-functions at $s = 1$. Later, Kowalski and Michel [10] obtained a general density theorem for automorphic L-functions with large conductors. In 2004, Cogdell and Michel [1] proved a density theorem for arbitrary symmetric power L-functions of primitive holomorphic cusp forms of large level and applied it to study the distribution of values at $s = 1$ of arbitrary symmetric power L-functions of primitive holomorphic cusp forms of large level under a certain assumption. In 2005, inspired by these works, Lau and Wu [14] showed a density theorem for the mth symmetric power L-functions attached to primitive holomorphic cusp forms and used it to investigate the extreme values of the mth symmetric power L-functions attached to primitive holomorphic cusp forms at $s = 1$, for $m = 1, 2, 3, 4$.

In this paper, we prove that Lau and Wu's density theorem [14] also holds for symmetric power L-functions attached to primitive Maass forms (see Theorem 1.1), and apply this result to study the distribution of the values of the mth symmetric power L-functions attached to primitive Maass forms at 1, for $m = 1, 2, 3, 4$. One major obstacle is that the generalized Ramanujan conjecture is still unknown for primitive Maass forms and we use the Rankin–Selberg method to get around the difficulty caused by the possible "exceptional" Hecke eigenvalues.

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{H}$ be the open upper plane in $\mathbb{C}$. Denote the space spanned by the Maass cusp forms for Γ by $\mathcal{C}(\Gamma\backslash\mathbb{H})$. Let $\{u_j : j \ge 0\}$ be a complete orthonormal basis for $\mathcal{C}(\Gamma\backslash\mathbb{H})$ with

$$\Delta u_j = (1/4 + t_j^2)u_j, \qquad T_n u_j = \lambda_j(n) u_j$$

where Δ is the non-Euclidean Laplace operator and T_n is the nth Hecke operator. Then $0 < t_1 \le t_2 \le \cdots$, $\lambda_j(n) \in \mathbb{R}$ and we have the Fourier expansion

$$u_j(z) = \sqrt{y}\,\rho_j(1)\sum_{n\neq 0}\lambda_j(n)K_{it_j}(2\pi|n|y)e(nx) \qquad (z = x+iy \in \mathbb{H})$$

where $\rho_j(1) \neq 0$ and K_ν is the K-Bessel function of order ν. By a *primitive Maass form* we mean $\rho_j(1)^{-1}u_j(z)$.

The generalized Ramanujan conjecture asserts that

$$|\lambda_j(p)| \le 2 \quad \text{for all primes } p.$$

However, this conjecture is out of reach at present . The best result in this direction is due to Kim and Sarnak [7], who proved that for all primes p,

$$|\lambda_j(p)| \le p^{\theta} + p^{-\theta} \tag{1.1}$$

where $\theta = 7/64$. The "exceptional" eigenvalues (whose absolute values are greater than 2) raise extra difficulties compared with the case of primitive holomorphic cusp forms. Moreover, we know that (*Weyl's law*)

$$r(T) = \#\{j : 0 < t_j \le T\} = \frac{1}{12}T^2 + O(T\log T). \tag{1.2}$$

Define two parameters, $\alpha_{u_j}(p)$ and $\beta_{u_j}(p)$, by

$$\alpha_{u_j}(p) + \beta_{u_j}(p) = \lambda_j(p) \quad \text{and} \quad \alpha_{u_j}(p)\beta_{u_j}(p) = 1.$$

For $m \in \mathbb{N}$, the *symmetric mth power L-function* associated to u_j is defined by

$$L(s, \operatorname{sym}^m u_j) = \prod_p \prod_{k=0}^{m}(1-\alpha_{u_j}(p)^{m-2k}p^{-s})^{-1} = \sum_{n=1}^{\infty}\lambda_{\operatorname{sym}^m u_j}(n)n^{-s} \tag{1.3}$$

for $\Re s \gg 1$. For $m = 1,2,3,4$, the above Dirichlet series converges absolutely for $\Re s > 1$ and $L(s, \operatorname{sym}^m u_j)$ is the L-function attached to a cuspidal automorphic form of $\mathrm{GL}_{m+1}(\mathbb{Q})$. Moreover, $L(s, \operatorname{sym}^m u_j)$ can be analytically continued to the entire complex plane and satisfies the functional equation (see [3] for $m = 1$, [20] for $m = 2$ and [6], [8], [9] for $m = 3,4$)

$$\begin{aligned} \Lambda(s, \operatorname{sym}^m u_j) &= L_\infty(s, \operatorname{sym}^m u_j)L(s, \operatorname{sym}^m u_j) \\ &= \varepsilon_{\operatorname{sym}^m u_j}\Lambda(1-s, \operatorname{sym}^m u_j), \end{aligned} \tag{1.4}$$

where $\varepsilon_{\operatorname{sym}^m u_j} = \pm 1$ and

$$L_\infty(s, \operatorname{sym}^m u_j) = \prod_{k=0}^{m}\Gamma_{\mathbb{R}}(s - i(m-2k)t_j) \quad \text{with} \quad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

Our main objective is to prove the following density theorem on symmetric power L-functions attached to primitive Maass forms.

THEOREM 1.1. *Let $m = 1,2,3,4$ and $r \ge 1$ be given. Let $N(\alpha, H, \operatorname{sym}^m u_j)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s, \operatorname{sym}^m u_j)$ with $\beta \ge \alpha$ and $0 \le \gamma \le H$.*

Then for any $\epsilon > 0$, we have

$$\sum_{t_j \le T} N(\alpha, H, \operatorname{sym}^m u_j) \ll_{r,\epsilon} H^{1+1/r} T^{\frac{8((m+1)(m+r+1)+64)(1-\alpha)}{17-16\alpha}+\epsilon}$$

uniformly for $15/16 + \epsilon \le \alpha \le 1$ and $H \ge 1$. The implied constant depends on ϵ and r only.

Remark 1.2. (i) A similar result has been proved for the case of holomorphic cusp forms by Lau and Wu [14].

(ii) The key points here are the convexity bounds in Lemma 2.3 and the large sieve inequality in Lemma 2.5, which handles the difficulties arising from the possible "exceptional" eigenvalues (of absolute value more than 2).

Now we deduce some consequences of Theorem 1.1. Let $s = \sigma + i\tau$ and $m = 1, 2, 3, 4$. For any $\eta \in (0, 1/2)$, we define

$$H^+_{T,\operatorname{sym}^m}(\eta) = \{0 < t_j \le T : L(s, \operatorname{sym}^m u_j) \ne 0,\ s \in \mathcal{S}\},$$

where $\mathcal{S} = \{s : \sigma \ge 1 - \eta,\ |\tau| \le 100T^\eta\} \cup \{s : \sigma \ge 1\}$. Then we define

$$H^-_{T,\operatorname{sym}^m}(\eta) = \{0 < t_j \le T : t_j \notin H^+_{T,\operatorname{sym}^m}(\eta)\}.$$

By Theorem 1.1 with $r = 1$, it is easy to see

$$\text{(1.5)} \qquad |H^-_{T,\operatorname{sym}^m}(\eta)| \le \sum_{t_j \in H^-_{T,\operatorname{sym}^m}(\eta)} N(1-\eta, 100T^\eta, \operatorname{sym}^m u_j)$$
$$\le \sum_{0 < t_j \le T} N(1-\eta, 100T^\eta, \operatorname{sym}^m u_j) \ll T^{8\eta((m+1)(m+2)+66)}.$$

Therefore, by Weyl's law (1.2), we have

$$|H^+_{T,\operatorname{sym}^m}(\eta)| = r(T) + O(T^{8\eta((m+1)(m+2)+66)})$$

for $m = 1, 2, 3, 4$ and $\eta \in (1, 1/1000)$. This implies that for almost u_j, the functions $L(s, \operatorname{sym}^m u_j)$ with $m \le 4$ satisfy a weak form of GRH.

Next, we apply Theorem 1.1 to study the distribution of the values of the mth symmetric power L-functions of primitive Maass forms at $s = 1$, for $m = 1, 2, 3, 4$. As the first application, we generalize Lau and Wu's result [14, Theorem 2] on the extreme values of symmetric power L-functions attached to primitive holomorphic cusp forms to the case of primitive Maass forms.

Theorem 1.3. *Let $\eta \in (0, 1/1000)$ be fixed and $m = 1, 2, 3, 4$. Then there exist $t_{j_1}, t_{j_2} \in H^+_{T,\operatorname{sym}^m}(\eta)$ such that, for $T \to \infty$,*

$$L(1, \operatorname{sym}^m u_{j_1}) \ge \{1 + o(1)\}(B^+_m \log_2 T)^{A^+_m},$$
$$L(1, \operatorname{sym}^m u_{j_2}) \le \{1 + o(1)\}(B^-_m \log_2 T)^{-A^-_m},$$

where the constants $A_m^{\pm}$ *and* $B_m^{\pm}$ *are given by*

$$\begin{cases} A_m^+ = m+1, & B_m^+ = e^{\gamma} \qquad\qquad (m=1,2,3,4),\\ A_m^- = m+1, & B_m^- = e^{\gamma}\zeta(2)^{-1} \quad (m=1,3),\\ A_2^- = 1, & B_2^- = e^{\gamma}\zeta(2)^{-2},\\ A_4^- = \frac{5}{4}, & B_4^- = e^{\gamma}B_{4,*}^-, \end{cases}$$

and

$$B_{4,*}^- = \prod_p \biggl\{\Bigl(1-\frac{1}{p}\Bigr)\biggl[\frac{2p}{27}\Bigl(-1+\frac{10}{p}+\frac{6}{p^2}+\frac{3}{p^3}-\frac{3}{p^4}-\frac{6}{p^5}-\frac{10}{p^6}+\frac{1}{p^7}\Bigr) + \frac{2p}{27}\Bigl(1+\frac{2}{p}+\frac{5}{p^2}-\frac{5}{p^3}-\frac{2}{p^4}-\frac{1}{p^5}\Bigr)\sqrt{1+\frac{3}{p}+\frac{8}{p^2}+\frac{3}{p^3}+\frac{1}{p^4}}\biggr]^{4/5}\biggr\}.$$

Here γ *is the Euler constant.*

The proof of Theorem 1.3 is very similar to the proof of [14, Theorem 2] with obvious modifications, and so we shall omit it.

Next we study the analogues of the first two Montgomery–Vaughan conjectures for automorphic L-functions associated to primitive Maass forms. These conjectures were first proposed by Montgomery and Vaughan [17] for Dirichlet L-functions $L(s,\chi_d)$ associated to a primitive real character χ_d in 1999. For a survey on these conjectures, the interested readers may refer to [12, Section 3].

Let $1 \le G \le T$ and

$$H(G,T) = \{t_j : T-G \le t_j \le T+G\}.$$

Define

$$F_{T,G}^+(t,\mathrm{sym}^m) = \frac{1}{|H(G,T)|}\sum_{\substack{t_j\in H(G,T)\\ L(1,\mathrm{sym}^m u_j) > (B_m^+ t)^{A_m^+}}} 1,$$

$$F_{T,G}^-(t,\mathrm{sym}^m) = \frac{1}{|H(G,T)|}\sum_{\substack{t_j\in H(G,T)\\ L(1,\mathrm{sym}^m u_j) < (B_m^- t)^{A_m^-}}} 1,$$

where the constants $A_m^{\pm}$ and $B_m^{\pm}$ are defined as in Theorem 1.3. The analogues of the first two conjectures of Montgomery and Vaughan for automorphic L-functions associated to primitive Maass forms are as follows. For each $m \in \mathbb{Z}$, there exist positive constants $T_0 = T_0(m)$, $C_m > c_m > 0$, and $0 < \Theta_m < \Theta_m < 1$ such that for $T > T_0$,

$$e^{-C_m(\log T)/\log_2 T} \le F_{T,G}^{\pm}(\log_2 T,\mathrm{sym}^m) \le e^{-c_m(\log T)/\log_2 T}, \tag{1.6}$$

$$T^{-\Theta_m} \le F_{T,G}^{\pm}(\log_2 T + \log_3 T,\mathrm{sym}^m) \le T^{-\theta_m}. \tag{1.7}$$

Lau and Wu [12] proved the upper-bound part of the analogue of the first conjecture for automorphic L-functions associated to holomorphic modular forms and a weaker upper bound for the second one in 2008. Unfortunately, we cannot prove the same results for primitive Maass forms due to the absence of the generalized Ramanujan conjecture. Applying Theorem 1.1, we prove weaker upper bounds for (1.6) and (1.7).

THEOREM 1.4. *Let $m = 1, 2, 3, 4$. Then for any $\epsilon > 0$, there are positive constants $c = c(\epsilon)$ and $T_0 = T_0(\epsilon)$ such that*

$$F^{\pm}_{T,G}(\log_2 T + r, \operatorname{sym}^m) \leq \exp\left(-c(r+1)\frac{\log T}{(\log_2 T)(\log_3 T)^2(\log_4 T)^2}\right)$$

for $T \geq T_0$ and $\log \epsilon \leq r \leq (9-\epsilon)\log_2 T$.

2. Preliminaries

2.1. Convexity bounds for symmetric power L-functions. The following result should be well-known and the interested readers may refer to [24, Lemma 2.1.1] for a proof.

LEMMA 2.1. *Let $s = \sigma + i\tau$. Then*

$$\frac{\Gamma(1/2-s)}{\Gamma(s)} \ll_\epsilon (1+|\tau|)^{1/2-2\sigma}$$

uniformly for $-1+\epsilon \leq \sigma \leq -\epsilon$, where the implied constant only depends on ϵ.

COROLLARY 2.2. *Let $s = \sigma + i\tau$. Then*

$$\frac{L_\infty(1-s, \operatorname{sym}^m u_j)}{L_\infty(s, \operatorname{sym}^m u_j)} \ll_\epsilon \prod_{k=0}^{m} (1+|\tau|+|(m-2k)t_{j_1}|)^{1/2-\sigma}$$

and

$$\frac{L_\infty(1-s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2})}{L_\infty(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2})} \ll_\epsilon \prod_{k,\ell=0}^{m} \bigl(1+|\tau|+|(m-2k)t_{j_1}|+|(m-2\ell)t_{j_2}|\bigr)^{1/2-\sigma}$$

uniformly for $-1+\epsilon \leq \sigma \leq -\epsilon$, where the implied constants only depend on ϵ.

Define the *Rankin–Selberg L-function* of $\operatorname{sym}^m u_{j_1}$ and $\operatorname{sym}^m u_{j_2}$ by

$$L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) = \prod_p \prod_{k,\ell=0}^{m} \bigl(1-\alpha_{u_{j_1}}(p)^{m-2k}\alpha_{u_{j_2}}(p)^{m-2\ell}p^{-s}\bigr)^{-1}$$

for $\Re s \gg 1$. For $m = 1,2,3,4$, the above product is absolutely convergent for $\Re s > 1$. Furthermore, by (1.4) and Rudnick and Sarnak's work [18], $L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2})$ has the functional equation

$$
\begin{aligned}
(2.1)\qquad & \Lambda(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \\
&= L_\infty(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \\
&= \varepsilon_{\operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}} \Lambda(1-s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}),
\end{aligned}
$$

where $\varepsilon_{\operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}} = \pm 1$ and

$$L_\infty(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) = \prod_{k,\ell=0}^{m} \Gamma_{\mathbb{R}}\big(s - i(m-2k)t_{j_1} - i(m-2\ell)t_{j_2}\big).$$

If $u_{j_1} \neq u_{j_2}$, then $L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2})$ can be analytically continued to the entire complex plane. If $u_{j_1} = u_{j_2}$, it has an analytic continuation with a simple pole at 1.

Lemma 2.3. *Let $m = 1,2,3,4$ and $s = \sigma + i\tau$. For any $\epsilon > 0$, we have*

$$L(s, \operatorname{sym}^m u_j) \ll_\epsilon \prod_{k=0}^{m} (1 + |\tau| + |(m-2k)t_j|)^{(1-\sigma)/2+\epsilon}$$

in the strip $-\epsilon \le \sigma \le 1 + \epsilon$ and

$$
\begin{aligned}
& s(s-1)L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \\
&\qquad \ll_\epsilon \prod_{k,\ell=0}^{m} \big(1 + |\tau| + |(m-2k)t_{j_1}| + |(m-2\ell)t_{j_2}|\big)^{(1-\sigma)/2+\epsilon}
\end{aligned}
$$

in the strip $-\epsilon \le \sigma \le 1 + \epsilon$.

Proof. Let $\epsilon_1 = \frac{1}{3}\epsilon$. By (1.1) and a direct computation, we have

$$L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \ll_{\epsilon_1} 1$$

on the line $\sigma = 15/8 + \epsilon_1$. Then by the functional equation (2.1) and Corollary 2.2, we have

$$
\begin{aligned}
& L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \\
&\qquad \ll_{\epsilon_1} \prod_{k,\ell=0}^{m} \big(1 + |\tau| + |(m-2k)t_{j_1}| + |(m-2\ell)t_{j_2}|\big)^{11/8+\epsilon_1}
\end{aligned}
$$

on the line $\sigma = -7/8 - \epsilon_1$. By Property (**RS** 3) in Rudnick and Sarnak [18], we know $s(s-1)L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2})$ is entire. By the Phragmén–

Lindelöf principle [23, §5.65],

$$s(s-1)L(s,\operatorname{sym}^m u_{j_1}\times\operatorname{sym}^m u_{j_2}) \ll_{\epsilon_1} \prod_{k,\ell=0}^{m}\big(1+|\tau|+|(m-2k)t_{j_1}|+|(m-2\ell)t_{j_2}|\big)^{(15-8\sigma+8\epsilon_1)/16} \tag{2.2}$$

in the strip $-7/8-\epsilon_1\le\sigma\le 15/8+\epsilon_1$.

Jacquet and Shalika [5] proved that $L(s,\operatorname{sym}^m u_{j_1}\times\operatorname{sym}^m u_{j_2})$ does not vanish on the half-plane $\sigma>1$. We can define $\log L(s,\operatorname{sym}^m u_{j_1}\times\operatorname{sym}^m u_{j_2})$ in the half-plane $\sigma\ge 1+\epsilon_1$ in the usual way; we then have $\log L(s,\operatorname{sym}^m u_{j_1}\times\operatorname{sym}^m u_{j_2})\ll_{\epsilon_1} 1$ on the half-plane $\sigma\ge 15/8+\epsilon_1$. The Borel–Carathéodory theorem [23, §5.5] with the convex bounds (2.2) implies that if $\sigma\ge 1+2\epsilon_1$, then

$$\log L(s,\operatorname{sym}^m u_{j_1}\times\operatorname{sym}^m u_{j_2})\ll_{\epsilon_1}\log(t_{j_1}+t_{j_2}+|\tau|).$$

If $\tau\ge 0$, let

$$k(s)=\frac{\sigma_2-s}{\sigma_2-\sigma_1}\quad\text{and}\quad \varrho(s)=e^{k(s)\log_2(-is+t_{j_1}+t_{j_2})},$$

where $\sigma_1=1+2\epsilon_1$ and $\sigma_2=15/8+\epsilon_1$. Then

$$\begin{aligned}\log_2(&-is+t_{j_1}+t_{j_2})\\ &=\frac{1}{2}\log\left(\frac{1}{4}\log^2(\sigma^2+(\tau+t_{j_1}+t_{j_2})^2)+\arctan^2\frac{\sigma}{\tau+t_{j_1}+t_{j_2}}\right)\\ &\quad-i\arctan\frac{2\arctan\frac{\sigma}{\tau+t_{j_1}+t_{j_2}}}{\log(\sigma^2+(\tau+t_{j_1}+t_{j_2})^2)},\end{aligned}$$

where $\log_r$ is the r-fold iterated logarithm. A direct computation leads to

$$\Re k(s)\log_2(-is+t_{j_1}+t_{j_2})=k(\sigma)\log_2(\tau+t_{j_1}+t_{j_2})+O(1)$$

and

$$|\varrho(s)|=\log(\tau+t_{j_1}+t_{j_2})^{k(\sigma)}e^{O(1)}.$$

Therefore,

$$\frac{\log L(s,\operatorname{sym}^m u_{j_1}\times\operatorname{sym}^m u_{j_2})}{\varrho(s)}\ll_{\epsilon_1} 1$$

on the two parallel upper half-lines $\sigma=1+2\epsilon_1$, $\tau\ge 0$ and $\sigma=15/8+\epsilon_1$, $\tau\ge 0$. Hence by the Phragmén–Lindelöf principle again, we have

$$\frac{\log L(s,\operatorname{sym}^m u_{j_1}\times\operatorname{sym}^m u_{j_2})}{\varrho(s)}\ll_{\epsilon_1} 1$$

in the upper half-strip $1+2\epsilon_1\le\sigma\le 15/8+\epsilon_1$, $\tau\ge 0$. Therefore, on the upper half-line $\sigma=1+3\epsilon_1=1+\epsilon$, $\tau\ge 0$,

$$\log L(s,\operatorname{sym}^m u_{j_1}\times\operatorname{sym}^m u_{j_2})\ll_{\epsilon_1}\log(|\tau|+t_{j_1}+t_{j_2})^{\frac{7-16\epsilon_1}{7-8\epsilon_1}}$$

and

$$L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \ll_{\epsilon_1} e^{\log(|\tau|+t_{j_1}+t_{j_2})^{1-\epsilon_1}} \ll_{\epsilon_1} (|\tau| + t_{j_1} + t_{j_2})^{\epsilon_1}. \tag{2.3}$$

Similarly, we can prove (2.3) for the lower half-line $\sigma = 1+3\epsilon_1 = 1+\epsilon$, $\tau \leq 0$. Therefore, by the functional equation (2.1) and Corollary 2.2, we have

$$L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \ll_\epsilon \prod_{k,\ell=0}^{m} \big(1+|\tau|+|(m-2k)t_{j_1}|+|(m-2\ell)t_{j_2}|\big)^{1/2+2\epsilon}$$

on the line $\sigma = -\epsilon$. By the Phragmén–Lindelöf principle,

$$s(s-1)L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \ll_\epsilon \prod_{k,\ell=0}^{m} \big(1+|\tau|+|(m-2k)t_{j_1}|+|(m-2\ell)t_{j_2}|\big)^{\frac{1-\sigma+8\epsilon}{2+4\epsilon}}$$

in the strip $-\epsilon \leq \sigma \leq 1+\epsilon$.

On the other hand, Lau and Lü [11] proved that if $j_1 = j_2 = j$, then the coefficients of $L(s, \operatorname{sym}^m u_j \times \operatorname{sym}^m u_j)$ are nonnegative and

$$|\lambda_{\operatorname{sym}^m u_j}(n)|^2 \leq \lambda_{\operatorname{sym}^m u_j \times \operatorname{sym}^m u_j}(n). \tag{2.4}$$

Hence, by Cauchy's inequality, $L(s, \operatorname{sym}^m u_j)$ converges absolutely for $\sigma > 1$ and

$$L(s, \operatorname{sym}^m u_j) \ll L(1+\epsilon, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2})^{1/2} \zeta(1+\epsilon)^{1/2} \ll_\epsilon t_j^\epsilon \tag{2.5}$$

for $\sigma \geq 1+\epsilon$. Hence by the functional equation (1.4) and Corollary 2.2, we have

$$L(s, \operatorname{sym}^m u_j) \ll_\epsilon \prod_{k=0}^{m} (1+|\tau|+|(m-2k)t_j|)^{1/2+2\epsilon}$$

on the line $\sigma = -\epsilon$. By the Phragmén–Lindelöf principle,

$$L(s, \operatorname{sym}^m u_j) \ll_\epsilon \prod_{k=0}^{m} (1+|\tau|+|(m-2k)t_j|)^{\frac{1-\sigma+8\epsilon}{2+4\epsilon}}$$

in the strip $-\epsilon \leq \sigma \leq 1+\epsilon$. Replacing $\frac{1-\sigma+8\epsilon}{2+4\epsilon}$ by $(1-\sigma)/2+\epsilon$, the proof is complete. ∎

Note that $L(s, \operatorname{sym}^m u_j) \ll 1$ and $L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \ll 1$ for $\Re s \geq 2$. By the above lemma with $\Re s = 1$ and the Phragmén–Lindelöf principle, we have the following corollary.

Corollary 2.4. *Let $m = 1, 2, 3, 4$ and $s = \sigma + i\tau$. For any $\epsilon > 0$, we have*

$$L(s, \operatorname{sym}^m u_j) \ll_\epsilon \prod_{k=0}^{m} (1+|\tau|+|(m-2k)t_j|)^{(2-\sigma)\epsilon}$$

in the strip $1 \le \sigma \le 2$, *and*

$$s(s-1)L(s, \operatorname{sym}^m u_{j_1} \times \operatorname{sym}^m u_{j_2}) \ll_\epsilon \prod_{k,\ell=0}^{m} \big(1 + |\tau| + |(m-2k)t_{j_1}| + |(m-2\ell)t_{j_2}|\big)^{(2-\sigma)\epsilon}$$

in the strip $1 \le \sigma \le 2$.

2.2. A large sieve inequality. The following lemma is an analogue of [14, Proposition 4.1]. Since the generalized Ramanujan conjecture is unknown for primitive Maass forms, we apply Duke and Kowalski's method [2] and the convexity bounds proved above to get around the difficulty caused by the possible "exceptional" Hecke eigenvalues.

LEMMA 2.5. *Let* $m = 1, 2, 3, 4$, $L \ge 1$, *and let* $\{a_\ell\}_{\ell \le L}$ *be a sequence of complex numbers. Then for any* $\epsilon > 0$, *we have*

$$\sum_{t_j \le T} \Big| \sum_{\ell \le L} a_\ell \lambda_{\operatorname{sym}^m u_j}(\ell) \Big|^2 \ll_\epsilon T^\epsilon (L + T^{(m+1)^2/32+2} L^{15/16+\epsilon}) \sum_{\ell \le L} |a_\ell|^2.$$

Proof. By the duality principle, we only have to prove

$$\sum_{\ell \le L} \Big| \sum_{t_j \le T} b_j \lambda_{\operatorname{sym}^m u_j}(\ell) \Big|^2 \ll_\epsilon T^\epsilon (L + T^{(m+1)^2/32+2} L^{15/16+\epsilon}) \sum_{t_j \le T} |b_j|^2 \tag{2.6}$$

for any sequence $\{b_j\}_{t_j \le T}$ of complex numbers.

The left hand side of (2.6) is bounded by

$$\sum_{\ell \ge 1} \Big| \sum_{t_j \le T} b_j \lambda_{\operatorname{sym}^m u_j}(\ell) \Big|^2 e^{-\ell/L} = \sum_{t_{j_1}, t_{j_2} \le T} b_{j_1} \overline{b_{j_2}} \sum_{\ell \ge 1} \lambda_{\operatorname{sym}^m u_{j_1}}(\ell) \lambda_{\operatorname{sym}^m u_{j_2}}(\ell) e^{-\ell/L}. \tag{2.7}$$

On the other hand, it is well-known that for $c > 0$, we have

$$e^{-1/y} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s) y^s \, ds. \tag{2.8}$$

Thus

$$\sum_{\ell \ge 1} \lambda_{\operatorname{sym}^m u_{j_1}}(\ell) \lambda_{\operatorname{sym}^m u_{j_2}}(\ell) e^{-\ell/L} = \frac{1}{2\pi i} \int_{(2)} \sum_{\ell=1}^{\infty} \frac{\lambda_{\operatorname{sym}^m u_{j_1}}(\ell) \lambda_{\operatorname{sym}^m u_{j_2}}(\ell)}{\ell^s} \Gamma(s) L^s \, ds. \tag{2.9}$$

Note that

$$\sum_{\ell=1}^{\infty} \frac{\lambda_{\mathrm{sym}^m u_{j_1}}(\ell)\lambda_{\mathrm{sym}^m u_{j_2}}(\ell)}{\ell^s} = \prod_p \Big(\sum_{n=0}^{\infty} \frac{\lambda_{\mathrm{sym}^m u_{j_1}}(p^n)\lambda_{\mathrm{sym}^m u_{j_2}}(p^n)}{p^{ns}} \Big).$$

By Duke and Kowalski's work [2, (24)], for $\Re s = 2$, we have

$$\sum_{n=0}^{\infty} \frac{\lambda_{\mathrm{sym}^m u_{j_1}}(p^n)\lambda_{\mathrm{sym}^m u_{j_2}}(p^n)}{p^{ns}} = L_p(s, \mathrm{sym}^m u_{j_1} \times \mathrm{sym}^m u_{j_2}) H_p(s; \mathrm{sym}^m u_{j_1}, \mathrm{sym}^m u_{j_2})$$

where

$$L_p(s, \mathrm{sym}^m u_{j_1} \times \mathrm{sym}^m u_{j_2}) = \prod_{k,\ell=0}^{m} \big(1 - \alpha_{u_{j_1}}(p)^{m-2k} \alpha_{u_{j_2}}(p)^{m-2\ell} p^{-s}\big)^{-1}$$

and $H_p(s; \mathrm{sym}^m u_{j_1}, \mathrm{sym}^m u_{j_2})$ is a polynomial in p^{-s} of degree not higher than $(m+1)^2-1$. Moreover, the coefficient of p^{-s} in $H_p(s; \mathrm{sym}^m u_{j_1}, \mathrm{sym}^m u_{j_2})$ vanishes and the coefficients of p^{-ns}, $2 \le n \le (m+1)^2 - 1$, are

$$\sum_{h=0}^{n} \lambda_{\mathrm{sym}^m u_{j_1}}(p^{n-h}) \lambda_{\mathrm{sym}^m u_{j_2}}(p^{n-h}) f_m^{(h)}(0), \tag{2.10}$$

where

$$f_m(x) = \prod_{k,\ell=0}^{m} \big(1 - \alpha_{u_{j_1}}(p)^{m-2k} \alpha_{u_{j_2}}(p)^{m-2\ell} x\big).$$

Set

$$H(s; \mathrm{sym}^m u_{j_1}, \mathrm{sym}^m u_{j_2}) = \prod_p H_p(s; \mathrm{sym}^m u_{j_1}, \mathrm{sym}^m u_{j_2}),$$

$$F(s; m, u_{j_1}, u_{j_2}) = L(s, \mathrm{sym}^m u_{j_1} \times \mathrm{sym}^m u_{j_2}) H(s; \mathrm{sym}^m u_{j_1}, \mathrm{sym}^m u_{j_2}).$$

Then (2.9) can be rewritten as

$$\sum_{\ell \ge 1} \lambda_{\mathrm{sym}^m u_{j_1}}(\ell) \lambda_{\mathrm{sym}^m u_{j_2}}(\ell) e^{-\ell/L} = \frac{1}{2\pi i} \int_{(2)} F(s; m, u_{j_1}, u_{j_2}) \Gamma(s) L^s \, ds. \tag{2.11}$$

Let $s = \sigma + i\tau$. If $m = 1$, then it is easy to see that

$$H_p(s; \mathrm{sym}^1 u_{j_1}, \mathrm{sym}^1 u_{j_2}) = 1 - p^{-2s}.$$

If $m=2$, then by direct calculation, we have

$$\begin{aligned}H_p(s;\operatorname{sym}^2 u_{j_1},\operatorname{sym}^2 u_{j_2}) &= 1-\lambda_{\operatorname{sym}^2 u_{j_1}}(p)\lambda_{\operatorname{sym}^2 u_{j_2}}(p)p^{-2s}\\ &\quad+\big(\lambda_{\operatorname{sym}^4 u_{j_1}}(p)+\lambda_{\operatorname{sym}^4 u_{j_2}}(p)+\lambda_{\operatorname{sym}^2 u_{j_1}}(p)+\lambda_{\operatorname{sym}^2 u_{j_2}}(p)\big)p^{-3s}\\ &\quad-\lambda_{\operatorname{sym}^2 u_{j_1}}(p)\lambda_{\operatorname{sym}^2 u_{j_2}}(p)p^{-4s}+p^{-6s}\\ &=1+O(p^{-2\sigma+7/16}).\end{aligned}$$

Hence, for $m=1,2$, $H(s;\operatorname{sym}^m u_{j_1},\operatorname{sym}^m u_{j_2})$ converges absolutely for $\sigma>23/32$ and is bounded by an absolute constant.

For $m=3,4$, we use Wolfram Mathematica 7 to determine the coefficients of $H_p(s;\operatorname{sym}^m u_{j_1},\operatorname{sym}^m u_{j_2})$. The details are given in the Appendix. In what follows, we assume that $\sigma>15/16$ and p is sufficiently large. If $m=3$, by the Appendix and (1.1), we have

$$H_p(s;\operatorname{sym}^3 u_{j_1},\operatorname{sym}^3 u_{j_2})=1+O(p^{-2\sigma+7/8}).$$

If $m=4$, by the Appendix and (1.1), we have

$$H_p(s;\operatorname{sym}^4 u_{j_1},\operatorname{sym}^4 u_{j_2})=1-\lambda_{\operatorname{sym}^3 u_{j_1}}(p^2)\lambda_{\operatorname{sym}^3 u_{j_2}}(p^2)p^{-2s}+O(p^{-3\sigma+7/4}).$$

Since

$$\lambda_{\operatorname{sym}^3 u_j}(p^2)=\lambda_{\operatorname{sym}^3 u_j}(p)^2-\lambda_{\operatorname{sym}^4 u_j}(p)-1,$$

by (1.1) again, we have

$$\begin{aligned}|H_p(s;\operatorname{sym}^4 u_{j_1},\operatorname{sym}^4 u_{j_2})| &\le 1+\lambda_{\operatorname{sym}^3 u_{j_1}}(p)^2\lambda_{\operatorname{sym}^3 u_{j_2}}(p)^2p^{-2\sigma}\\ &\quad+5\big(\lambda_{\operatorname{sym}^3 u_{j_1}}(p)^2+\lambda_{\operatorname{sym}^3 u_{j_2}}(p)^2\big)p^{-2\sigma+7/16}+O(p^{-2\sigma+7/8}+p^{-3\sigma+7/4})\\ &\le\big(1+16\lambda_{\operatorname{sym}^3 u_{j_1}}(p)^2p^{-2\sigma+21/32}\big)\big(1+5\lambda_{\operatorname{sym}^3 u_{j_1}}(p)^2p^{-2\sigma+7/8}\big)\\ &\quad\times\big(1+5\lambda_{\operatorname{sym}^3 u_{j_2}}(p)^2p^{-2\sigma+7/8}\big)\big(1+O(p^{-2\sigma+7/8}+p^{-3\sigma+7/4})\big).\end{aligned}$$

Therefore, by (2.4) and the analytic properties of $L(s,\operatorname{sym}^m u_j\times\operatorname{sym}^m u_j)$, we conclude that for $m=3,4$, $H(s;\operatorname{sym}^m u_{j_1},\operatorname{sym}^m u_{j_2})$ converges absolutely for $\sigma>15/16$. In addition, by Corollary 2.4, for $\sigma\ge 15/16+\epsilon$, we get

$$H(s;\operatorname{sym}^m u_{j_1},\operatorname{sym}^m u_{j_2})\ll_\epsilon(1+|\tau|+t_{j_1})^\epsilon(1+|\tau|+t_{j_2})^\epsilon. \tag{2.12}$$

Hence, for $m=1,2,3,4$,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(2)}F(s;m,u_{j_1},u_{j_2})\Gamma(s)L^s\,ds \\ &\quad=\operatorname{Res}_{s=1}F(s;m,u_{j_1},u_{j_2})L+\frac{1}{2\pi i}\int_{(15/16+\epsilon)}F(s;m,u_{j_1},u_{j_2})\Gamma(s)L^s\,ds,\end{aligned} \tag{2.13}$$

By Lemma 2.3 and (2.12), we have

$$\operatorname{Res}_{s=1}F(s;m,u_{j_1},u_{j_2})L=\lim_{s\to1+}(s-1)F(s;m,u_{j_1},u_{j_2})L\ll_\epsilon\delta_{j_1,j_2}T^\epsilon L,$$

where $\delta_{j_1,j_2} = 1$ if $j_1 = j_2$ and $\delta_{j_1,j_2} = 0$ otherwise. By Stirling's formula [23, §4.42], Lemma 2.3 and (2.12), the last integral in (2.13) is

$$\ll T^{(m+1)^2/32+\epsilon} L^{15/16+\epsilon}.$$

Inserting the above bounds into (2.13) and combining with (2.11), we have

$$\sum_{\ell\geq 1} \lambda_{\mathrm{sym}^m u_{j_1} \times \mathrm{sym}^m u_{j_2}}(\ell) e^{-\ell/L} \ll T^{\epsilon}(\delta_{j_1,j_2} L + T^{(m+1)^2/32} L^{15/16+\epsilon}). \tag{2.14}$$

Combining (2.7) and (2.14) with Weyl's law, we get the statement. ∎

2.3. Two preliminary lemmas. By Lemma 2.3 and (2.5), we obtain

$$\log |L(2 + i\ell + 3e^{i\theta}, \mathrm{sym}^m u_j)| \ll_\epsilon \log \ell t_j$$

for any $\theta \in [0, 2\pi)$. Combining the above formula with an argument similar to [22, §9.2], we have the following lemma.

LEMMA 2.6. *For $\ell > 2$ and any $\epsilon > 0$, we have*

$$N(1/2, \ell + 1, \mathrm{sym}^m u_j) - N(1/2, \ell, \mathrm{sym}^m u_j) \ll_\epsilon \log \ell t_j.$$

The following lemma is an analogue of [14, Lemma 5.1] with a similar proof.

LEMMA 2.7. *Let $m \in \mathbb{N}$, let $z > (m+1)^2$ be any fixed number and define $P(z) = \prod_{p\leq z} p$. For $\Re s > 1$, we have*

$$L(s, \mathrm{sym}^m u_j)^{-1} = G_j(s) \sum_{(n,P(z))=1} \lambda_{\mathrm{sym}^m u_j}(n)\mu(n)n^{-s},$$

where the Dirichlet series $G_j(s)$ converges absolutely for $\Re s > 15/16$ and $G_j(s) \ll_{m,z,\epsilon} 1$ uniformly for $\Re s > 15/16 + \epsilon$.

Proof. By (1.3) and (1.1) we have $\lambda_{\mathrm{sym}^m u_j}(p) \leq (m + 1)p^{7/64}$. Hence $1 - \lambda_{\mathrm{sym}^m u_j}(p)p^{-s}$ is nonzero for $\sigma > 15/16$ and $p > z$. Formally, we can write $G_j(s)$ as

$$\prod_{p\leq z} \prod_{i=0}^{m} (1-\alpha_{u_j}(p)^{m-2i}p^{-s}) \prod_{p>z} (1-\lambda_{\mathrm{sym}^m u_j}(p)p^{-s})^{-1} \prod_{i=0}^{m} (1-\alpha_{u_j}(p)^{m-2i}p^{-s}).$$

If $p > z$, the p-local factor of $G_j(s)$ is of the form $1 + O_m(p^{-2(\sigma-7/16)})$ and the statements follow provided $\sigma > 15/16 + \epsilon$. ∎

3. Proof of Theorem 1.1. According to Lemma 2.6 and Weyl's law (1.2), Theorem 1.1 holds if $H \geq T^{2r}$. Moreover, we observe that the particular case $H = (\log T)^3$ implies the case $1 \leq H \leq (\log T)^3$. Hence we only have to consider the case

$$(\log T)^3 \leq H \leq T^{2r}.$$

Inspired by Lemma 2.6, we cut the rectangle region $\alpha \leq \Re s \leq 1$ and $0 \leq \Im s \leq H$ vertically into boxes of height $2(\log T)^3$, and each box $\alpha \leq \Re s \leq 1$ and $Y \leq \Im s \leq Y + 2(\log T)^3$ contains at most $O((\log T)^4)$ zeros of $L(s, \operatorname{sym}^m u_j)$. Denote by $n_{\operatorname{sym}^m u_j}$ the number of boxes which contain at least one zero ρ of $L(s, \operatorname{sym}^m u_j)$. Then it is easy to see

$$N(\alpha, H, \operatorname{sym}^m u_j) \ll n_{\operatorname{sym}^m u_j} (\log T)^4.$$

Therefore, it is sufficient to prove that

$$\sum_{t_j \leq T} n_{\operatorname{sym}^m u_j} \ll_{\epsilon, r} HT^{\frac{8((m+1)(m+r+1)+64)(1-\alpha)}{17-16\alpha}+\epsilon}.$$

We assume that $15/16 + 2\epsilon \leq \alpha \leq 1$ in the remainder of this section. Let $\rho = \beta + i\gamma$ with $\beta \geq \alpha$ and $0 \leq \gamma \leq H$ be a zero of $L(s, \operatorname{sym}^m u_j)$, and define

$$\kappa = 1/\log T, \qquad \kappa_1 = 1 - \beta + \kappa, \qquad \kappa_2 = 15/16 - \beta + \epsilon.$$

Then it is obvious that $\kappa_1 > 0$ and $\kappa_2 < 0$ by our assumption on α. Let $x, y \in [1, T^{40m^2(1+r)}]$ and $G_j(s)$ and $P(z)$ be as in Lemma 2.7. Then define

$$M_x(s, \operatorname{sym}^m u_j) = G_j(s) \sum_{\substack{\ell \leq x \\ (\ell, P(z))=1}} \mu(\ell) \lambda_{\operatorname{sym}^m u_j}(\ell) \ell^{-s},$$
$$\varPhi(s, x; \operatorname{sym}^m u_j) = L(s, \operatorname{sym}^m u_j) M_x(s, \operatorname{sym}^m u_j),$$

for $\Re s > 15/16$. By Lemma 2.7, we have

$$1 = (1 - \varPhi(s, x; \operatorname{sym}^m u_j)) + \varPhi(s, x; \operatorname{sym}^m u_j)$$

for $\Re s > 15/16$. Combining the above formula with (2.8), we have

$$\begin{aligned} e^{-1/y} = \frac{1}{2\pi i} &\int_{(\kappa_1)} (1 - \varPhi(\rho + \omega, x; \operatorname{sym}^m u_j)) \varGamma(\omega) y^\omega \, d\omega \\ &+ \frac{1}{2\pi i} \int_{(\kappa_1)} \varPhi(\rho + \omega, x; \operatorname{sym}^m u_j) \varGamma(\omega) y^\omega \, d\omega. \end{aligned}$$

Since the zero of $L(\rho + \omega, \operatorname{sym}^m u_j)$ cancels the simple pole of $\varGamma(\omega)$ at $\omega = 0$, we shift the line of integration of the second integral in the above formula to $\Re \omega = \kappa_2$ without introducing extra terms. Hence we obtain

$$\begin{aligned} e^{-1/y} = \frac{1}{2\pi i} &\int_{(\kappa_1)} (1 - \varPhi(\rho + \omega, x; \operatorname{sym}^m u_j)) \varGamma(\omega) y^\omega \, d\omega \\ &+ \frac{1}{2\pi i} \int_{(\kappa_2)} \varPhi(\rho + \omega, x; \operatorname{sym}^m u_j) \varGamma(\omega) y^\omega \, d\omega. \end{aligned} \tag{3.1}$$

By Lemma 2.3, for $\Re\omega = \kappa_2$ we have

$$L(\rho+\omega, \mathrm{sym}^m u_j) \ll (T + H + |\Im\omega|)^{(m+1)/4+\epsilon} \tag{3.2}$$

and by Lemma 2.7, Cauchy's inequality, (2.4) and Lemma 2.3 again, for $\Re\omega = \kappa_2$ we have

$$\begin{aligned} M_x(\rho+\omega, \mathrm{sym}^m u_j) &\ll_\epsilon \Big(\sum_{\ell_1 \le x} \lambda_{\mathrm{sym}^m u_j}(\ell_1)^2 \ell_1^{-1-\epsilon}\Big)^{1/2} \Big(\sum_{\ell_2 \le x} \ell_2^\epsilon\Big)^{1/2} \\ &\ll_\epsilon x^{(1+\epsilon)/2} L(1+\epsilon, \mathrm{sym}^m u_j \times \mathrm{sym}^m u_j)^{1/2} \ll_\epsilon x^{1/2+\epsilon}. \end{aligned} \tag{3.3}$$

Thus the contribution of $|\Im\omega| \ge (\log T)^3$ to the second integral of (3.1) is

$$\begin{aligned} &\ll_\epsilon x^{1/2+\epsilon} y^{1/2-\alpha} \int_{\omega| \ge (\log T)^3} (T + H + |\Im\omega|)^{(m+1)/4+\epsilon} |\Gamma(\omega)|\,|d\omega| \\ &\ll_\epsilon x^{1/2+\epsilon} y^{1/2-\alpha} (T+H)^{(m+1)/4+\epsilon} e^{-(\log T)^3} \ll_{\epsilon,r} 1/T \end{aligned} \tag{3.4}$$

by our assumption that $H \le T^r$.

By Lemma 2.7, for $\Re\omega = \kappa_1 = 1 - \beta + \kappa$ and $x \ge 1$, we have

$$\begin{aligned} 1 - \varPhi(\rho+\omega, x; \mathrm{sym}^m u_j) &= L(\rho+\omega, \mathrm{sym}^m u_j) G_j(\rho+\omega) \sum_{\ell > x,\, (\ell, P(z))=1} \frac{\mu(\ell)\lambda_{\mathrm{sym}^m u_j}(\ell)}{\ell^{\rho+\omega}} \\ &\ll_\epsilon \Big(\sum_{\ell=1}^{\infty} \frac{|\lambda_{\mathrm{sym}^m u_j}(\ell)|}{\ell^{1+\kappa}}\Big)^2. \end{aligned}$$

By Cauchy's inequality, (2.4) and Corollary 2.4, the above term is

$$\ll_\epsilon L(1+\kappa, \mathrm{sym}^m u_j \times \mathrm{sym}^m u_j)\zeta(1+\kappa) \ll_\epsilon T^{((m+1)^2+1)\epsilon}. \tag{3.5}$$

Therefore, as in (3.4), the contribution of $|\Im\omega| \ge (\log T)^3$ to the first integral of (3.1) is $\ll_{\epsilon,r} 1/T$. Inserting this upper bound and (3.4) into (3.1) and applying the fact that

$$1 \le C(a+b) \ \Rightarrow\ 1 \le 2C^2(a+b^2) \qquad \text{for } a>0,\, b>0,\, C\ge 1$$

and Cauchy–Schwarz's inequality, we have

$$\begin{aligned} 1 &\ll T^\epsilon y^{2(1-\alpha)} \int_{-K}^{K} |1 - \varPhi(1+\kappa+i(\gamma+\nu), x; \mathrm{sym}^m u_j)|^2\, d\nu \\ &\quad + y^{15/16-\alpha} \int_{-K}^{K} |\varPhi(15/16+\epsilon+i(\gamma+\nu), x; \mathrm{sym}^m u_j)|\, d\nu, \end{aligned}$$

where $K = (\log T)^3$. Here we have used the fact that $1 \ll e^{-1/y}$.

Now we label the boxes from bottom to top by consecutive natural numbers and split them into two groups depending on the parity of their indices.

This ensures that two zeros from different boxes in the same group are a distance at least $2(\log T)^3$ apart. Therefore, the number of boxes in any group which contain at least one zero is

$$\begin{aligned}&\ll T^{\epsilon} y^{2(1-\alpha)} \int_0^{2H} |1-\Phi(1+\kappa+i\nu,x;\operatorname{sym}^m u_j)|^2\,d\nu\\&\quad+y^{16/17-\alpha}\int_0^{2H}|\Phi(15/16+\epsilon+i\nu,x;\operatorname{sym}^m u_j)|\,d\nu\\&=: T^{\epsilon}(y^{2(1-\alpha)}I'_{\operatorname{sym}^m u_j}+y^{15/16-\alpha}I''_{\operatorname{sym}^m u_j}).\end{aligned}$$

This implies that

$$n_{\operatorname{sym}^m u_j} \ll T^{\epsilon}(y^{2(1-\alpha)}I'_{\operatorname{sym}^m u_j}+y^{15/16-\alpha}I''_{\operatorname{sym}^m u_j}). \tag{3.6}$$

By (3.2) and (3.3), for $H \le T^r$ we have

$$I''_{\operatorname{sym}^m u_j} \ll_{\epsilon,r} Hx^{1/2+\epsilon}T^{r(m+1)/4+r\epsilon}. \tag{3.7}$$

To deal with $I'_{\operatorname{sym}^m u_j}$, we apply Lemma 2.7 and Corollary 2.4 to obtain

$$\begin{aligned}&1-\Phi(1+\kappa+i\nu,x;\operatorname{sym}^m u_j)\\&\quad\ll_\epsilon \Big|L(1+\kappa+i\nu,\operatorname{sym}^m u_j)\sum_{\ell>x,\,(\ell,P(z))=1}\frac{\mu(\ell)\lambda_{\operatorname{sym}^m u_j}(\ell)}{\ell^{1+\kappa+i\nu}}\Big|\\&\quad\ll_\epsilon T^{(m+1)\epsilon}\Big|\sum_{\substack{x<\ell\le X\\(\ell,P(z))=1}}\frac{\mu(\ell)\lambda_{\operatorname{sym}^m u_j}(\ell)}{\ell^{1+\kappa+i\nu}}\Big|+T^{\epsilon}\Big|\sum_{\substack{\ell>X\\(\ell,P(z))=1}}\frac{\mu(\ell)\lambda_{\operatorname{sym}^m u_j}(\ell)}{\ell^{1+\kappa+i\nu}}\Big|,\end{aligned}$$

where $X=e^{4(\log T)^3}$. As in (3.5), the second term is

$$\ll_{\epsilon,m} T^{\epsilon}X^{-\kappa/2}(L(1+\kappa/2,\operatorname{sym}^m u_j\times\operatorname{sym}^m u_j))^{1/2}(\zeta(1+\kappa/2))^{1/2}\ll_{\epsilon,m}T^{-10}.$$

This implies that by Cauchy–Schwarz's inequality,

$$\begin{aligned}\sum_{t_j\le T}I'_{\operatorname{sym}^m u_j}&\ll_\epsilon T^{\epsilon}\int_0^{2H}\sum_{t_j\le T}\Big|\sum_{\substack{x<\ell\le X\\(\ell,P(z))=1}}\frac{\mu(\ell)\lambda_{\operatorname{sym}^m u_j}(\ell)}{\ell^{1+\kappa+i\nu}}\Big|^2 d\nu+H\\&\ll_\epsilon T^{\epsilon}\sum_{n=0}^{[(\log X)/\log 2]}\int_0^{2H}\sum_{t_j\le T}\Big|\sum_{\substack{2^n x<\ell\le 2^{n+1}x\\(\ell,P(z))=1}}\frac{\mu(\ell)\lambda_{\operatorname{sym}^m u_j}(\ell)}{\ell^{1+\kappa+i\nu}}\Big|^2 d\nu+H.\end{aligned}\tag{3.8}$$

For $\ell\in\mathbb{N}$, let $a_\ell=\mu(\ell)\ell^{-(1+\kappa+i\nu)}$ for $2^n x<\ell\le 2^{n+1}x$ with $(\ell,P(z))=1$ and 0 otherwise. Then by Lemma 2.5,

$$\sum_{t_j\le T}\Bigg|\sum_{\substack{2^n x<\ell\le 2^{n+1}x\\ (\ell,P(z))=1}}\frac{\mu(\ell)\lambda_{\mathrm{sym}^m u_j}(\ell)}{\ell^{1+\kappa+i\nu}}\Bigg|^2 \ll T^{\epsilon}(2^n x+T^{(m+1)^2/32+2}(2^n x)^{15/16+\epsilon})(2^n x)^{-1-2\kappa}.$$

After inserting the above formula into (3.8), a direct calculation leads to

$$\sum_{t_j\le T} I'_{\mathrm{sym}^m u_j}\ll_{\epsilon,r} T^{\epsilon}H(1+T^{(m+1)^2/32+2}x^{-1/16}).$$

Combining this with (3.6) and (3.7), we obtain

$$\sum_{t_j\le T} n_{\mathrm{sym}^m u_j} \ll_{\epsilon,r} T^{r\epsilon}x^{\epsilon}H\big(y^{2(1-\alpha)}(1+T^{(m+1)^2/32+2}x^{-1/16})+y^{15/16-\alpha}x^{1/2}T^{r(m+1)/4}\big).$$

Taking $x=T^{(m+1)^2/2+32}$ and $y=T^{\frac{4((m+1)(m+r+1)+64)}{17-16\alpha}}$ completes the proof of Theorem 1.1 with $\epsilon/(r+(m+1)^2/2+32)$ in place of ϵ.

4. Further preparation. This section contains preparatory material for the proof of Theorem 1.4 to be carried out in Section 5.

As mentioned at the beginning, the generalized Ramanujan conjecture for primitive Maass forms is out of reach. Then it is natural to consider the number of "exceptional" Hecke eigenvalues whose absolute values are greater than 2. In this direction, the first result was due to Sarnak [19].

Lemma 4.1 (Sarnak). *Let p be a fixed prime. Then*

$$\#\{t_j\le T:|\lambda_j(p)|\ge a\ge 2\}\ll T^{2-\log(a/2)/\log p},$$

where the implied constant is absolute.

In the thesis [24], the author generalized a weaker form of Sarnak's result (see [24, Lemma 3.4.2]).

Lemma 4.2. *Let $1\le G\le T$. For any prime p and any positive real number $a\ge 2$, define*

$$E(a;T,G,p)=\{T-G\le t_j\le T+G:\lambda_j(p)\ge a\ge 2\}.$$

Then

$$|E(a;T,G,p))|\ll (GT)^{1-\frac{\kappa_0\log(a/2)}{2\log p}}, \tag{4.1}$$

where $\kappa_0=11/155$ and the implied constant is absolute.

To prove Theorem 1.4, we also need a more general result:

PROPOSITION 4.3. *Let $m = 1, 2, 3, 4$. There is a positive constant $c_0 = c_0(\epsilon)$ such that uniformly for $\epsilon \log T \le z \le (\log T)^{10}$, we have*

$$L(1, \mathrm{sym}^m u_j) = \left\{1 + O\left(\frac{1}{\log_2 T}\right)\right\} \prod_{p \le z} \prod_{0 \le k \le m} \left(1 - \frac{\alpha_{u_j}(p)^{m-2k}}{p}\right)^{-1}$$

for all but $O_\epsilon\big(GT \exp\{-\frac{c_0 \log T}{\log_2 T} \log\big(\frac{2z}{\epsilon \log T}\big)\}\big)$ of the t_j's in $H(G,T)$. The implied constant in the O-term is absolute.

Before we prove Proposition 4.3, we need to establish two preliminary lemmas.

The first lemma is an analogue of [13, Lemma 3.1] with a similar proof. To begin with, we introduce some notation. Define, for $m = 1, 2, 3, 4$,

$$\Lambda_{\mathrm{sym}^m u_j}(n) = \begin{cases} [\alpha_{u_j}(p)^{m\nu} + \alpha_{u_j}(p)^{(m-2)\nu} + \cdots + \alpha_{u_j}(p)^{-m\nu}] \log p & \text{if } n = p^\nu, \\ 0 & \text{otherwise.} \end{cases}$$

By the Cauchy inequality and (2.4), we have

$$\sum_p \frac{\lambda_{\mathrm{sym}^m u_j}(p) \log p}{p^s} \le \left(\sum_p \frac{\lambda_{\mathrm{sym}^m u_j \times \mathrm{sym}^m u_j}(p)}{p^\sigma}\right)^{1/2} \left(\sum_p \frac{(\log p)^2}{p^\sigma}\right)^{1/2}, \tag{4.2}$$

which implies that the infinite sum on the left-hand side converges absolutely for $\sigma > 1$. Hence we can define

$$L(s) = \sum_{n=1}^{\infty} \frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n^s} = \sum_p \frac{\lambda_{\mathrm{sym}^m u_j}(p) \log p}{p^s} + O(1)$$

for $\sigma > 1$. Suppose $t_j \in H^+_{T,\mathrm{sym}^m}(\eta)$. Then we know that $L(s)$ is holomorphic and zero-free in the region $\mathcal{S}$. Therefore, we can define the logarithm $\log L(s, \mathrm{sym}^m u_j)$ in $\mathcal{S}$ by the integral of $L(s)$ from 2 to $s \in \mathcal{S}$ with the initial value taken as the usual natural logarithm of $L(2, \mathrm{sym}^m u_j)$. In particular, we have the absolutely convergent series

$$\log L(s, \mathrm{sym}^m u_j) = \sum_{n=2}^{\infty} \frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n^s \log n} \qquad (\sigma > 1).$$

Similarly to (4.2), by Lemma 2.3, we obtain a rather crude estimate for $\sigma > 1$,

$$\begin{aligned} |\log L(s, \mathrm{sym}^m u_j)| &\ll_{\epsilon,m} |L(\sigma, \mathrm{sym}^m u_j \times \mathrm{sym}^m u_j)|^{1/2} \zeta(\sigma)^{1/2} \\ &\ll_{\epsilon,m} T^\epsilon (\sigma - 1). \end{aligned}$$

set $\sigma_0 = 1 - \eta$. Combining the Borel–Carathéodory theorem [23, §5.5] with Lemma 2.3, we find that for $\sigma > \sigma_0$ and $|t| \le 100T^\eta$,

$$\log L(s, \operatorname{sym}^m u_j) \ll \frac{\log T}{\sigma - \sigma_0}, \tag{4.3}$$

where the implied constant is absolute.

LEMMA 4.4. *Let $m = 1, 2, 3, 4$, $\eta \in (1, 1/1000)$ and $x = T^{4\eta/(4+\eta)}$. Let $H^+_{T,\operatorname{sym}^m}(\eta)$ be defined as in Section 6.1. Then for any $t_j \in H^+_{T,\operatorname{sym}^m}(\eta)$,*

$$\log L(1, \operatorname{sym}^m u_j) = \sum_{p \le x} \sum_{0 \le k \le m} \log\left(1 - \frac{\alpha_{u_j}(p)^{m-2k}}{p}\right)^{-1} + O(T^{-\eta^2/(4+\eta)+\epsilon}).$$

The implied constant depends on ϵ, η and m only.

Proof. Let $t_j \in H^+_{T,\operatorname{sym}^m}(\eta)$, $H \ge 1$ and $x \ge 1$. By the Perron formula (see [21, Theorem II.2.2]),

$$\sum_{2 \le n \le x} \frac{\Lambda_{\operatorname{sym}^m u_j}(n)}{n \log n} = \frac{1}{2\pi i} \int_{1/\log x - iH}^{1/\log x + iH} \log L(s+1, \operatorname{sym}^m u_j) \frac{x^s}{s}\, ds + O\biggl(\sum_{n \ge 1} \frac{|\Lambda_{\operatorname{sym}^m u_j}(n)/\log n|}{n^{1+1/\log x}(1 + H|\log(x/n)|)}\biggr).$$

If $n \ge 3x/2$ or $n \le x/2$, then by the Cauchy–Schwarz inequality, (2.4) and Corollary 2.4, the sum in the error term is

$$\begin{aligned}
&\ll \sum_{p \le x/2} \frac{|\lambda_{\operatorname{sym}^m u_j}(p)|}{Hp^{1+1/\log x}} + \sum_{p \ge 3x/2} \frac{|\lambda_{\operatorname{sym}^m u_j}(p)|}{Hp^{1+1/\log x}} \\
&\ll \frac{1}{H}\biggl(\sum_{n=1}^{\infty} \frac{1}{n^{1+1/\log x}}\biggr)^{1/2} \biggl(\sum_{n=1}^{\infty} \frac{|\lambda_{\operatorname{sym}^m u_j \times \operatorname{sym}^m u_j}(n)|}{n^{1+1/\log x}}\biggr)^{1/2} \ll \frac{x^\epsilon T^\epsilon}{H}.
\end{aligned}$$

If $x/2 < n < 3x/2$ and $n \ne x$, the sum in the error term is

$$\begin{aligned}
&\ll \sum_{\substack{x/2 \le p \le 3x/2 \\ p \ne x}} \frac{|\lambda_{\operatorname{sym}^m u_j}(p)|}{p + H|p - x|} \ll x^{7/16} \sum_{\substack{x/2 \le p \le 3x/2 \\ p \ne x}} \frac{1}{p + H|p - x|} \\
&\ll x^{7/16}\left(\frac{1}{x} + \frac{1}{H}\right).
\end{aligned}$$

By (4.3), $\log L(s, \operatorname{sym}^m u_j) \ll_\eta \log T$ for $\sigma \ge 1 - \eta/4$ and $|t| \le 100T^\eta$. We move the line of integration to $\sigma = -\eta/4$ and we have

$$\sum_{2\le n\le x}\frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n\log n}$$
$$=\log L(1,\mathrm{sym}^m u_j)+O\bigg(\frac{x^{7/16+\epsilon}T^\epsilon}{H}+\frac{\log T}{H}+\frac{(\log T)(\log H)}{x^{\eta/4}}\bigg).$$

If we take $H=T^\eta$ and $x=T^{4\eta/(4+\eta)}$, the error term is $\ll T^{-\eta^2/(4+\eta)+\epsilon}$.

On the other hand, note that

$$\sum_{p\le\sqrt{x}}\sum_{\nu>(\log x)/\log p}\frac{|\alpha_{u_j}(p)|^{m\nu}}{\nu p^\nu}\ll\sum_{p\le\sqrt{x}}\sum_{\nu>(\log x)/\log p}\frac{1}{\nu p^{9\nu/16+\epsilon}}\ll x^{-1/16},$$
$$\sum_{\sqrt{x}<p\le x}\sum_{\nu>(\log x)/\log p}\frac{|\alpha_{u_j}(p)|^{m\nu}}{\nu p^\nu}\ll\sum_{\sqrt{x}<p\le x}\sum_{\nu\ge2}\frac{1}{\nu p^{9\nu/16+\epsilon}}\ll x^{-1/16}.$$

We have

$$\sum_{2\le n\le x}\frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n\log n}=\sum_{p\le x}\sum_{0\le k\le m}\sum_{\nu\le(\log x)/\log p}\frac{\alpha_{u_j}(p)^{(m-2k)\nu}}{\nu p^\nu}$$
$$=\sum_{p\le x}\sum_{0\le k\le m}\bigg(\log\bigg(1-\frac{\alpha_{u_j}(p)^{m-2k}}{p}\bigg)^{-1}-\sum_{\nu>(\log x)/\log p}\frac{\alpha_{u_j}(p)^{(m-2k)\nu}}{\nu p^\nu}\bigg)$$
$$=\sum_{p\le x}\sum_{0\le k\le m}\log\bigg(1-\frac{\alpha_{u_j}(p)^{m-2k}}{p}\bigg)^{-1}+O(x^{-1/16}).$$

This completes the proof of Lemma 4.4. ∎

To prove the second preliminary lemma, we need the following large sieve inequality (see [24, Theorem 7.1.1]).

Proposition 4.5. *Let $1\le G\le T$ and*

$$H(G,T)=\{t_j:T-G\le t_j\le T+G\}.$$

Let $\nu\ge1$ be a fixed integer and $\{b_p\}_p$ be a sequence of real numbers indexed by prime numbers such that $|b_p|\le B$ for some constant B and for all primes p. Then

$$\sum_{t_j\in H(G,T)}\bigg|\sum_{P<p\le Q}\frac{b_p\lambda_j(p^\nu)}{p}\bigg|^{2k}\tag{4.4}$$
$$\ll GT\bigg(\frac{96B^2(\nu+1)^2k}{P\log P}\bigg)^k\bigg\{1+\bigg(\frac{40k\log P}{P}\bigg)^{k/3}\bigg\}+F(G,T,B,P,\nu,k)$$

uniformly for $B>0$, $k\ge1$ and $2\le P<Q\le2P$. Here

$$F(G,T,B,P,\nu,k)=\begin{cases} G^{\frac{112}{155}}T^{\frac{144}{155}+\epsilon}\left(\dfrac{10BQ^{\nu/4+\epsilon}}{\log P}\right)^{2k} & \text{if } 1\le G\le T^{1-\epsilon},\\ T^{\frac{299}{155}+\epsilon}\left(\dfrac{10BQ^{43\nu/620+\epsilon}}{\log P}\right)^{2k} & \text{if } T^{1-\epsilon}\le G\le T,\end{cases}$$

and the implied constant depends on ν and ϵ only.

Applying Proposition 4.5, we obtain the following lemma.

Lemma 4.6. *Let $\nu=1,2,3,4$ and $\eta\in(0,1/10^{10}]$ be fixed.*

(i) *Define*

$$E_\nu^1(P,Q)=\left\{t_j\in H(G,T):\left|\sum_{P<p\le Q}\frac{\lambda_j(p^\nu)}{p}\right|>\frac{10(\nu+1)}{(\log T)(\log P)}\right\}. \tag{4.5}$$

We have

$$|E_\nu^1(P,Q)|\ll_\nu GT^{1-1/(150\nu)}$$

for

$$(\log T)^{10}\le P\le Q\le 2P\le T^\eta. \tag{4.6}$$

The implied constant depends on ν at most.

(ii) *Let*

$$E_\nu^2(P,Q)=\left\{t_j\in H(G,T):\left|\sum_{P<p\le Q}\frac{\lambda_j(p^\nu)}{p}\right|>\left(\frac{96(\nu+1)^2z}{(\log_2 T)^2P}\right)^{1/2}\right\}. \tag{4.7}$$

There is a positive constant $c_0(\epsilon,\nu)$ such that if

$$\epsilon\log T\le z\le P\le Q\le 2P\le(\log T)^{10}, \tag{4.8}$$

then

$$|E_\nu^2(P,Q)|\ll_{\nu,\epsilon} GT\exp\left\{-c_0(\epsilon,\nu)\frac{\log T}{\log_2 T}\log\left(\frac{2z}{\epsilon\log T}\right)\right\},$$

where the implied constant depends on ϵ and ν at most.

Proof. We can assume that $T\ge T_0$ ($T_0=e^{(200\nu)^2}$ for assertion (i) and $T_0=e^{(200\nu/\epsilon)^2}$ for assertion (ii)), as the remaining cases are trivial. We shall apply (4.4) with $b_p=1$, $B=1$ and

$$k=\begin{cases}\left[\dfrac{\log T}{100\nu\log P}\right] & \text{if (4.5) holds,}\\ \left[\dfrac{\epsilon\log T}{100\nu\log_2 T}\right] & \text{if (4.8) holds,}\end{cases}$$

to count $|E_\nu^1(P,Q)|$ and $|E_\nu^2(P,Q)|$. The right-hand side of (4.4) leads to

$$GT\left\{\left(\frac{96(\nu+1)^2k}{P\log P}\right)^k+T^{-\frac{11}{155}+\epsilon}Q^{k\nu/2+\epsilon}\right\}. \tag{4.9}$$

By (4.5), (4.6) and (4.9), we have

$$|E_\nu^1(P,Q)| \ll GT\bigg\{\bigg(\frac{96(\nu+1)^2k}{P\log P}\bigg)^k + T^{-\frac{11}{155}+\epsilon}Q^{k\nu/2+\epsilon}\bigg\}\bigg(\frac{\log^2 T(\log P)^2}{100(\nu+1)^2}\bigg)^k$$
$$\ll GT\bigg\{\bigg(\frac{(\log T)^3}{P}\bigg)^k + T^{-\frac{11}{155}+\epsilon}Q^{k\nu+\epsilon}\bigg\} \ll T^{1-1/(150\nu)}G.$$

If (4.8) holds, we have $\log P \ge \frac{1}{2}\log_2 T$ and $z \ge (\log_2 T)^2$. Then

$$|E_\nu^2(P,Q)| \ll GT\bigg\{\bigg(\frac{96(\nu+1)^2k}{P\log P}\bigg)^k + T^{-\frac{11}{155}+\epsilon}Q^{k\nu/2+\epsilon}\bigg\}\bigg(\frac{(\log_2 T)^2 P}{96(\nu+1)^2 z}\bigg)^k$$
$$\ll GT\bigg\{\bigg(\frac{k\log_2 T}{z}\bigg)^k + T^{-\frac{11}{155}+\epsilon}Q^{k\nu+\epsilon}\bigg\}$$
$$\ll GT\bigg\{\bigg(\frac{\epsilon\log T}{2z}\bigg)^k + T^{-\frac{11}{155}+\epsilon}Q^{k\nu+\epsilon}\bigg\}$$
$$\ll GT\exp\bigg\{-\frac{\epsilon\log T}{101\nu\log_2 T}\log\bigg(\frac{2z}{\epsilon\log T}\bigg)\bigg\}.$$

The proof is complete. ■

Now we prove Proposition 4.3.

Let $s=\sigma+i\tau$ and $m=1,2,3,4$. For any $\eta\in(0,1/10^{10}]$, we define

$$H^+_{G,T;\mathrm{sym}^m}(\eta) = \{t_i \in H(G,T) : L(s,\mathrm{sym}^m u_j)\neq 0,\ s\in\mathcal{S}\},$$

where $\mathcal{S}=\{s:\sigma\ge 1-\eta,\ |\tau|\le 100T^\eta\}\cup\{s:\sigma\ge 1\}$. Then we define

$$H^-_{G,T;\mathrm{sym}^m}(\eta) = H(G,T)\setminus H^+_{G,T;\mathrm{sym}^m}(\eta).$$

By Theorem 1.1, we have $|H^-_{G,T;\mathrm{sym}^m}(\eta)| \ll_\eta T^{1000\eta}$. Define

$$y_0 = T^{4\eta/(4+\eta)},\qquad y_1=(\log T)^{10},\qquad y_2=\epsilon\log T.$$

By Lemma 4.4, for any $t_j\in H^+_{G,T;\mathrm{sym}^m}(\eta)$, we have

$$\log L(s,\mathrm{sym}^m u_j) = \sum_{p\le y_0}\sum_{0\le k\le m}\log\bigg(1-\frac{\alpha_{u_j}(p)^{m-2k}}{p}\bigg)^{-1} + O(T^{-\eta^2/(4+\eta)+\epsilon}).$$

Then, for each $t_j\in H^+_{G,T;\mathrm{sym}^m}(\eta)$, we truncate the sum in the above formula to

$$\sum_{p\le y_1}\sum_{0\le k\le m}\log\bigg(1-\frac{\alpha_{u_j}(p)^{m-2k}}{p}\bigg)^{-1} + R_1(\mathrm{sym}^m u_j),$$

where

$$\begin{aligned}
R_1(\operatorname{sym}^m u_j) &= \sum_{y_1<p\le y_0}\sum_{0\le k\le m}\log\Big(1-\frac{\alpha_{u_j}(p)^{m-2k}}{p}\Big)^{-1}\\
&= \sum_{y_1<p\le y_0}\sum_{0\le k\le m}\sum_{\ell=1}^{\infty}\frac{\alpha_{u_j}(p)^{(m-2k)\ell}}{\ell p^{\ell}}\\
&= \sum_{y_1<p\le y_0}\sum_{0\le k\le m}\frac{\alpha_{u_j}(p)^{m-2k}}{p}+\sum_{y_1<p\le y_0}\sum_{\ell=2}^{\infty}\sum_{0\le k\le m}\frac{\alpha_{u_j}(p)^{(m-2k)\ell}}{\ell p^{\ell}}\\
&= \sum_{y_1<p\le y_0}\frac{\lambda_j(p^m)}{p}+O(y_1^{-1/8}).
\end{aligned}$$

Here we have used (1.1). To deal with the last sum, we divide it dyadically. Define

$$\begin{gathered}
P_\ell(y_1)=2^{\ell-1}y_1,\qquad Q_\ell(y_1,y_0)=\min\{2^{\ell}y_1,y_0\},\\
E_m^1=H^-_{G,T;\operatorname{sym}^m}(\eta)\cup\bigcup_{\ell}E_m^1(P_\ell(y_1),Q_\ell(y_1,y_0)),
\end{gathered}$$

where $E_m^1(P,Q)$ is defined as in (4.5). There are at most $(\log y_0)/\log 2+1$ values of ℓ which occur in the union. By Lemma 4.6(i), we have

$$\begin{aligned}
|E_m^1| &\ll T^{1000\eta}+\sum_{\ell}|E_m^1(P_\ell(y_1),Q_\ell(y_1,y_0))| \\
&\ll T^{1000\eta}+GT^{1-1/(150m)}\log T\ll GT^{1-1/(151m)}.
\end{aligned}\tag{4.10}$$

For all $t_j\in H(G,T)\setminus E_m^1$, we have

$$\begin{aligned}
R_1(\operatorname{sym}^m u_j) &\ll \sum_{\ell}\Bigg|\sum_{P_\ell(y_1)<p\le Q_\ell(y_1,y_0)}\frac{\lambda_j(p^m)}{p}\Bigg|+y_1^{-1/8}\\
&\ll \sum_{\ell}\frac{10(m+1)}{(\log T)(\log P_\ell(y_1))}+y_1^{-1/8}\ll\frac{\log_2 T}{\log T}.
\end{aligned}$$

Therefore, for all $t_j\in H(G,T)\setminus E_m^1$,

$$\log L(s,\operatorname{sym}^m u_j)=\sum_{p\le y_1}\sum_{0\le k\le m}\log\Big(1-\frac{\alpha_{u_j}(p)^{m-2k}}{p}\Big)^{-1}+O(\frac{1}{\log_2 T}),$$

where the implied constant is absolute.

Finally, we consider $y_2\le z\le y_1$. Similarly, it remains to evaluate

$$R_2(\operatorname{sym}^m u_j)=\sum_{z<p\le y_1}\sum_{0\le k\le m}\log\Big(1-\frac{\alpha_{u_j}(p)^{m-2k}}{p}\Big)^{-1}.$$

Set

$$E_m^2(z) = E_m^1 \cup \bigcup_{\ell} E_m^2(P_\ell(y_1), Q_\ell(y_1, y_0); z),$$

where $E_m^2(P, Q; z)$ is defined as in (4.7). Here the number of sets in the union over ℓ is at most $(\log y_1)/\log 2 + 1 \ll \log_2 T$. By (4.10) and Lemma 4.6(ii), we have

$$\begin{aligned}|E_m^2(z)| &\ll_\epsilon GT^{1-1/(151m)} + GT \exp\biggl\{-2c_0 \frac{\log T}{\log_2 T} \log\biggl(\frac{2z}{\epsilon \log T}\biggr)\biggr\} \log_2 T \\ &\ll_\epsilon GT \exp\biggl\{-c_0 \frac{\log T}{\log_2 T} \log\biggl(\frac{2z}{\epsilon \log T}\biggr)\biggr\},\end{aligned}$$

where $c_0 = c_0(\epsilon)$ is a positive constant depending on ϵ.

For all $t_j \in H(G, T) \setminus E_m^2(z)$, we have

$$\begin{aligned}R_2(\operatorname{sym}^m u_j) &\ll \sum_\ell \biggl| \sum_{P_\ell(z) < p \le Q_\ell(z, y_1)} \frac{\lambda_j(p^m)}{p} \biggr| + z^{-1/8} \\ &\ll \sum_\ell \biggl(\frac{z}{P_\ell(z)(\log_2 T)^2}\biggr)^{1/2} + z^{-1/8} \ll \sum_\ell \frac{1}{2^{\ell/2} \log_2 T} + z^{-1/8} \ll \frac{1}{\log_2 T}.\end{aligned}$$

Hence

$$\log L(s, \operatorname{sym}^m u_j) = \sum_{p \le z} \sum_{0 \le k \le m} \log\biggl(1 - \frac{\alpha_{u_j}(p)^{m-2k}}{p}\biggr)^{-1} + O\biggl(\frac{1}{\log_2 T}\biggr)$$

for all $t_j \in H(G, T) \setminus E_m^2(z)$, where the implied constant is absolute. The proof of Proposition 4.3 is complete.

5. Proof of Theorem 1.4. Assume that $|\alpha_{u_j}(p)| \le \alpha'$, where $\alpha' > 1$ is a parameter to be determined later. Then $|\lambda_j(p)| \le \alpha' + (\alpha')^{-1} =: a$. By (4.1), we have

$$\biggl|\bigcup_{p \le z} E(a; T, G, p)\biggr| \ll (GT)^{1 - \frac{\kappa_0 \log(a/2)}{2 \log z}} z \ll (GT)^{1 - \frac{\kappa_0 \log(a/2)}{3 \log_2 T}}. \tag{5.1}$$

For $t_j \in H(G, T) \setminus (E_m^2(z) \cup \bigcup_{p \le z} |E(a; T, G, p)|)$, we have

$$\begin{aligned}L(1, \operatorname{sym}^m u_j) &\le \biggl\{1 + O\biggl(\frac{1}{\log_2 T}\biggr)\biggr\} \prod_{p \le z} \biggl(1 - \frac{\alpha'}{p}\biggr)^{-m-1} \\ &\le \biggl\{1 + O\biggl(\frac{1}{\log_2 T}\biggr)\biggr\} (e^\gamma \log z)^{\alpha'(m+1)} \\ &\le \bigl\{e^\gamma \bigl((e^{\gamma(1-1/\alpha')} \log z)^{\alpha'} + C_0 (\log_2 T)^{\alpha' - 1}\bigr)\bigr\}^{m+1},\end{aligned}$$

where C_0 is an absolute constant. We take $\alpha' = 1 + \{(\log_3 T) \log_4 T\}^{-1}$ and

$$z = e^{e^{-\gamma(1-1/\alpha')}(\log_2 T + r - C_0(\log_2 T)^{\alpha'-1})^{1/\alpha'}}$$
$$= e^{(1+O((\log_4 T)^{-1}))(\log_2 T + r - C_0(\log_2 T)^{\alpha'-1})}.$$

Noting (5.1) and taking $\epsilon = \epsilon e^{-C_0}$ proves the assertion of Theorem 1.4 for $F^+_{T,G}$.

Similarly, for $t_j \in H(G,T) \setminus (E^2_m(z) \cup \bigcup_{p\le z} |E(a;T,G,p)|)$, we obtain

$$L(1, \mathrm{sym}^m u_j) \ge \big\{B^-_m\big((B^-_m)^{1-1/\alpha'} \log z\big)^{\alpha'} + C'_0(\log_2 T)^{\alpha'-1}\big)\big\}^{-A^-_m},$$

where C'_0 is an absolute positive constant. We take

$$\alpha' = 1 + \{(\log_3 T)\log_4 T\}^{-1}$$

and

$$z = e^{(B^-_m)^{-(1-1/\alpha')}(\log_2 T + r - C'_0(\log_2 T)^{\alpha'-1})^{1/\alpha'}}$$
$$= e^{(1+O((\log_4 T)^{-1}))(\log_2 T + r - C_0(\log_2 T)^{\alpha'-1})}.$$

Noting (5.1) and taking $\epsilon = \epsilon e^{-C'_0}$ proves the assertion of Theorem 1.4 for $F^-_{T,G}$.

Appendix. Let $\Re s = 2$ and $H_p(s; \mathrm{sym}^m u_{j_1}, \mathrm{sym}^m u_{j_2})$ be the polynomial in p^{-s} defined as in Section 2.2 whose degree is not higjer than $(m+1)^2 - 1$, the coefficient of p^{-s} vanishes and the coefficients of p^{-ns}, $2 \le n \le (m+1)^2 - 1$, given by (2.10). Now, we use Wolfram Mathematica 7 to determine these coefficients.

Let $x = \alpha_{u_{j_1}}(p)$, $y = \alpha_{u_{j_2}}(p)$ and $z = p^{-s}$. We first consider the polynomial $H_p(s; \mathrm{sym}^3 u_{j_1}, \mathrm{sym}^3 u_{j_2})$. The procedure is as follows.

```
m = 3;    h = 25;
P1[x_, z_]:=Product[Sum[x^((m-2*i)*n) * z^n, {n, 0, h}], {i, 0, m}];
P2[y_, z_]:=Product[Sum[y^((m-2*j)*n) * z^n, {n, 0, h}], {j, 0, m}];
P3[x_, y_, z_]:=Product[1 - x^(m-2*i) * y^(m-2*j) * z, {i, 0, m}, {j, 0, m}];
a[n_, x_]:=Coefficient[P1[x, z], z, n];
b[n_, y_]:=Coefficient[P2[y, z], z, n];
c[n_, x_, y_]:=Coefficient[P3[x, y, z], z, n];
d[n_, x_, y_]:=Sum[a[n - t, x] * b[n - t, y]c[t, x, y], {t, 0, n}];
For[i = 0, i < 16, i + +, Print[Simplify[d[i, x, y]]]]
```

Here $a[n,x] = \lambda_{\mathrm{sym}^3 u_{j_1}}(p^n)$, $b[n,x] = \lambda_{\mathrm{sym}^3 u_{j_2}}(p^n)$, $c[n,x,y] = f_3^{(n)}(0)$ and $d[n,x,y]$ is the coefficient of p^{-ns} in $H_p(s; \mathrm{sym}^3 u_{j_1}, \mathrm{sym}^3 u_{j_2})$. The results are as follows:

$$d[0,x,y] = 1, \quad d[1,x,y] = 0,$$
$$d[2,x,y] = -x^{-4}y^{-4}(1 + x^2 + 2x^4 + x^6 + x^8)(1 + y^2 + 2y^4 + y^6 + y^8),$$

$$
\begin{aligned}
d[3,x,y] &= x^{-7}y^{-7}(1+x^2+x^4+x^6)(1+y^2+y^4+y^6)\\
&\quad \times (y^4+x^2y^4+x^6y^4+x^8y^4+x^4(1+y^2+2y^4+y^6+y^8)),\\
d[4,x,y] &= -x^{-10}y^{-10}\Big(y^{10}+2x^2y^{10}+2x^{18}y^{10}+x^{20}y^{10}\\
&\quad + x^4y^4(1+2y^2+3y^4+7y^6+3y^8+2y^{10}+y^{12})\\
&\quad + x^{16}y^4(1+2y^2+3y^4+7y^6+3y^8+2y^{10}+y^{12})\\
&\quad + x^6y^4(2+4y^2+6y^4+11y^6+6y^8+4y^{10}+2y^{12})\\
&\quad + x^{14}y^4(2+4y^2+6y^4+11y^6+6y^8+4y^{10}+2y^{12})\\
&\quad + x^8y^4(3+6y^2+9y^4+16y^6+9y^8+6y^{10}+3y^{12})\\
&\quad + x^{12}y^4(3+6y^2+9y^4+16y^6+9y^8+6y^{10}+3y^{12})\\
&\quad + x^{10}(1+2y^2+7y^4+11y^6+16y^8+23y^{10}+16y^{12}\\
&\qquad\qquad + 11y^{14}+7y^{16}+2y^{18}+y^{20})\Big),\\
d[5,x,y] &= x^{-9}y^{-9}(1+x^2+x^4+x^6)(1+y^2+y^4+y^6)\\
&\quad \times (y^6+x^2y^6+2x^4y^6+2x^8y^6+x^{10}y^6+x^{12}y^6\\
&\qquad\qquad + x^6(1+y^2+2y^4+2y^6+2y^8+y^{10}+y^{12})),\\
d[6,x,y] &= 0, \quad d[7,x,y]=-d[5,x,y], \quad d[8,x,y]=-d[4,x,y],\\
d[9,x,y] &= -d[3,x,y], \quad d[10,x,y]=-d[2,x,y], \quad d[11,x,y]=-d[1,x,y],\\
d[12,x,y] &= -d[0,x,y], \quad d[13,x,y]=d[14,x,y]=d[15,x,y]=0.
\end{aligned}
$$

Next, we consider $H_p(s;\mathrm{sym}^4 u_{j_1},\mathrm{sym}^4 u_{j_2})$. The procedure is as follows.

```
m = 4;      h = 25;
P1[x_, z_]:=Product[Sum[x^((m-2*i)*n) * z^n, {n, 0, h}], {i, 0, m}];
P2[y_, z_]:=Product[Sum[y^((m-2*j)*n) * z^n, {n, 0, h}], {j, 0, m}];
P3[x_, y_, z_]:=Product[1 - x^(m-2*i) * y^(m-2*j) * z, {i, 0, m}, {j, 0, m}];
a[n_, x_]:=Coefficient[P1[x, z], z, n];
b[n_, y_]:=Coefficient[P2[y, z], z, n];
c[n_, x_, y_]:=Coefficient[P3[x, y, z], z, n];
d[n_, x_, y_]:=Sum[a[n - t, x] * b[n - t, y]c[t, x, y], {t, 0, n}];
For[i = 0, i < 25, i + +, Print[Simplify[d[i, x, y]]]]
```

Here $a[n,x]=\lambda_{\mathrm{sym}^4 u_{j_1}}(p^n)$, $b[n,x]=\lambda_{\mathrm{sym}^4 u_{j_2}}(p^n)$, $c[n,x,y]=f_4^{(n)}(0)$ and $d[n,x,y]$ is the coefficient of p^{-ns} in $H_p(s;\mathrm{sym}^4 u_{j_1},\mathrm{sym}^4 u_{j_2})$. The results are as follows.

$$
\begin{aligned}
d[0,x,y] &= 1, \qquad d[1,x,y]=0,\\
d[2,x,y] &= -x^{-6}y^{-6}(1+x^2+2x^4+2x^6+2x^8+x^{10}+x^{12})\\
&\quad \times (1+y^2+2y^4+2y^6+2y^8+y^{10}+y^{12}),\\
d[3,x,y] &= x^{-10}y^{-10}(1+x^2+2x^4+2x^6+2x^8+x^{10}+x^{12})\\
&\quad \times (1+y^2+2y^4+2y^6+2y^8+y^{10}+y^{12})\\
&\quad \times (y^4+x^2y^4+x^6y^4+x^8y^4+x^4(1+y^2+y^6+y^8)),
\end{aligned}
$$

$$
\begin{aligned}
d[4,x,y] = {} & -x^{-14}y^{-14}(1+x^2+x^4+x^6+x^8)(1+y^2+y^4+y^6+y^8) \\
& \times (y^{10}+x^2y^{10}+x^{18}y^{10}+x^{20}y^{10}+(x^4y^4+x^6y^4+x^{14}y^4+x^{16}y^4) \\
& \times \big(1+y^2+2y^4+3y^6+2y^8+y^{10}+y^{12}) \\
& +(x^8y^4+x^{12}y^4)(2+2y^2+4y^4+5y^6+4y^8+2y^{10}+2y^{12}) \\
& +x^{10}(1+y^2+3y^4+3y^6+5y^8+7y^{10}+5y^{12}+3y^{14}+3y^{16}+y^{18}+y^{20})), \\
d[5,x,y] = {} & x^{-18}y^{-18}\Big(y^{18}+2x^2y^{18}+2x^{34}y^{18}+x^{36}y^{18} \\
& +5x^8y^{10}(1+2y^2+3y^4+4y^6+6y^8+4y^{10}+3y^{12}+2y^{14}+y^{16}) \\
& +5x^{28}y^{10}(1+2y^2+3y^4+4y^6+6y^8+4y^{10}+3y^{12}+2y^{14}+y^{16}) \\
& +x^4y^{10}(1+2y^2+3y^4+4y^6+8y^8+4y^{10}+3y^{12}+2y^{14}+y^{16}) \\
& +x^{32}y^{10}(1+2y^2+3y^4+4y^6+8y^8+4y^{10}+3y^{12}+2y^{14}+y^{16}) \\
& +x^6y^{10}(2+4y^2+6y^4+8y^6+15y^8+8y^{10}+6y^{12}+4y^{14}+2y^{16}) \\
& +x^{30}y^{10}(2+4y^2+6y^4+8y^6+15y^8+8y^{10}+6y^{12}+4y^{14}+2y^{16}) \\
& +x^{10}y^4\big(1+2y^2+5y^4+14y^6+25y^8+34y^{10}+44y^{12}+58y^{14} \\
& +44y^{16}+34y^{18}+25y^{20}+14y^{22}+5y^{24}+2y^{26}+y^{28}) \\
& +x^{26}y^4(1+2y^2+5y^4+14y^6+25y^8+34y^{10}+44y^{12}+58y^{14} \\
& \qquad +44y^{16}+34y^{18}+25y^{20}+14y^{22}+5y^{24}+2y^{26}+y^{28}) \\
& +x^{12}y^4(2+4y^2+10y^4+25y^6+44y^8+59y^{10}+76y^{12}+94y^{14} \\
& \qquad +76y^{16}+59y^{18}+44y^{20}+25y^{22}+10y^{24}+4y^{26}+2y^{28}) \\
& +x^{24}y^4(2+4y^2+10y^4+25y^6+44y^8+59y^{10}+76y^{12}+94y^{14} \\
& \qquad +76y^{16}+59y^{18}+44y^{20}+25y^{22}+10y^{24}+4y^{26}+2y^{28}) \\
& +x^{14}y^4(3+6y^2+15y^4+34y^6+59y^8+78y^{10}+100y^{12}+121y^{14} \\
& \qquad +100y^{16}+78y^{18}+59y^{20}+34y^{22}+15y^{24}+6y^{26}+3y^{28}) \\
& +x^{22}y^4(3+6y^2+15y^4+34y^6+59y^8+78y^{10}+100y^{12}+121y^{14} \\
& \qquad +100y^{16}+78y^{18}+59y^{20}+34y^{22}+15y^{24}+6y^{26}+3y^{28}) \\
& +x^{16}y^4(4+8y^2+20y^4+44y^6+76y^8+100y^{10}+128y^{12}+151y^{14} \\
& \qquad +128y^{16}+100y^{18}+76y^{20}+44y^{22}+20y^{24}+8y^{26}+4y^{28}) \\
& +x^{20}y^4(4+8y^2+20y^4+44y^6+76y^8+100y^{10}+128y^{12}+151y^{14} \\
& \qquad +128y^{16}+100y^{18}+76y^{20}+44y^{22}+20y^{24}+8y^{26}+4y^{28}) \\
& +x^{18}(1+y^2)^2(1+7y^4+y^6+21y^8+15y^{10}+43y^{12}+20y^{14}+68y^{16} \\
& \qquad +20y^{18}+43y^{20}+15y^{22}+21y^{24}+y^{26}+7y^{28}+y^{32})\Big), \\
d[6,x,y] = {} & -x^{-18}y^{-18}(1+x^2+x^4+x^6+x^8)(1+y^2+y^4+y^6+y^8) \\
& \times \Big(y^{14}+x^2y^{14}+3x^4y^{14}+3x^{24}y^{14}+x^{26}y^{14}+x^{28}y^{14} \\
& +x^6y^8(1+y^4+3y^6+y^8+y^{12})+x^{22}y^8(1+y^4+3y^6+y^8+y^{12}) \\
& +x^{10}y^8(2+2y^4+5y^6+2y^8+2y^{12})+x^{18}y^8(2+2y^4+5y^6+2y^8+2y^{12}) \\
& +x^8y^6(1+y^2+2y^4+2y^6+7y^8+2y^{10}+2y^{12}+y^{14}+y^{16})
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{20}y^6(1 + y^2 + 2y^4 + 2y^6 + 7y^8 + 2y^{10} + 2y^{12} + y^{14} + y^{16})\\
&+ x^{12}y^6(1 + 2y^2 + 2y^4 + 3y^6 + 9y^8 + 3y^{10} + 2y^{12} + 2y^{14} + y^{16})\\
&+ x^{16}y^6(1 + 2y^2 + 2y^4 + 3y^6 + 9y^8 + 3y^{10} + 2y^{12} + 2y^{14} + y^{16})\\
&+ x^{14}(1 + y^2 + 3y^4 + 3y^6 + 7y^8 + 5y^{10} + 9y^{12} + 12y^{14} + 9y^{16} + 5y^{18}\\
&\qquad + 7y^{20} + 3y^{22} + 3y^{24} + y^{26} + y^{28})\Big),
\end{aligned}
$$

$$
\begin{aligned}
d[7,x,y] = {}& -x^{-16}y^{-16}(1 + x^4 + x^6 + x^8 + x^{12})(1 + y^4 + y^6 + y^8 + y^{12})\\
&\times \Big(y^6(1 + y^2 + y^4 + y^6 + y^8) + 2x^2y^6(1 + y^2 + y^4 + y^6 + y^8)\\
&+ 2x^{18}y^6(1 + y^2 + y^4 + y^6 + y^8) + x^{20}y^6(1 + y^2 + y^4 + y^6 + y^8)\\
&+ x^4y^6(4 + 5y^2 + 5y^4 + 5y^6 + 4y^8) + x^{16}y^6(4 + 5y^2 + 5y^4 + 5y^6 + 4y^8)\\
&+ x^8(1 + y^2)^2(1 + 4y^4 + 4y^6 + 4y^8 + 4y^{10} + 4y^{12} + y^{16})\\
&+ x^{10}(1 + y^2)^2(1 + 4y^4 + 4y^6 + 4y^8 + 4y^{10} + 4y^{12} + y^{16})\\
&+ x^{12}(1 + y^2)^2(1 + 4y^4 + 4y^6 + 4y^8 + 4y^{10} + 4y^{12} + y^{16})\\
&+ (x^6 + x^{14})(1 + 2y^2 + 4y^4 + 10y^6 + 12y^8 + 12y^{10} + 12y^{12}\\
&\qquad + 10y^{14} + 4y^{16} + 2y^{18} + y^{20})\Big),
\end{aligned}
$$

$$
\begin{aligned}
d[8,x,y] = {}& x^{-20}y^{-20}(1 + x^2 + x^4 + x^6 + x^8)(1 + y^2 + y^4 + y^6 + y^8)\\
&\times \Big(y^{14} + y^{18} + x^2y^{12}(1 + y^2 + y^4 + y^6 + y^8)\\
&\quad + x^{30}y^{12}(1 + y^2 + y^4 + y^6 + y^8) + x^{32}(y^{14} + y^{18})\\
&\quad + x^4y^{12}(1 + 4y^2 + y^4 + 4y^6 + y^8) + x^{28}y^{12}(1 + 4y^2 + y^4 + 4y^6 + y^8)\\
&\quad + x^6y^6(1 + y^2)^2(1 + 2y^4 + 3y^6 + y^8 + 3y^{10} + 2y^{12} + y^{16})\\
&\quad + x^{26}y^6(1 + y^2)^2(1 + 2y^4 + 3y^6 + y^8 + 3y^{10} + 2y^{12} + y^{16})\\
&\quad + x^8y^6(2 + 3y^2 + 6y^4 + 10y^6 + 17y^8 + 12y^{10} + 17y^{12} + 10y^{14}\\
&\qquad + 6y^{16} + 3y^{18} + 2y^{20})\\
&\quad + x^{24}y^6(2 + 3y^2 + 6y^4 + 10y^6 + 17y^8 + 12y^{10} + 17y^{12} + 10y^{14}\\
&\qquad + 6y^{16} + 3y^{18} + 2y^{20})\\
&\quad + x^{10}y^6(3 + 6y^2 + 9y^4 + 18y^6 + 23y^8 + 21y^{10} + 23y^{12} + 18y^{14}\\
&\qquad + 9y^{16} + 6y^{18} + 3y^{20})\\
&\quad + x^{22}y^6(3 + 6y^2 + 9y^4 + 18y^6 + 23y^8 + 21y^{10} + 23y^{12} + 18y^{14}\\
&\qquad + 9y^{16} + 6y^{18} + 3y^{20})\\
&\quad + x^{12}(y^2 + y^4 + 7y^6 + 10y^8 + 18y^{10} + 27y^{12} + 39y^{14} + 32y^{16} + 39y^{18}\\
&\qquad + 27y^{20} + 18y^{22} + 10y^{24} + 7y^{26} + y^{28} + y^{30})\\
&\quad + x^{20}(y^2 + y^4 + 7y^6 + 10y^8 + 18y^{10} + 27y^{12} + 39y^{14} + 32y^{16} + 39y^{18}\\
&\qquad + 27y^{20} + 18y^{22} + 10y^{24} + 7y^{26} + y^{28} + y^{30})\\
&\quad + x^{16}(y^2 + y^4 + 8y^6 + 12y^8 + 21y^{10} + 32y^{12} + 46y^{14} + 38y^{16} + 46y^{18}\\
&\qquad + 32y^{20} + 21y^{22} + 12y^{24} + 8y^{26} + y^{28} + y^{30})
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{14}(1 + y^2 + 4y^4 + 9y^6 + 17y^8 + 23y^{10} + 39y^{12} + 46y^{14} + 46y^{16} \\
&\qquad + 46y^{18} + 39y^{20} + 23y^{22} + 17y^{24} + 9y^{26} + 4y^{28} + y^{30} + y^{32}) \\
&+ x^{18}(1 + y^2 + 4y^4 + 9y^6 + 17y^8 + 23y^{10} + 39y^{12} + 46y^{14} + 46y^{16} \\
&\qquad + 46y^{18} + 39y^{20} + 23y^{22} + 17y^{24} + 9y^{26} + 4y^{28} + y^{30} + y^{32})\Big),
\end{aligned}
$$

$$
\begin{aligned}
d[9,x,y] = &- x^{-22}y^{-22}(1 + x^2 + x^4 + x^6 + x^8)(1 + y^2 + y^4 + y^6 + y^8) \\
&\times \Big(y^{18} + 2x^2y^{18} + 2x^{34}y^{18} + x^{36}y^{18} \\
&\quad + x^4y^{12}(1 + y^2 + 2y^4 + 5y^6 + 2y^8 + y^{10} + y^{12}) \\
&\quad + x^{32}y^{12}(1 + y^2 + 2y^4 + 5y^6 + 2y^8 + y^{10} + y^{12}) \\
&\quad + 2x^8y^{10}(1 + 3y^2 + 4y^4 + 6y^6 + 9y^8 + 6y^{10} + 4y^{12} + 3y^{14} + y^{16}) \\
&\quad + 2x^{28}y^{10}(1 + 3y^2 + 4y^4 + 6y^6 + 9y^8 + 6y^{10} + 4y^{12} + 3y^{14} + y^{16}) \\
&\quad + x^6y^{10}(1 + 3y^2 + 4y^4 + 6y^6 + 11y^8 + 6y^{10} + 4y^{12} + 3y^{14} + y^{16}) \\
&\quad + x^{30}y^{10}(1 + 3y^2 + 4y^4 + 6y^6 + 11y^8 + 6y^{10} + 4y^{12} + 3y^{14} + y^{16}) \\
&\quad + x^{10}y^6(1 + 2y^2 + 6y^4 + 14y^6 + 18y^8 + 25y^{10} + 34y^{12} + 25y^{14} \\
&\qquad\quad + 18y^{16} + 14y^{18} + 6y^{20} + 2y^{22} + y^{24}) \\
&\quad + x^{26}y^6(1 + 2y^2 + 6y^4 + 14y^6 + 18y^8 + 25y^{10} + 34y^{12} + 25y^{14} \\
&\qquad\quad + 18y^{16} + 14y^{18} + 6y^{20} + 2y^{22} + y^{24}) \\
&\quad + x^{12}y^4(1 + 3y^2 + 6y^4 + 14y^6 + 26y^8 + 34y^{10} + 44y^{12} + 56y^{14} \\
&\qquad\quad + 44y^{16} + 34y^{18} + 26y^{20} + 14y^{22} + 6y^{24} + 3y^{26} + y^{28}) \\
&\quad + x^{24}y^4(1 + 3y^2 + 6y^4 + 14y^6 + 26y^8 + 34y^{10} + 44y^{12} + 56y^{14} \\
&\qquad\quad + 44y^{16} + 34y^{18} + 26y^{20} + 14y^{22} + 6y^{24} + 3y^{26} + y^{28}) \\
&\quad + x^{14}y^4(1 + 4y^2 + 8y^4 + 18y^6 + 34y^8 + 44y^{10} + 57y^{12} + 71y^{14} \\
&\qquad\quad + 57y^{16} + 44y^{18} + 34y^{20} + 18y^{22} + 8y^{24} + 4y^{26} + y^{28}) \\
&\quad + x^{22}y^4(1 + 4y^2 + 8y^4 + 18y^6 + 34y^8 + 44y^{10} + 57y^{12} + 71y^{14} \\
&\qquad\quad + 57y^{16} + 44y^{18} + 34y^{20} + 18y^{22} + 8y^{24} + 4y^{26} + y^{28}) \\
&\quad + x^{16}y^4(2 + 6y^2 + 12y^4 + 25y^6 + 44y^8 + 57y^{10} + 72y^{12} + 88y^{14} \\
&\qquad\quad + 72y^{16} + 57y^{18} + 44y^{20} + 25y^{22} + 12y^{24} + 6y^{26} + 2y^{28}) \\
&\quad + x^{20}y^4(2 + 6y^2 + 12y^4 + 25y^6 + 44y^8 + 57y^{10} + 72y^{12} + 88y^{14} \\
&\qquad\quad + 72y^{16} + 57y^{18} + 44y^{20} + 25y^{22} + 12y^{24} + 6y^{26} + 2y^{28}) \\
&\quad + x^{18}(1 + 2y^2 + 5y^4 + 11y^6 + 18y^8 + 34y^{10} + 56y^{12} + 71y^{14} \\
&\qquad\quad + 88y^{16} + 104y^{18} + 88y^{20} + 71y^{22} + 56y^{24} + 34y^{26} \\
&\qquad\quad + 18y^{28} + 11y^{30} + 5y^{32} + 2y^{34} + y^{36})\Big),
\end{aligned}
$$

$$
\begin{aligned}
d[10,x,y] = x^{-24}y^{-24}\Big(&y^{24} + 2x^2y^{24} + 2x^{46}y^{24} + x^{48}y^{24} \\
&+ (x^4y^{16} + x^{44}y^{16})(1 + 2y^2 + 3y^4 + 4y^6 + 10y^8 + 4y^{10} + 3y^{12} + 2y^{14} + y^{16}) \\
&+ 2(x^6y^{16} + x^{42}y^{16})(2 + 4y^2 + 6y^4 + 8y^6 + 13y^8 + 8y^{10} + 6y^{12} + 4y^{14} + 2y^{16})
\end{aligned}
$$

$$
\begin{aligned}
&+ (x^8y^{12} + x^{40}y^{12})(1 + 2y^2 + 13y^4 + 24y^6 + 36y^8 + 46y^{10} + 66y^{12} + 46y^{14}\\
&+ 36y^{16} + 24y^{18} + 13y^{20} + 2y^{22} + y^{24})\\
&+ 2(x^{10}y^{10} + x^{38}y^{10})(1 + 3y^2 + 7y^4 + 20y^6 + 35y^8 + 49y^{10} + 62y^{12} + 77y^{14}\\
&+ 62y^{16} + 49y^{18} + 35y^{20} + 20y^{22} + 7y^{24} + 3y^{26} + y^{28})\\
&+ (x^{12}y^8 + x^{36}y^8)(1 + 6y^2 + 18y^4 + 38y^6 + 92y^8 + 152y^{10}\\
&+ 211y^{12} + 260y^{14} + 310y^{16} + 260y^{18} + 211y^{20} + 152y^{22}\\
&+ 92y^{24} + 38y^{26} + 18y^{28} + 6y^{30} + y^{32}) + 2(x^{14}y^8 + x^{34}y^8)(1 + 7y^2\\
&+ 19y^4 + 41y^6 + 87y^8 + 141y^{10} + 191y^{12} + 234y^{14}\\
&+ 268y^{16} + 234y^{18} + 191y^{20} + 141y^{22} + 87y^{24} + 41y^{26} + 19y^{28} + 7y^{30} + y^{32})\\
&+ 2(x^{18}y^4 + x^{30}y^4)(1 + 4y^2 + 12y^4 + 35y^6 + 76y^8 + 141y^{10} + 253y^{12}\\
&+ 381y^{14} + 497y^{16} + 592y^{18} + 653y^{20} + 592y^{22} + 497y^{24} + 381y^{26} + 253y^{28}\\
&+ 141y^{30} + 76y^{32} + 35y^{34} + 12y^{36} + 4y^{38} + y^{40})\\
&+ (x^{16}y^4 + x^{32}y^4)(1 + 4y^2 + 13y^4 + 40y^6 + 92y^8 + 174y^{10} + 330y^{12}\\
&+ 506y^{14} + 671y^{16} + 804y^{18} + 903y^{20} + 804y^{22} + 671y^{24} + 506y^{26} + 330y^{28}\\
&+ 174y^{30} + 92y^{32} + 40y^{34} + 13y^{36} + 4y^{38} + y^{40})\\
&+ 2(x^{22}y^4 + x^{26}y^4)(2 + 8y^2 + 23y^4 + 62y^6 + 130y^8 + 234y^{10} + 402y^{12}\\
&+ 592y^{14} + 763y^{16} + 900y^{18} + 982y^{20} + 900y^{22} + 763y^{24} + 592y^{26} + 402y^{28}\\
&+ 234y^{30} + 130y^{32} + 62y^{34} + 23y^{36} + 8y^{38} + 2y^{40})\\
&+ (x^{20}y^4 + x^{28}y^4)(3 + 12y^2 + 36y^4 + 98y^6 + 211y^8 + 382y^{10} + 671y^{12}\\
&+ 994y^{14} + 1292y^{16} + 1526y^{18} + 1678y^{20} + 1526y^{22} + 1292y^{24} + 994y^{26}\\
&+ 671y^{28} + 382y^{30} + 211y^{32} + 98y^{34} + 36y^{36} + 12y^{38} + 3y^{40}) + x^{24}(1 + 2y^2\\
&+ 10y^4 + 26y^6 + 66y^8 + 154y^{10} + 310y^{12} + 536y^{14} + 903y^{16} + 1306y^{18}\\
&+ 1678y^{20} + 1964y^{22} + 2144y^{24} + 1964y^{26} + 1678y^{28} + 1306y^{30} + 903y^{32}\\
&+ 536y^{34} + 310y^{36} + 154y^{38} + 66y^{40} + 26y^{42} + 10y^{44} + 2y^{46} + y^{48})\Big).
\end{aligned}
$$

For $11 \le n \le 20$, $d[n,x,y] = d[20-n,x,y]$. For $21 \le n \le 24$, $d[n,x,y] = 0$.

References

[1] J. Cogdell and P. Michel, *On the complex moments of symmetric power L-functions at $s = 1$*, Int. Math. Res. Notices 2004, no. 31, 1561–1618.

[2] W. Duke and E. Kowalski, *A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations*, Invent. Math. 139 (2000), 1–39.

[3] D. Goldfeld, *Automorphic Forms and L-Functions for the Group $GL(n,\mathbb{R})$*, Cambridge Stud. Adv. Math. 99, Cambridge Univ. Press, Cambridge, 2006.

[4] A. Granville and K. Soundararajan, *The distribution of values of $L(1,\chi_d)$*, Geom. Funct. Anal. 13 (2003), 992–1028.

[5] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations I*, Amer. J. Math. 103 (1981), 499-558.

[6] H. Kim, *Functoriality for the exterior square of* GL_4 *and symmetric fourth of* GL_2 (with Appendix 1 by D. Ramakrishnan and Appendix 2 by H. Kim and P. Sarnak), J. Amer. Math. Soc. 16 (2003), 139–183.
[7] H. Kim and P. Sarnak, Appendix 2 in [6].
[8] H. Kim and F. Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and functorial symmetric cube for* GL_2 (with appendix by C. J. Bushnell and G. Henniart), Ann. of Math. 155 (2002), 837–893.
[9] H. Kim and F. Shahidi, *Cuspidality of symmetric power with applications*, Duke Math. J. 112 (2002), 177–197.
[10] E. Kowalski and P. Michel, *Zeros of families of automorphic L-functions close to* 1 *and applications*, Pacific J. Math. 207 (2002), 411–431.
[11] Y.-K. Lau and G. Lü, *Sums of Fourier coefficients of cusp forms*, Quart. J. Math. 62 (2011), 687–716.
[12] Y.-K. Lau and J. Wu, *A large sieve inequality of Elliott–Montgomery–Vaughan type for automorphic forms and two applications*, Int. Math. Res. Notices 2008, no. 5, art. ID rnm 162, 35 pp.
[13] Y.-K. Lau and J. Wu, *Extreme values of symmetric power L-functions at* 1, Acta Arith. 126 (2007), 57–76.
[14] Y.-K. Lau and J. Wu, *A density theorem on automorphic L-functions and some applications*, Trans. Amer. Math. Soc. 358 (2006), 441–472.
[15] J. E. Littlewood, *On the class-number of the corpus* $P(\sqrt{-k})$, Proc. London Math. Soc. 27 (1928), 358–372.
[16] W. Luo, *Values of symmetric square L-functions at* 1, J. Reine Angew. Math. 506 (1999), 215–235.
[17] H. L. Montgomery and R. C. Vaughan, *Extreme values of Dirichlet L-functions at* 1, in: Number Theory in Progress, K. Győry et al. (eds.), Vol. 2, de Gruyter, Berlin, 1999, 1039–1052.
[18] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. 81 (1996), 269–322.
[19] P. Sarnak, *Statistical properties of eigenvalues of the Hecke operators*, in: Analytic Number Theory and Diophantine Problems (Stillwater, 1984), Progr. Math. 70, Birkhäuser, Basel, 1987, 321–331.
[20] G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. 31 (1975), 79–98.
[21] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, 1995.
[22] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford Univ. Press, 1951.
[23] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford Univ. Press, 1939.
[24] Y. Wang, *Trace formulas and their applications on Hecke eigenvalues*, Ph.D. thesis, The Univ. of Hong Kong; http://hub.hku.hk/handle/10722/173871.

Yingnan Wang
College of Mathematics and Computational Science
Shenzhen University
Shenzhen, Guangdong 518060, P.R. China
E-mail: ynwang@szu.edu.cn

Received on 11.7.2013
and in revised form on 4.11.2014 (7517)