APPLICATIONES MATHEMATICAE
30,1 (2003), pp. 11–37

K. W. Morris (Adelaide)
D. Szynal (Lublin)

TESTS DERIVED FROM CHARACTERIZATIONS IN TERMS OF MOMENTS OF RECORD VALUES

Abstract. We derive tests of fit from characterizations of continuous distributions via moments of the kth upper record values.

1. Characterization conditions via moments of record values. Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. random variables with cdf F and pdf f. For a fixed integer $k \geq 1$ we define the sequence $U_k(1), U_k(2), \ldots$ of kth (*upper*) *record times* of $X_1, X_2, \ldots$ as follows:

$$U_k(1) = 1,$$
$$U_k(n) = \min\{j > U_k(n-1) : X_{j:j+k-1} > X_{U_k(n-1):U_k(n-1)+k-1}\},$$

for $n = 2, 3, \ldots$ Write

$$Y_n^{(k)} := X_{U_k(n):U_k(n)+k-1}, \quad n \geq 1.$$

The sequence $\{Y_n^{(k)} : n \geq 1\}$ is called the sequence of kth (*upper*) *record values* of the above sequence. For convenience we also take $Y_0^{(k)} = 0$ and note that $Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k)$ (cf. [3]). Thus the finite sequence $X_1, \ldots, X_k$ is enough to determine $Y_1^{(k)}$, whereas if $n > 1$ then in general an infinite sequence is needed for $Y_n^{(k)}$.

We see that for $k = 1, 2, \ldots,$ the sequences $\{Y_n^{(k)} : n \geq 1\}$ of kth record values can be obtained from $\{X_n : n \geq 1\}$ by inspecting successively the samples X_1, (X_1, X_2), (X_1, X_2, X_3), and so on. For $k = 1$, $Y_1^{(1)} = X_1$, and the following terms are obtained by looking at the maxima of the successive samples: $Y_2^{(1)}$ is the first maximum that exceeds $Y_1^{(1)}$, $Y_3^{(1)}$ is the

2000 *Mathematics Subject Classification*: Primary 62E10, 62F03.

Key words and phrases: record times, record values, moments, uniform, exponential, Weibull, Pareto, logistic, Cauchy, normal, extreme-value distributions.

first maximum that exceeds $Y_2^{(1)}$, and so on. For $k=2$, $Y_1^{(2)}=\min(X_1,X_2)$, and the following terms are obtained by looking at the next-to-largest values in the successive samples: $Y_2^{(2)}$ is the first next-to-largest value that exceeds $Y_1^{(2)}$, $Y_3^{(2)}$ is the first next-to-largest value that exceeds $Y_2^{(2)}$, and so on. And generally, $Y_1^{(k)}=\min(X_1,\dots,X_k)=X_{1:k}$, and the following kth record values are obtained by looking at the kth largest values in successive samples, i.e., looking at the order statistics $X_{2:k+1}$ from $(X_1,\dots,X_{k+1})$, $X_{3:k+2}$ from $(X_1,\dots,X_{k+2})$, and so on.

We have the following characterizations.

THEOREM 1.1 (cf. [5]). *Let $\{X_n : n\ge 1\}$ be a sequence of i.i.d. random variables with cdf F. Assume that G is a nondecreasing right-continuous function from $\mathbb{R}$ to $(-\infty,1]$, and let n, k, l be given integers such that $k\ge 1$ and $n\ge l\ge 1$. Then $F(x)=G(x)$ on $I(F)$ iff*

$$k^{2l}(n-l)!E[-\ln(1-G(Y_{n-l+1}^{(k)}))]^{2l} - 2n!k^lE[-\ln(1-G(Y_{n+1}^{(k)}))]^l+(n+l)!=0.$$

THEOREM 1.2 (cf. [5], [8]). *Under the assumptions of Theorem* 1.1, *$F(x)=G(x)$ on $I(F)$ iff*

$$E[-\ln(1-G(Y_{n+1}^{(k)}))]^l=\frac{(n+l)!}{n!k^l},$$
$$E[-\ln(1-G(Y_{n-l+1}^{(k)}))]^{2l}=\frac{(n+l)!}{(n-l)!k^{2l}}.$$

COROLLARY 1.1. *$X\sim F$ and F is continuous iff*

$$E[-\ln(1-F(Y_1^{(k)}))]^2-\frac{2}{k}E[-\ln(1-F(Y_2^{(k)}))]+\frac{2}{k^2}=0$$

or

$$E[-\ln(1-F(Y_2^{(k)}))]=\frac{2}{k},\qquad E[-\ln(1-F(Y_1^{(k)}))]^2=\frac{2}{k^2}.$$

2. Goodness-of-fit tests derived from characterizations in Section 1

(A) *Parameters of F are specified.* To simplify the notation we write

$$g(x)=1-F(x),\qquad h(x)=\begin{cases}-\ln(g(x)) & \text{if } F(x)<1,\\ 0 & \text{otherwise.}\end{cases}$$

Then Theorem 1.1 says that $X\sim F$ iff

$$k^{2l}(n-l)!Eh^{2l}(Y_{n-l+1}^{(k)})-2n!k^lEh^l(Y_{n+1}^{(k)})+(n+l)!=0.$$

These characterizations cannot be used to construct tests since in practice only a finite sample is available, whereas in general information about $Y_n^{(k)}$ can be obtained only from an infinite sample. As noted above, the exception is $Y_1^{(k)} = X_{1:k}$, and the method used here is to base the tests on consequences of these characterizations that involve order statistics only. These are no longer characterizations, and so they test only certain aspects of the distribution. Our procedure is as follows.

We consider the special case $l = n$. Then $X \sim F$ iff

$$Eh^{2n}(X_{1:k}) - \frac{2n!}{k^n}\, Eh^n(Y_{n+1}^{(k)}) + \frac{(2n)!}{k^{2n}} = 0. \tag{2.1}$$

We now show that if $X \sim F$ then

$$Eh^{2n}(X_{1:k}) = \frac{n!}{k^n}\, Eh^n(Y_{n+1}^{(k)}) = \frac{(2n)!}{k^{2n}} \tag{2.2}$$

and it then follows from (2.1) that $X \sim F$ implies

$$Eh^{2n}(X_{1:k}) - \frac{2n!}{k^n}\, Eh^n(X_{1:k}) = 0. \tag{2.3}$$

Now (2.2) follows from the fact that if $X \sim F$ then the pdf of $Y_n^{(k)}$ is

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}\, h^{n-1}(x) g^{k-1}(x) f(x) \qquad \text{(cf. [3])}.$$

In this paper we use (2.3) with $n = 1$. Then

$$Eh^2(X_{1:k}) - \frac{2}{k}\, Eh(X_{1:k}) = 0. \tag{2.4}$$

For given $k = 1, 2, \ldots$ we construct tests of $H : X \sim F$ based on (2.4). Write

$$U_k := X_{1:k} = \min(X_1, \ldots, X_k),$$
$$R_k = kh(U_k), \qquad W_k = R_k^2 - 2R_k.$$

Then (2.4) can be written as $EW_k = 0$. The test is constructed by obtaining an estimate EW_k, $\widehat{E}$ say, and rejecting H if $\widehat{E}$ is distant from 0, and so if $|\widehat{E}|$ or equivalently $\widehat{E}^2$ is large.

Suppose now that we have a sample $X_1, \ldots, X_n$ of size $n = kN$. This provides the sample $R_{k1}, \ldots, R_{kN}$ where

$$R_{ki} = kh(U_{ki}), \qquad U_{ki} := \min(X_{k(i-1)+1}, \ldots, X_{ki}).$$

Since $R_k \sim \mathrm{Exp}(1)$, $k \geq 1$, we have

$$\mathrm{Var}(W_k) = E(R_k^4 - 4R_k^3 + 4R_k^2) = 4! - 4 \cdot 3! + 4 \cdot 2! = 8$$

and by the Central Limit Theorem,

$$\sqrt{N}\, \overline{W}_{kN} \xrightarrow{D} \mathrm{N}(0, 8),$$

where

$$\overline{W}_{kN} = \overline{R^2}_{kN} - 2\overline{R}_{kN}, \qquad \overline{R^2}_{kN} = \frac{1}{N}\sum_{i=1}^{N} R_{ki}^2, \qquad \overline{R}_{kN} = \frac{1}{N}\sum_{i=1}^{N} R_{ki}.$$

Thus a simple asymptotic test of $H : X \sim F$ is provided by

$$T_{kN} = N(\overline{R^2}_{kN} - 2\overline{R}_{kN})^2/8 \xrightarrow{D} \chi^2(1). \tag{2.5}$$

For completeness we state here the test when $k = n$.

PROPOSITION 1 (cf. [9]). *The significance probability of the test using* $T_n := ((nh(U_n))^2 - 2nh(U_n))^2$, *where* $U_n := U_{n1} = \min(X_1, \ldots, X_n)$, *is*

$$P_t := P[T_n > t]$$
$$= \begin{cases} e^{-1-\sqrt{1+\sqrt{t}}} + e^{-1+\sqrt{1-\sqrt{t}}} - e^{-1-\sqrt{1-\sqrt{t}}} & \text{if } 0 < t \le 1, \\ e^{-1-\sqrt{1+\sqrt{t}}} & \text{if } t \ge 1. \end{cases}$$

Now suppose that the distribution function has the form $F(x; \boldsymbol{\lambda})$ where $\boldsymbol{\lambda}$ is an $m \times 1$ vector of unknown parameters. In this case we need the following

THEOREM 2.1 (cf. [11]). *Let* $\widehat{\boldsymbol{T}}_n = \boldsymbol{T}_n(X_1, \ldots, X_n; \widehat{\boldsymbol{\lambda}}_n)$, *where* $\widehat{\boldsymbol{\lambda}}_n = \widehat{\boldsymbol{\lambda}}_n(X_1, \ldots, X_n)$ *is an estimator of a parameter* $\boldsymbol{\lambda}$, *and moreover let* $\boldsymbol{T}_n = \boldsymbol{T}_n(X_1, \ldots, X_n; \boldsymbol{\lambda})$ (*here* $\boldsymbol{T}_n$, $\boldsymbol{\lambda}$ *and* $\widehat{\boldsymbol{\lambda}}_n$ *may be vectors*). *Suppose that*:

(i) *For each* $\boldsymbol{\lambda}$,

$$\sqrt{n}\begin{pmatrix} \boldsymbol{T}_n \\ \widehat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda} \end{pmatrix} \xrightarrow{D} \boldsymbol{T} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{V}),$$

where

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix}$$

and $\boldsymbol{V}_{22}$ *is nonsingular.*

(ii) *There is a matrix* $\boldsymbol{B}$, *possibly depending continuously on* $\boldsymbol{\lambda}$, *such that*

$$\sqrt{n}\,\widehat{\boldsymbol{T}}_n = \sqrt{n}\,\boldsymbol{T}_n + \boldsymbol{B}\sqrt{n}\,(\widehat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}) + o_p(1).$$

(iii) $\widehat{\boldsymbol{\lambda}}_n$ *is asymptotically efficient* (*cf.* [11]).

Then

$$\sqrt{n}\,\widehat{\boldsymbol{T}}_n \xrightarrow{D} \boldsymbol{T}^* \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{V}_{11} - \boldsymbol{B}\boldsymbol{V}_{22}\boldsymbol{B}').$$

Note that (ii) is satisfied when T_n is differentiable in $\boldsymbol{\lambda}$, and then

$$\boldsymbol{B} = \lim_{n\to\infty} E\left[\frac{\partial}{\partial \boldsymbol{\lambda}} \boldsymbol{T}_n\right].$$

THEOREM 2.2 (cf. [11], [9]). *Let* $(X_1, \ldots, X_n)$ *be a sample of size* $n = kN$ *with an absolutely continuous distribution function* $F(x; \boldsymbol{\lambda})$ *differentiable with respect to the* $m \times 1$ *vector* $\boldsymbol{\lambda}$. *Set*

$$\overline{W}_{kN} := \overline{W}_{kN}(\boldsymbol{\lambda}) = \overline{R^2}_{kN}(\boldsymbol{\lambda}) - 2\overline{R}_{kN}(\boldsymbol{\lambda}) = \frac{1}{N}\sum_{i=1}^N R_{ki}^2(\boldsymbol{\lambda}) - \frac{2}{N}\sum_{i=1}^N R_{ki}(\boldsymbol{\lambda}),$$

where

$$R_{ki}(\boldsymbol{\lambda}) = kh(U_{ki}, \boldsymbol{\lambda}), \qquad h(U_{ki}, \boldsymbol{\lambda}) = -\ln(1 - F(U_{ki}, \boldsymbol{\lambda})).$$

Write

$$\widehat{\overline{W}}_{kN} := \overline{W}_{kN}(\widehat{\boldsymbol{\lambda}}_n) = \widehat{\overline{R^2}}_{kN} - 2\widehat{\overline{R}}_{kN}$$

where

$$\widehat{\overline{R^2}}_{kN} = \overline{R^2}_{kN}(\widehat{\boldsymbol{\lambda}}_n), \qquad \widehat{\overline{R}}_{kN} = \overline{R}_{kN}(\widehat{\boldsymbol{\lambda}}_n).$$

Suppose that F is such that MLE $\widehat{\boldsymbol\lambda}_n$ is "regular", in the sense that

$$\sqrt{n}\,(\widehat{\boldsymbol\lambda}_n - \boldsymbol\lambda) \xrightarrow{D} \mathrm{N}(\mathbf{0}, \mathcal{I}^{-1}),$$

where $\mathcal{I} = \mathcal{I}(\boldsymbol\lambda)$ is the expected information matrix for $\boldsymbol\lambda$ based on a single observation. Then

$$\sqrt{N}\,\widehat{\overline{W}}_{kN} \xrightarrow{D} \mathrm{N}(0, \sigma_k^2),$$

and

$$\widehat{T}_{kN} := N(\widehat{\overline{R^2}}_{kN} - 2\widehat{\overline{R}}_{kN})^2/\sigma_k^2 \xrightarrow{D} \chi^2(1),$$

where

$$\sigma_k^2 = 8 - \frac{1}{k}\,\boldsymbol{d}_k'\mathcal{I}^{-1}\boldsymbol{d}_k, \qquad \boldsymbol{d}_k = E\biggl(\frac{\partial W_k}{\partial \boldsymbol\lambda}\biggr).$$

NOTE. It follows from the invariance property of MLE that $\widehat{T}_{kN}$ does not depend on the parameterization used to specify the distributions.

This theorem is obtained by applying Theorem 2.1 with $n = N$ and $T_N = \widehat{\overline{W}}_{kN}$. Then $V_{11} = 8$, $V_{22} = (1/k)\mathcal{I}^{-1}$ and

$$\boldsymbol{B} = E\biggl(\frac{\partial \widehat{\overline{W}}_{kN}}{\partial \lambda}\biggr) =: \boldsymbol{d}_k.$$

Furthermore

$$\begin{aligned} d_{ki} &= 2E(R_k - 1)\,\frac{\partial R_k}{\partial \lambda_i} \\ &= -2kE(-k\ln(1 - F(U_k;\boldsymbol\lambda)) - 1)\,\frac{\partial \ln(1 - F(U_k;\boldsymbol\lambda))}{\partial \lambda_i} \\ &= 2k^2 \int [-k\ln(1 - F(x;\boldsymbol\lambda)) - 1]\,\frac{1}{1 - F(x;\boldsymbol\lambda)}\,\frac{\partial F(x;\boldsymbol\lambda)}{\partial \lambda_i} \\ &\qquad \times (1 - F(x;\boldsymbol\lambda))^{k-1} f(x;\boldsymbol\lambda)\,dx, \end{aligned}$$

since U_k has pdf $k(1 - F(x;\boldsymbol\lambda))^{k-1} f(x;\boldsymbol\lambda)$. Hence

$$d_{ki} = -2k^2 E\biggl[(1 - F(X;\boldsymbol\lambda))^{k-1}(k\ln(1 - F(X;\boldsymbol\lambda)) + 1)\,\frac{\partial F}{\partial \lambda_i}\biggr].$$

Letting $\mathcal{K}_k = \boldsymbol{d}_k'\mathcal{I}^{-1}\boldsymbol{d}_k$ we have $\sigma_k^2 = 8 - (1/k)\mathcal{K}_k$ and the results appear as

$$\begin{aligned} \widehat{T}_{kN} &:= \frac{kN}{8k - \mathcal{K}_k}(\widehat{\overline{R^2}}_{kN} - 2\widehat{\overline{R}}_{kN})^2 \\ &= \frac{k^5}{N(8k - \mathcal{K}_k)}\biggl(\sum_{i=1}^{N} \ln^2(1 - F(U_{ki};\widehat{\boldsymbol\lambda}_n)) \\ &\qquad + \frac{2}{k}\sum_{i=1}^{N} \ln(1 - F(U_{ki};\widehat{\boldsymbol\lambda}_n))\biggr)^2 \xrightarrow{D} \chi^2(1). \end{aligned} \tag{2.6}$$

NOTE. Since $\boldsymbol{d}_k$ and $\mathcal{I}$ may depend on $\boldsymbol{\lambda}$, $\mathcal{K}_k$ may also depend on $\boldsymbol{\lambda}$. In this case $\widehat{T}_{kN}$ cannot be used to test $H : X \sim F$, $\boldsymbol{\lambda}$ unknown. But if $\mathcal{K}_k = \mathcal{K}_k(\boldsymbol{\lambda})$ is replaced by $\widehat{\mathcal{K}}_k = \mathcal{K}_k(\widehat{\boldsymbol{\lambda}}_n)$ then the resulting statistic $\widehat{T}_{kN}$ can be used. And $\widehat{T}_{kN}$ converges in distribution to $\chi^2(1)$ since $\widehat{\boldsymbol{\lambda}}_n$ converges in probability to $\boldsymbol{\lambda}$.

Special cases:

1°. Let $F \in \mathrm{Exp}(\alpha)$, i.e. $F(x) = 1 - e^{-\alpha x}$, $x > 0$; $\alpha > 0$, $f(x) = \alpha e^{-\alpha x}$. Here

$$h(x) = -\alpha x, \qquad R_k = \alpha k U_k, \qquad \widehat{\alpha}_n = 1/\overline{X}_n, \qquad \mathcal{I}^{-1} = \alpha^2,$$

and

$$d_k = E\left(\frac{\partial W_k}{\partial \alpha}\right) = -2k^2 E e^{-(k-2)\alpha X}(-k\alpha X + 1) X e^{-\alpha X}$$
$$= -2k^2 \alpha \int_0^\infty e^{-(k-2)\alpha x}(-k\alpha x + 1) x e^{-2\alpha x}\, dx = \frac{2}{\alpha}.$$

Thus $\mathcal{K}_k = 4$ and from (2.6) we have

PROPOSITION 2. *A goodness-of-fit test for $F \in \mathrm{Exp}(\alpha)$ is given by*

$$\widehat{T}_{kN} = \frac{k^5}{4N(2k-1)}\left(\frac{\sum_{i=1}^N U_{ki}^2}{(\overline{X}_{kN})^2} - \frac{2}{k}\,\frac{\sum_{i=1}^N U_{ki}}{\overline{X}_{kN}}\right)^2 \xrightarrow{D} \chi^2(1).$$

For the special cases $k = 1, 2, 3, 4$ we have

$$\widehat{T}_{1N} = \frac{N}{4}\left(\frac{\overline{X}_N^2}{(\overline{X}_N)^2} - 2\right)^2, \qquad \widehat{T}_{2N} = \frac{8}{3N}\left(\frac{\sum_{i=1}^N U_{2i}^2}{(\overline{X}_{2N})^2} - \frac{\sum_{i=1}^N U_{2i}}{\overline{X}_{2N}}\right)^2,$$
$$\widehat{T}_{3N} = \frac{243}{20N}\left(\frac{\sum_{i=1}^N U_{3i}^2}{(\overline{X}_{3N})^2} - \frac{2}{3}\,\frac{\sum_{i=1}^N U_{3i}}{\overline{X}_{3N}}\right)^2,$$
$$\widehat{T}_{4N} = \frac{256}{7N}\left(\frac{\sum_{i=1}^N U_{4i}^2}{(\overline{X}_{4N})^2} - \frac{1}{2}\,\frac{\sum_{i=1}^N U_{4i}}{\overline{X}_{4N}}\right)^2.$$

2°. Let $F \in \mathrm{W}(\alpha, \beta)$, i.e. $F(x) = 1 - \exp(-\alpha x^\beta)$, $x > 0$; $\alpha > 0$, $\beta > 0$, $f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}$. Then

$$\frac{\partial F}{\partial \alpha} = \frac{x}{\alpha\beta} f(x), \qquad \frac{\partial F}{\partial \beta} = \frac{x \ln x}{\beta} f(x)$$

and

$$d_k(\alpha) = -2k^2 E(1 - F(X))^{k-2}(k \ln(1 - F(X)) + 1)\,\frac{\partial F}{\partial \alpha}$$
$$= 2k^3 \alpha \int_0^\infty x^{2\beta} e^{-\alpha(k-1)x^\beta} \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}\, dx$$

$$- 2k^2 \int_0^\infty x^\beta e^{-\alpha(k-1)x^\beta} \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}\, dx$$

$$= 2k^2\alpha \int_0^\infty x^{2\beta} (\alpha k)\beta x^{\beta-1} e^{-(\alpha k)x^\beta}\, dx$$

$$- 2k \int_0^\infty x^\beta (\alpha k)\beta x^{\beta-1} e^{-\alpha k x^\beta}\, dx$$

$$= 2k^2\alpha E X^{2\beta} - 2kEX^\beta, \quad \text{where} \quad X \sim W(\alpha k, \beta).$$

But if $X \in \mathrm{W}(\alpha,\beta)$ then $EX^\beta = 1/\alpha$ and $EX^{2\beta} = 2/\alpha^2$, whence

$$d_k(\alpha) = 2/\alpha.$$

Now

$$d_k(\beta) = 2k^3\alpha^3\beta \int_0^\infty x^{3\beta-1} e^{-\alpha k x^\beta} \ln x\, dx + 2k^2\alpha^2\beta \int_0^\infty x^{2\beta-1} e^{-\alpha k x^\beta} \ln x\, dx.$$

Since

$$- \int_0^\infty e^{-y} \ln y\, dy = \gamma \qquad \text{(cf. [12, 3.711.2])}$$

is the Euler constant, integrating by parts gives

$$2k^2\alpha^2\beta \int_0^\infty x^{2\beta-1} e^{-\alpha k x^\beta} \ln x\, dx = 2k\alpha \int_0^\infty \frac{y}{\alpha k}\, e^{-y}\, \frac{1}{\beta} \ln \frac{y}{\alpha k}\, dy$$

$$\text{(substituting } y = \alpha k x^\beta\text{)}$$

$$= \frac{2}{\beta}\Big(\int_0^\infty y e^{-y} \ln y dy - \ln \alpha k \Big) = \frac{2}{\beta}(1 - \gamma - \ln \alpha k)$$

$$\text{(integrating by parts).}$$

Similarly

$$2k^3\alpha^3\beta \int_0^\infty x^{3\beta-1} \ln x e^{-\alpha k x^\beta}\, dx = \frac{2}{\beta}\,[3 - 2\gamma - 2\ln \alpha k].$$

Then

$$d_k(\beta) = \frac{2}{\beta}\,(2 - \gamma - \ln \alpha k)$$

and

$$\boldsymbol{d}_k = 2 \begin{pmatrix} 1/\alpha \\ (1/\beta)(2 - \gamma - \ln \alpha k) \end{pmatrix}.$$

Since

$$\mathcal{I}^{-1} = \frac{6}{\pi^2} \begin{bmatrix} \alpha^2(\pi^2/6 + (1-\gamma-\ln\alpha)^2) & -\alpha\beta(1-\gamma-\ln\alpha) \\ -\alpha\beta(1-\gamma-\ln\alpha) & \beta^2 \end{bmatrix}$$

we obtain

$$\mathcal{K}_k = \boldsymbol{d}_k' \mathcal{I}^{-1} \boldsymbol{d}_k = \frac{24}{\pi^2}\Big[\frac{\pi^2}{6} + (1-\ln k)^2\Big].$$

Thus from (2.6) we have

PROPOSITION 3. *A goodness-of-fit test for $F \in W(\alpha,\beta)$ is given by*

$$\widehat{T}_{kN} = \frac{k^5 \widehat{\alpha}_n^2}{4N(2k-1-(6/\pi^2)(\ln k - 1)^2)} \times \Big(\widehat{\alpha}_n \sum_{i=1}^{N} U_{ki}^{2\widehat{\beta}_n} - \frac{2}{k}\sum_{i=1}^{N} U_{ki}^{\widehat{\beta}_n}\Big)^2 \xrightarrow{D} \chi^2(1),$$

where $\widehat{\alpha}_n$ and $\widehat{\beta}_n$ are obtained by numerical solution of the equation

$$\frac{d}{d\beta} L_n\Big(n/\sum_{i=1}^{n} x_i^{\beta},\, \beta\Big) = 0$$

for

$$L_n(\alpha,\beta) = n\ln\alpha + n\ln\beta + (\beta-1)\sum_{i=1}^{n}\ln x_i - \alpha\sum_{i=1}^{n} x_i^{\beta}.$$

Numerical evaluation of $\mathcal{K}_k$

k	1	2	3	4	5
$\mathcal{K}_k$	6.43171	4.22897	4.02365	4.36287	4.90317

3°. Let $F \in \mathrm{Par}_S(\alpha,\sigma)$, i.e. $F(x) = 1 - (\sigma/x)^{\alpha}$, $x > \sigma$; $\alpha > 0$, $\sigma > 0$, $f(x) = \alpha\sigma^{\alpha}/x^{\alpha+1}$.

We consider first the case when σ is known, which occurs frequently in practice. Since $Y = \log(X/\sigma) \sim \mathrm{Exp}(\alpha)$, from Proposition 2 we have

PROPOSITION 4. *A goodness-of-fit test for $F \in \mathrm{Par}_S(\alpha,\sigma)$ when σ is known is given by*

$$\widehat{T}_{kN} = \frac{k^5}{4N(2k-1)}\bigg(\frac{\sum_{i=1}^{N}(U_{ki}')^2}{(\overline{Y}_{kN})^2} - \frac{2}{k}\,\frac{\sum_{i=1}^{N} U_{ki}'}{\overline{Y}_{kN}}\bigg)^2 \xrightarrow{D} \chi^2(1),$$

where $U_{ki}' = \min(Y_{k(i-1)+1},\ldots,Y_{ki}) = \ln(U_{ki}/\sigma)$.

When both σ and α are unknown we cannot apply Theorem 2.2 since this is a situation where the MLE are not regular. But then we can use

PROPOSITION 4′. *A goodness-of-fit test for $F \in \mathrm{Par}_S(\alpha,\sigma)$ when α and σ are unknown is given by*

$$\widehat{T}_{kN} = \frac{k^5}{4N(2k-1)}\bigg(\frac{\sum_{i=1}^{N}(\widehat{U}_{ki}')^2}{(\widehat{\overline{Y}}_{kN})^2} - \frac{2}{k}\,\frac{\sum_{i=1}^{N} \widehat{U}_{ki}'}{\widehat{\overline{Y}}_{kN}}\bigg)^2 \xrightarrow{D} \chi^2(1),$$

where $\widehat{Y}_j = \ln(X_j/\widehat{\sigma}_n)$, $\widehat{U}'_{ki} = \min(\widehat{Y}_{k(i-1)+1}, \ldots, \widehat{Y}_{ki}) = \ln(U_{ki}/\widehat{\sigma}_n)$, $\widehat{\sigma}_n = \min(X_1, \ldots, X_n)$.

This follows from Proposition 4 and the fact that $\sqrt{n}(\widehat{\sigma}_n - \sigma)$ converges in probability to zero.

4°. Let $F \in \mathrm{Par}_T(\alpha, \theta)$, i.e. $F(x) = 1 - (\theta/(x+\theta))^\alpha$, $x > 0$; $\alpha > 0$, $\theta > 0$, $f(x) = \alpha\theta^\alpha/(x+\theta)^{\alpha+1}$. Then

$$\frac{\partial F}{\partial \alpha} = -\left(\frac{\theta}{x+\theta}\right)^\alpha \ln\frac{\theta}{x+\theta}, \qquad \frac{\partial F}{\partial \theta} = -\frac{x}{\theta}\, f(x) \quad \text{(cf. [7, p. 576])}.$$

Hence

$$\begin{aligned} d_k(\alpha) = & -2k^2 E(1-F(X))^{k-2}(k\ln(1-F(X))+1)\,\frac{\partial F}{\partial \alpha} \\ &= 2k^2 E\left(\frac{\theta}{X+\theta}\right)^{\alpha(k-2)}\left(k\ln\left(\frac{\theta}{X+\theta}\right)^\alpha + 1\right) \\ &\quad \times \left(\frac{\theta}{X+\theta}\right)^\alpha \ln\frac{\theta}{X+\theta} \\ &= 2k^3\,\frac{\alpha^2}{\theta}\int_0^\infty \left(\frac{\theta}{x+\theta}\right)^{k\alpha+1} \ln^2\left(\frac{\theta}{x+\theta}\right) dx \\ &\quad + 2k^2\,\frac{\alpha}{\theta}\int_0^\infty \left(\frac{\theta}{x+\theta}\right)^{k\alpha+1} \ln\left(\frac{\theta}{x+\theta}\right) dx \\ &= 2k^3\alpha^2 \int_0^1 y^{k\alpha-1}\ln^2 y\,dy + 2k^2\alpha\int_0^1 y^{k\alpha-1}\ln y\,dy \\ &= 2k^3\alpha^2\cdot\frac{2}{(k\alpha)^3} - 2k^2\alpha\cdot\frac{1}{(k\alpha)^2} = 2/\alpha. \end{aligned}$$

Similarly

$$\begin{aligned} d_k(\theta) = & -2k^2 E(1-F(X))^{k-2}(k\ln(1-F(X))+1)\,\frac{\partial F}{\partial \theta} \\ &= 2k^3\frac{\alpha^3}{\theta^2}\left[\int_0^\infty \left(\frac{\theta}{x+\theta}\right)^{k\alpha+1}\ln\frac{\theta}{x+\theta}\,dx \right. \\ &\quad \left. - \int_0^\infty \left(\frac{\theta}{x+\theta}\right)^{k\alpha+2}\ln\frac{\theta}{x+\theta}\,dx\right] \\ &\quad + 2k^2\,\frac{\alpha^2}{\theta^2}\left[\int_0^\infty \left(\frac{\theta}{x+\theta}\right)^{k\alpha+1} dx - \int_0^\infty \left(\frac{\theta}{x+\theta}\right)^{k\alpha+2} dx\right] \end{aligned}$$

$$= 2k^3\,\frac{\alpha^3}{\theta}\Big[\int_0^1 y^{k\alpha-1}\ln y\,dy - \int_0^1 y^{k\alpha}\ln y\,dy\Big]$$
$$+ 2k^2\,\frac{\alpha^2}{\theta}\left[\frac{1}{k\alpha} - \frac{1}{k\alpha+1}\right]$$
$$= -\frac{2k^2\alpha^2}{\theta(k\alpha+1)^2}.$$

Thus

$$\boldsymbol{d}_k = 2\begin{pmatrix} 1/\alpha \\ -k^2\alpha^2/(\theta(k\alpha+1)^2)\end{pmatrix},$$

and since

$$\mathcal{I}^{-1} = \alpha(\alpha+1)^2(\alpha+2)\begin{bmatrix} \alpha/(\alpha+2) & \theta/(\alpha+1) \\ \theta/(\alpha+1) & \theta^2/\alpha^2 \end{bmatrix},$$

we obtain

$$\mathcal{K}_k = \mathcal{K}_k(\alpha) = 4\left[1 + \frac{\alpha(\alpha+2)}{(k\alpha+1)^4}\,(1+2k\alpha-k^2\alpha)^2\right].$$

Here $\mathcal{K}_k$ depends on α, and from the Note preceding 1° we then have

PROPOSITION 5. *A goodness-of-fit test for $F \in \mathrm{Par}_T(\alpha,\theta)$ is given by*

$$\widehat{T}_{kN} = \frac{k^5\widehat{\alpha}_n^2}{N(8k-\mathcal{K}_k(\widehat{\alpha}_n))}\left[\widehat{\alpha}_n\sum_{i=1}^{N}\ln^2\frac{\widehat{\theta}_n}{U_{ki}+\widehat{\theta}_n} + \frac{2}{k}\sum_{i=1}^{N}\ln\frac{\widehat{\theta}_n}{U_{ki}+\widehat{\theta}_n}\right]^2$$
$$\xrightarrow{D} \chi^2(1),$$

where $\widehat{\alpha}_n$ and $\widehat{\theta}_n$ are obtained by numerical solution of the equation

$$\frac{d}{d\theta}L_n\Big(n\Big/\sum_{i=1}^{n}\ln(x_i/\theta+1),\,\theta\Big) = 0$$

for

$$L_n(\alpha,\theta) = n\ln\alpha + n\alpha\ln\theta - (\alpha+1)\sum_{i=1}^{n}\ln(x_i+\theta).$$

5°. Let $F \in \mathrm{Log}(\alpha,\beta)$, i.e.

$$F(x) = \frac{1}{1+\exp\big(-\frac{x-\alpha}{\beta}\big)}, \qquad -\infty < x < \infty;\ \alpha\in\mathbb{R},\ \beta>0,$$
$$f(x) = \frac{1}{\beta}\left[\exp\left(-\frac{x-\alpha}{\beta}\right)\right]\Big/\left[\left(1+\exp\left(-\frac{x-\alpha}{\beta}\right)\right)^2\right]$$
$$= \frac{1}{\beta}\,F(x)(1-F(x)).$$

Then

$$\frac{\partial F}{\partial \alpha} = -f(x), \qquad \frac{\partial F}{\partial \beta} = -\frac{x-\alpha}{\beta}\, f(x) = -\frac{x-\alpha}{\beta^2}\, F(x)(1-F(x)).$$

In what follows we use the integral

$$\begin{aligned}\int x^k \ln(a+bx)\,dx &= \frac{1}{k+1}\left(x^{k+1} - \frac{(-a)^{k+1}}{b^{k+1}}\right)\ln(a+bx)\\ &\quad + \frac{1}{k+1}\sum_{j=1}^{k+1}\frac{(-1)^j x^{k-j+2} a^{j+1}}{(k-j+1)b^{j-1}} \qquad \text{(cf. [12, 2.629.1])},\end{aligned}$$

from which we have

$$\int x^k \ln(1-x)\,dx = \frac{1}{k+1}\left[(x^{k+1}-1)\ln(1-x) - \sum_{j=1}^{k+1}\frac{x^j}{j}\right],$$

and

$$\int_0^1 x^k \ln(1-x)\,dx = -\frac{1}{k+1}\sum_{j=1}^{k+1}\frac{1}{j}.$$

Now integrating by parts we see that

$$\begin{aligned}\int x^k \ln(1-x)\ln x\,dx &= \frac{1}{k+1}\left[(x^{k+1}-1)\ln(1-x) - \sum_{j=1}^{k+1}\frac{x^j}{j}\right]\ln x\\ &\quad - \frac{1}{k+1}\int\left[(x^{k+1}-1)\ln(1-x) - \sum_{j=1}^{k+1}\frac{x^j}{j}\right]\frac{1}{x}\,dx\\ &= \frac{1}{k+1}\left[(x^{k+1}-1)\ln(1-x) - \sum_{j=1}^{k+1}\frac{x^j}{j}\right]\ln x\\ &\quad - \frac{1}{k+1}\left[\int x^k\ln(1-x)\,dx - \int\frac{\ln(1-x)}{x}\,dx - \sum_{j=1}^{k+1}\frac{x^j}{j}\right],\end{aligned}$$

and after using

$$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6} \qquad \text{(cf. [12, 3.621.2])},$$

we have

$$\int_0^1 x^k \ln(1-x)\ln x\,dx = \frac{1}{(k+1)^2}\sum_{j=1}^{k+1}\frac{1}{j} + \frac{1}{k+1}\int_0^1 \frac{\ln x}{1-x}\,dx + \frac{1}{k+1}\sum_{j=1}^{k+1}\frac{1}{j^2} = \frac{1}{(k+1)^2}\left(T_{k+1} - (k+1)\left(\frac{\pi^2}{6} - T^*_{k+1}\right)\right),$$

where

$$T_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}, \qquad T^*_k = 1 + \frac{1}{2^2} + \ldots + \frac{1}{k^2}.$$

Then

$$\begin{aligned} d_k(\alpha) &= -2k^2 E(1-F(X))^{k-2}(k\ln(1-F(X))+1)\,\frac{\partial F}{\partial \alpha} \\ &= 2k^3\,\frac{1}{\beta}\,E(1-F(X))^{k-1}F(X)\ln(1-F(X)) \\ &\quad + 2k^2\,\frac{1}{\beta}\,E(1-F(X))^{k-1}F(X) \\ &= 2k^3\,\frac{1}{\beta}\int_0^1 y^{k-1}(1-y)\ln y\,dy + 2k^2\,\frac{1}{\beta}\int_0^1 y^{k-1}(1-y)\,dy \\ &\qquad\qquad (\text{putting } y = 1-F(x)) \\ &= -\frac{2}{\beta}\,\frac{k^2}{(k+1)^2}. \end{aligned}$$

Similarly

$$\begin{aligned} d_k(\beta) &= 2k^3\,\frac{1}{\beta}\int_0^1 y^{k-1}(1-y)\ln\frac{1-y}{y}\ln y\,dy \\ &\quad + 2k^2\,\frac{1}{\beta}\int_0^1 y^{k-1}(1-y)\ln\frac{1-y}{y}\,dy \\ &= 2k^3\,\frac{1}{\beta}\Big[\int_0^1 y^{k-1}\ln y\ln(1-y)\,dy - \int_0^1 y^k\ln y\ln(1-y)\,dy \\ &\quad - \int_0^1 y^{k-1}\ln^2 y dy + \int_0^1 y^k\ln^2 y\,dy\Big] \\ &\quad + 2k^2\,\frac{1}{\beta}\Big[\int_0^1 y^{k-1}\ln(1-y)\,dy - \int_0^1 y^k\ln(1-y)\,dy \end{aligned}$$

$$
\begin{aligned}
&\quad - \int_0^1 y^{k-1} \ln y \, dy + \int_0^1 y^k \ln y \, dy \Big] \\
&= 2k^3 \, \frac{1}{\beta} \left[\frac{1}{k^2} \left(T_k - k \left(\frac{\pi^2}{6} - T_k^* \right) \right) \right. \\
&\quad - \frac{1}{(k+1)^2} \left(T_{k+1} - (k+1) \left(\frac{\pi^2}{6} - T_{k+1}^* \right) \right) \\
&\quad \left. - \frac{2}{k^3} + \frac{2}{(k+1)^3} \right] + \left[-T_k + T_{k+1} + \frac{1}{k^2} - \frac{1}{(k+1)^2} \right] \\
&\quad + 2k^2 \frac{1}{\beta} \left[- \left(\frac{1}{k} - \frac{1}{k+1} \right) T_k + \frac{1}{(k+1)^2} + \frac{1}{k^2} - \frac{1}{(k+1)^2} \right] \\
&= \frac{2}{\beta} \, \frac{k^2}{(k+1)^2} \left[T_k - (k+1) \left(\frac{\pi^2}{6} - T_k^* \right) - \frac{(k+1)^2}{k^2} \right].
\end{aligned}
$$

Thus

$$
\boldsymbol{d}_k = -\frac{2}{\beta} \, \frac{k^2}{(k+1)^2} \left[\begin{matrix} 1 \\ (k+1)^2/k^2 - T_k - (k+1)(T_k^* - \pi^2/6) \end{matrix} \right].
$$

Taking into account that

$$
\mathcal{I}^{-1} = 3\beta^2 \begin{bmatrix} 1 & 0 \\ 0 & 3/(3+\pi^2) \end{bmatrix}
$$

we obtain

$$
\mathcal{K}_k = \frac{12k^4}{(k+1)^4} \left[1 + \frac{3}{3+\pi^2} \left(\frac{(k+1)^2}{k^2} - T_k - (k+1) \left(T_k^* - \frac{\pi^2}{6} \right) \right)^2 \right].
$$

Thus we have

PROPOSITION 6. *A goodness-of-fit test for $F \in \mathrm{Log}(\alpha, \beta)$ is given by*

$$
\begin{aligned}
\widehat{T}_{kN} = \frac{k^5}{N(8k - \mathcal{K}_k)} & \left(\sum_{i=1}^{N} \ln^2 \frac{\exp(-(U_{ki} - \widehat{\alpha}_n)/\widehat{\beta}_n)}{1 + \exp(-(U_{ki} - \widehat{\alpha}_n)/\widehat{\beta}_n)} \right. \\
& \left. + \frac{2}{k} \sum_{i=1}^{N} \ln \frac{\exp(-(U_{ki} - \widehat{\alpha}_n)/\widehat{\beta}_n)}{1 + \exp(-(U_{ki} - \widehat{\alpha}_n)/\widehat{\beta}_n)} \right)^2,
\end{aligned}
$$

where $\widehat{\alpha}_n$ and $\widehat{\beta}_n$ are obtained by numerical solution of the equations

$$
\frac{\partial L_n}{\partial \alpha} = \frac{\partial L_n}{\partial \beta} = 0
$$

for

$$
L_n(\alpha, \beta) = -n \ln \beta - \frac{1}{\beta} \sum_{i=1}^{n} (x_i - \alpha) - 2 \sum_{i=1}^{n} \left(1 + \exp \left(-\frac{x_i - \alpha}{\beta} \right) \right).
$$

Numerical evaluation of $\mathcal{K}_k$

k	1	2	3	4	5
$\mathcal{K}_k$	3.96740	4.43882	4.82873	5.30836	5.86775

6°. Let $F \in C(\alpha,\beta)$, i.e.

$$F(x)=\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta},\qquad -\infty<x<\infty;\ \alpha\in\mathbb{R},\ \beta>0,$$

$$f(x)=\frac{1}{\pi\beta}\,\frac{1}{1+((x-\alpha)/\beta)^2},$$

and

$$\frac{\partial F}{\partial\alpha}=-f(x),\qquad \frac{\partial F}{\partial\beta}=-\frac{x-\alpha}{\beta}\,f(x).$$

Then

$$\begin{aligned}
d_k(\alpha)=&-2k^2E(1-F(X))^{k-2}(k\ln(1-F(X))+1)\,\frac{\partial F}{\partial\alpha}\\
=&\frac{2k^3}{\pi^2\beta^2}\int_{-\infty}^{\infty}\left(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}\right)^{k-2}\ln\left(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}\right)\\
&\times\frac{1}{(1+((x-\alpha)/\beta)^2)^2}\,dx\\
&+\frac{2k^2}{\pi^2\beta^2}\int_{-\infty}^{\infty}\left(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}\right)^{k-2}\frac{1}{(1+((x-\alpha)/\beta)^2)^2}\,dx.
\end{aligned}$$

Putting

$$y=\frac{1}{2}-\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta},\qquad \frac{x-\alpha}{\beta}=\tan\pi\left(\frac{1}{2}-y\right),$$

$$dy=-\frac{1}{\pi\beta}\,\frac{1}{1+(x-\alpha)^2/\beta^2}\,dx,$$

we get

$$\begin{aligned}
d_k(\alpha)&=\frac{2k^3}{\pi\beta}\int_0^1 y^{k-2}\ln y\cos^2\pi\left(\frac{1}{2}-y\right)dy+\frac{2k^2}{\pi\beta}\int_0^1 y^{k-2}\cos^2\pi\left(\frac{1}{2}-y\right)dy\\
&=\frac{2k^3}{\pi\beta}\int_0^1 y^{k-2}\ln y\,\frac{1+\cos\pi(1-2y)}{2}\,dy\\
&\quad+\frac{2k^2}{\pi\beta}\int_0^1 y^{k-2}\,\frac{1+\cos\pi(1-2y)}{2}\,dy
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{k^3}{\pi\beta}\int_0^1 (\cos 2\pi y - 1) y^{k-2}\ln y\, dy - \frac{k^2}{\pi\beta}\int_0^1 (\cos 2\pi y - 1) y^{k-2}\, dy \\
&= -\frac{k^3}{\pi\beta}\sum_{j=0}^{\infty}(-1)^j \frac{(2\pi)^{2j}}{(2j)!}\int_0^1 y^{k-2+2j}\ln y\, dy \\
&\quad -\frac{k^2}{\pi\beta}\sum_{j=1}^{\infty}(-1)^j \frac{(2\pi y)^{2j}}{(2j)!}\int_0^1 y^{k-2+2j}\, dy \\
&= \frac{k^3}{\pi\beta}\sum_{j=1}^{\infty}(-1)^j \frac{(2\pi)^{2j}}{(2j)!}\frac{1}{(k-1+2j)^2} - \frac{k^2}{\pi\beta}\sum_{j=1}^{\infty}(-1)^j \frac{(2\pi)^{2j}}{(2j)!}\frac{1}{k-1+2j} \\
&= -\frac{k^2}{\pi\beta}\sum_{j=1}^{\infty}(-1)^j \frac{(2\pi)^{2j}}{(2j)!}\frac{2j-1}{(k+2j-1)^2}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
d_k(\beta) &= -2k^2 E(1-F(X))^{k-2}(k\ln(1-F(X))+1)\frac{\partial F}{\partial\beta} \\
&= \frac{2k^3}{\pi\beta}\int_0^1 y^{k-2}\ln y \tan\pi\left(\frac{1}{2}-y\right)\cos^2\pi\left(\frac{1}{2}-y\right) dy \\
&\quad + \frac{2k^2}{\pi\beta}\int_0^1 y^{k-2}\tan\pi\left(\frac{1}{2}-y\right)\cos^2\pi\left(\frac{1}{2}-y\right) dy \\
&= \frac{k^3}{\pi\beta}\int_0^1 y^{k-2}\ln y \sin 2\pi y\, dy + \frac{k^2}{\pi\beta}\int_0^1 y^{k-2}\sin 2\pi y\, dy \\
&= \frac{k^3}{\pi\beta}\sum_{j=0}^{\infty}(-1)^j \frac{(2\pi)^{2j+1}}{(2j+1)!}\int_0^1 y^{k-1+2j}\ln y\, dy \\
&\quad + \frac{k^2}{\pi\beta}\sum_{j=0}^{\infty}(-1)^j \frac{(2\pi)^{2j+1}}{(2j+1)!}\int_0^1 y^{k-1+2j}\, dy \\
&= -\frac{k^3}{\pi\beta}\sum_{j=0}^{\infty}(-1)^j \frac{(2\pi)^{2j+1}}{(2j+1)!}\frac{1}{(k+2j)^2} \\
&\quad + \frac{k^2}{\pi\beta}\sum_{j=0}^{\infty}(-1)^j \frac{(2\pi)^{2j+1}}{(2j+1)!}\frac{1}{k+2j} \\
&= \frac{k^2}{\pi\beta}\sum_{j=1}^{\infty}(-1)^j \frac{(2\pi)^{2j+1}}{(2j+1)!}\frac{2j}{(k+2j)^2}.
\end{aligned}
$$

Therefore

$$\boldsymbol{d}_k = -\frac{k^2}{\pi\beta}\begin{bmatrix} -\sum_{j=1}^{\infty}(-1)^j \frac{(2\pi)^{2j}}{(2j)!}\frac{2j-1}{(k+2j-1)^2} \\ \sum_{j=1}^{\infty}(-1)^j \frac{(2\pi)^{2j+1}}{(2j+1)!}\frac{2j}{(k+2j)^2} \end{bmatrix}.$$

Taking into account that

$$\mathcal{I}^{-1} = 2\beta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have

$$\mathcal{K}_k = 8k^4\Bigg[\bigg(\sum_{j=1}^{\infty}(-1)^j \frac{(2\pi)^{2j-1}}{(2j)!}\frac{2j-1}{(k+2j-1)^2}\bigg)^2 + \bigg(\sum_{j=1}^{\infty}(-1)^j \frac{(2\pi)^{2j}}{(2j+1)!}\frac{2j}{(k+2j)^2}\bigg)^2\Bigg] =: 8k^4(S_{1k}^2 + S_{2k}^2).$$

Thus we have

Proposition 7. *A goodness-of-fit test for $F \in C(\alpha,\beta)$ is given by*

$$\widehat{T}_{kN} = \frac{k^4}{8N(1-k^3(S_{1k}^2+S_{2k}^2))}\bigg(\sum_{i=1}^{N}\ln^2\bigg(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{U_{ki}-\widehat{\alpha}_n}{\widehat{\beta}_n}\bigg) + \frac{2}{k}\sum_{i=1}^{N}\ln\bigg(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{U_{ki}-\widehat{\alpha}_n}{\widehat{\beta}_n}\bigg)\bigg)^2,$$

where $\widehat{\alpha}_n$ and $\widehat{\beta}_n$ are obtained by solving numerically the equations

$$\frac{\partial L_n}{\partial\alpha} = \frac{\partial L_n}{\partial\beta} = 0$$

for

$$L_n(\alpha,\beta) = -\sum_{i=1}^{n}(\ln\pi - \beta + \ln(\beta^2 + (x_i-\alpha)^2)).$$

Numerical evaluation of $\mathcal{K}_k$

k	1	2	3	4	5
$\mathcal{K}_k$	1.15379	2.92783	4.27882	5.28426	6.03719

7°. Let $F \in \mathrm{N}(\mu,\sigma^2)$, i.e.

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt, \qquad -\infty < x < \infty;\ \mu\in\mathbb{R},\ \sigma^2>0,$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

We see that

$$\frac{\partial F(x;\mu,\sigma^2)}{\partial \mu} = -f(x;\mu,\sigma^2), \qquad \frac{\partial F(x;\mu,\sigma^2)}{\partial \sigma^2} = -\frac{x-\mu}{2\sigma^2}\, f(x;\mu,\sigma^2).$$

Now we use the probability function

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy.$$

Then

$$F(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)$$

and

$$\begin{aligned}
d_k(\mu) =&\ -2k^2 E(1-F(X))^{k-2}(k\ln(1-F(X))+1)\,\frac{\partial F}{\partial \mu}\\
=&\ 2k^3 \int_{-\infty}^{\infty} \left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)^{k-2}\\
&\times \ln\left[\frac{1}{2}\left(1-\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)\right]\frac{1}{2\pi\sigma^2}\, e^{-(x-\mu)^2/\sigma^2} dx\\
&+ 2k^2 \int_{-\infty}^{\infty} \left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)^{k-2} \frac{1}{2\pi\sigma^2}\, e^{-(x-\mu)^2/\sigma^2}\, dx\\
=&\ \frac{2k^3}{\sqrt{2}\,\pi\sigma}\,\frac{1}{2^{k-2}} \int_{-\infty}^{\infty} (1-\operatorname{erf} z)^{k-2}\ln(1-\operatorname{erf} z) e^{-2z^2}\, dz\\
&- \frac{2k^2}{\sqrt{2}\,\pi\sigma}\,\frac{1}{2^{k-2}}(k\ln 2 - 1) \int_{-\infty}^{\infty} (1-\operatorname{erf} z)^{k-2} e^{-2z^2}\, dz\\
=&\ -\frac{4\sqrt{2}\,k^3}{\pi\sigma}\,\frac{1}{2^k} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \sum_{i=1}^{\infty} \frac{1}{i} \int_{-\infty}^{\infty} (\operatorname{erf} z)^{j+i} e^{-2z^2}\, dz\\
&- \frac{4\sqrt{2}\,k^3}{\pi\sigma}\,\frac{1}{2^k}\left(\ln 2 - \frac{1}{k}\right) \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \int_{-\infty}^{\infty} (\operatorname{erf} z)^j e^{-2z^2}\, dz,
\end{aligned}$$

for $k > 1$. Similarly

$$\begin{aligned}
d_k(\sigma^2) =&\ -2k^2 E(1-F(X))^{k-2}(k\ln(1-F(X)+1)\,\frac{\partial F}{\partial \sigma^2}\\
=&\ \frac{k^3}{\pi\sigma^2}\,\frac{1}{2^{k-2}} \int_{-\infty}^{\infty} (1-\operatorname{erf} z)^{k-2}\ln(1-\operatorname{erf} z) z e^{-2z^2}\, dz
\end{aligned}$$

$$-\frac{k^2}{\pi\sigma^2}\frac{1}{2^{k-2}}(k\ln 2-1)\int_{-\infty}^{\infty}(1-\operatorname{erf} z)^{k-2}ze^{-2z^2}\,dz$$

$$= -\frac{4k^3}{\pi\sigma^2}\frac{1}{2^k}\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\Big(\sum_{i=1}^{\infty}\frac{1}{i}\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j+i}ze^{-2z^2}\,dz\Big)$$

$$-\frac{4k^3}{\pi\sigma^2}\frac{1}{2^k}\Big(\ln 2-\frac{1}{k}\Big)\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j}ze^{-2z^2}\,dz,$$

for $k>1$. Hence

$$\boldsymbol{d}_k = -\frac{4k^3}{\sigma\pi 2^k}\begin{pmatrix} \sqrt{2}\big(\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\big(\sum_{i=1}^{\infty}\frac{1}{i}\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j+i}e^{-2z^2}\,dz \\ +\big(\ln 2-\frac{1}{k}\big)\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j}e^{-2z^2}\,dz\big)\big) \\ \frac{1}{\sigma}\big(\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\big(\sum_{i=1}^{\infty}\frac{1}{i}\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j+i}ze^{-2z^2}\,dz \\ +\big(\ln 2-\frac{1}{k}\big)\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j}ze^{-2z^2}\,dz\big)\big) \end{pmatrix}.$$

Using the fact that

$$\mathcal{I}^{-1}=\sigma^2\begin{bmatrix}1 & 0\\ 0 & 2\sigma^2\end{bmatrix},$$

we obtain

$$\mathcal{K}_k = 2\Big(\frac{4k^3}{\pi 2^k}\Big)^2\bigg(\bigg(\Big(\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\Big(\sum_{i=1}^{\infty}\frac{1}{i}\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j+i}e^{-2z^2}\,dz$$

$$+\Big(\ln 2-\frac{1}{k}\Big)\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j}e^{-2z^2}\,dz\Big)\Big)^2+\Big(\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}$$

$$\times\Big(\sum_{i=1}^{\infty}\frac{1}{i}\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j+i}ze^{-2z^2}\,dz$$

$$+\Big(\ln 2-\frac{1}{k}\Big)\int_{-\infty}^{\infty}(\operatorname{erf} z)^{j}ze^{-2z^2}\,dz\Big)\Big)^2\bigg).$$

Thus we have

PROPOSITION 8. *A goodness-of-fit test for $F\in \mathrm{N}(\mu,\sigma^2)$ is given by*

$$\widehat{T}_{kN}=\frac{k^5}{N(8k-\mathcal{K}_k)}\Big(\sum_{i=1}^{N}\ln^2(1-F(U_{ki};\widehat{\mu}_n,\widehat{\sigma}_n^2))$$

$$+\frac{2}{k}\sum_{i=1}^{N}\ln(1-F(U_{ki};\widehat{\mu}_n,\widehat{\sigma}_n^2))\Big)^2\xrightarrow{D}\chi^2(1)$$

for $k>1$, where $\widehat{\mu}_n=\overline{X}_n$ and $\widehat{\sigma}_n^2=\frac{1}{n}\sum_{j=1}^{n}(X_j-\overline{X}_n)^2$.

Now consider $k = 1$. Then as above

$$d_1(\mu) = \frac{2\sqrt{2}}{\pi\sigma} \int_{-\infty}^{\infty} \frac{\ln(1 - \operatorname{erf} z)}{1 - \operatorname{erf} z}\, e^{-2z^2}\, dz - \frac{2\sqrt{2}}{\pi\sigma}\,(\ln 2 - 1) \int_{-\infty}^{\infty} \frac{1}{1 - \operatorname{erf} z}\, e^{-2z^2}\, dz.$$

Taking into account that

$$-\frac{\ln(1-x)}{1-x} = \sum_{j=1}^{\infty} T_j x^j \qquad \text{(cf. [12, 1.513.6])},$$

where $T_j = 1 + \frac{1}{2} + \ldots + \frac{1}{j}$, we have

$$\begin{aligned} d_1(\mu) = &-\frac{4\sqrt{2}}{\pi\sigma} \sum_{j=1}^{\infty} T_{2j} \int_0^{\infty} (\operatorname{erf} z)^{2j} e^{-2z^2}\, dz \\ &-\frac{4\sqrt{2}}{\pi\sigma}\,(\log 2 - 1) \sum_{j=0}^{\infty} \int_0^{\infty} (\operatorname{erf} z)^{2j} e^{-2z^2}\, dz \\ = &-\frac{2}{\sqrt{\pi}\,\sigma}\,(\ln 2 - 1) - \frac{4\sqrt{2}}{\pi\sigma} \sum_{j=1}^{\infty} (T_{2j} + \ln 2 - 1) \int_0^{\infty} (\operatorname{erf} z)^{2j} e^{-2z^2}\, dz \\ = &-\frac{2}{\sqrt{\pi}\,\sigma}\left[\ln 2 - 1 + 2\sqrt{\frac{2}{\pi}} \sum_{j=1}^{\infty} (T'_{2j} + \ln 2) \int_0^{\infty} (\operatorname{erf} z)^{2j} e^{-2z^2}\, dz\right], \end{aligned}$$

where we used the fact that $\operatorname{erf} z$ is an odd function, and $T'_j = \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{j}$.

Similarly

$$\begin{aligned} d_1(\sigma^2) = &\frac{2}{\pi\sigma^2} \int_{-\infty}^{\infty} \frac{\ln(1 - \operatorname{erf} z)}{1 - \operatorname{erf} z}\, z e^{-2z^2}\, dz \\ &-\frac{2}{\pi\sigma^2}\,(\ln 2 - 1) \int_{-\infty}^{\infty} \frac{1}{1 - \operatorname{erf} z}\, z e^{-2z^2}\, dz \\ = &-\frac{4}{\pi\sigma^2} \sum_{j=1}^{\infty} (T'_{2j-1} + \ln 2) \int_0^{1} (\operatorname{erf} z)^{2j-1} z e^{-2z^2}\, dz. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{K}_1 = \frac{4}{\pi}\Bigg[&\left(\ln 2 - 1 + 2\sqrt{\frac{2}{\pi}} \sum_{j=1}^{\infty} (T'_{2j} + \ln 2) \int_0^{1} (\operatorname{erf} z)^{2j} e^{-2z^2}\right)^2 \\ &+ \frac{8}{\pi}\left(\sum_{j=1}^{\infty} (T'_{2j-1} + \ln 2) \int_0^{\infty} (\operatorname{erf} z)^{2j} e^{-2z^2}\, dz\right)^2\Bigg], \end{aligned}$$

and

$$\widehat{T}_{1N} = \frac{1}{N(8-\mathcal{K}_1)}\Big(\sum_{i=1}^{N}\ln^2(1-F(X_i;\widehat{\mu}_n,\widehat{\sigma}^2)) \\ + 2\sum_{i=1}^{N}\ln(1-F(X_i;\widehat{\mu}_n,\widehat{\sigma}_n^2))\Big)^2.$$

Similarly for $k=2$,

$$d_2(\mu) = -\frac{2\sqrt{2}}{\sqrt{\pi}\,\sigma}\left(2\ln 2 - 1 + \frac{4}{\sqrt{\pi}}\sum_{j=1}^{\infty}\frac{1}{j}\int_0^{\infty}(\operatorname{erf} z)^{2j}e^{-2z^2}\,dz\right),$$

$$d_2(\sigma^2) = -\frac{16}{\pi\sigma^2}\sum_{j=1}^{\infty}\frac{1}{2j-1}\int_0^{\infty}(\operatorname{erf} z)^{2j-1}ze^{-2z^2}\,dz,$$

and

$$\mathcal{K}_2 = \frac{8}{\pi}\Bigg[\left(2\ln 2 - 1 + \frac{4}{\sqrt{\pi}}\sum_{j=1}^{\infty}\frac{1}{j}\int_0^{\infty}(\operatorname{erf} z)^{2j}e^{-2z^2}\,dz\right)^2 \\ + \frac{64}{\pi}\Big(\sum_{j=1}^{\infty}\frac{1}{2j-1}\int_0^{\infty}(\operatorname{erf} z)^{2j}ze^{-2z^2}\,dz\Big)^2\Bigg].$$

Thus

$$\widehat{T}_{2N} = \frac{32}{N(16-\mathcal{K}_2)}\Big(\sum_{i=1}^{N}\ln^2(1-F(U_{2i};\widehat{\mu}_n,\widehat{\sigma}_n^2)) \\ + \sum_{i=1}^{N}\ln(1-F(U_{2i};\widehat{\mu}_n,\widehat{\sigma}^2))\Big)^2.$$

Numerical evaluation of $\mathcal{K}_k$

k	1	2	3	4	5
$\mathcal{K}_k$	5.084149	4.330119	4.231260	4.443297	4.827289

8°. Let $F \in \mathrm{EV}(\alpha,\beta)$ (the extreme-value distribution), i.e.

$$F(x) = \exp\left(-\exp\left(-\frac{x-\alpha}{\beta}\right)\right), \qquad -\infty < x < \infty;\ \alpha\in\mathbb{R},\ \beta>0,$$

$$f(x) = \frac{1}{\beta}\exp(-e^{-(x-\alpha)/\beta})e^{-(x-\alpha)/\beta} = -\frac{1}{\beta}F(x)\ln F(x).$$

Then

$$\frac{\partial F}{\partial \alpha} = -\frac{1}{\beta}\exp\left(-\exp\left(-\frac{x-\alpha}{\beta}\right)\right)e^{-(x-\alpha)/\beta} = \frac{1}{\beta}F(x)\ln F(x),$$

$$\begin{aligned}\frac{\partial F}{\partial \beta} &= \frac{1}{\beta}\,\frac{x-\beta}{\beta}\exp(-e^{-(x-\alpha)/\beta})e^{-(x-\alpha)/\beta}\\ &= \frac{1}{\beta}F(x)(-\ln F(x))\ln(-\ln F(x)).\end{aligned}$$

In what follows we use the following integrals:

$$\int_0^1 \left(\ln\frac{1}{x}\right)^p \ln(1-qx)\,\frac{dx}{x} = -\Gamma(p+1)\sum_{k=1}^{\infty}\frac{q^k}{k^{q+2}} \qquad \text{(cf. [12, 3.673.6])},$$

whence

$$\int_0^1 \frac{\ln x\cdot\ln(1-x)}{x}\,dx = \sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(3),$$

and

$$\int_0^1 x^{p-1}(-\ln x)^{q-1}\ln(-\ln x)\,dx = \frac{\Gamma(q)}{p^q}(\psi(q)-\ln p), \qquad p>0,\ q>0 \text{ (cf. [1])},$$

where

$$\psi(q) = \frac{\Gamma'(q)}{\Gamma(q)}, \qquad \psi(q+1) = \frac{1}{q}+\psi(q), \qquad \psi(1) = -\gamma,$$

hence

$$\int_0^1 x^{j+1}(-\ln x)\ln(-\ln x)\,dx = \frac{1}{(j+2)^2}(1-\gamma-\ln(j+2)).$$

Here we treat $k=1$ and $k>1$ separately. For $k>1$ we have

$$\begin{aligned}d_k(\alpha) = {} & -2k^3E(1-F(X))^{k-2}\ln(1-F(X))\,\frac{\partial F}{\partial\alpha}\\ & -2k^2E(1-F(X))^{k-2}\,\frac{\partial F}{\partial\alpha}\\ = {} & -\frac{2k^3}{\beta}\int_{-\infty}^{\infty}(1-F(x))^{k-2}\ln(1-F(x))F(x)\ln F(x)f(x)\,dx\\ & -\frac{2k^2}{\beta}\int_{-\infty}^{\infty}(1-F(x))^{k-2}F(x)\ln F(x)f(x)\,dx\\ = {} & -\frac{2k^3}{\beta}\int_0^1 y^{k-2}(1-y)\ln(1-y)\ln y\,dy\end{aligned}$$

$$
\begin{aligned}
&- \frac{2k^2}{\beta} \int\limits_0^1 y^{k-2}(1-y)\ln(1-y)\,dy \\
&= -\frac{2k^3}{\beta}\left[\frac{1}{(k-1)^2}\left(T_{k-1} - (k-1)\left(\frac{\pi^2}{6} - T^*_{k-1}\right)\right)\right. \\
&\quad \left. - \frac{1}{k^2}\left(T_k - k\left(\frac{\pi^2}{6} - T^*_k\right)\right)\right] - \frac{2k^2}{\beta}\left[-\frac{1}{k-1}\,T_{k-1} + \frac{1}{k}\,T_k\right] \\
&= \frac{2k^2}{(k-1)\beta}\left[\frac{\pi^2}{6} - T^*_k - \frac{1}{k-1}\,(T_k - 1)\right],
\end{aligned}
$$

where T_k and T^*_k are defined in 5°.

Similarly

$$
\begin{aligned}
d_k(\beta) &= -\frac{2k^3}{\beta}\int\limits_0^1 (1-y)^{k-2} y \ln(1-y)(-\ln y)\ln(-\ln y)\,dy \\
&\quad - \frac{2k^2}{\beta}\int\limits_0^1 (1-y)^{k-2} y(-\ln y)\ln(-\ln y)\,dy \\
&= -\frac{2k^3}{\beta}\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j \int\limits_0^1 y^{j+1}(-\ln y)\ln(1-y)\ln(-\ln y)\,dy \\
&\quad - \frac{2k^2}{\beta}\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j \int\limits_0^1 y^{j+1}(-\ln y)\ln(-\ln y)\,dy \\
&= -\frac{2k^3}{\beta}\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j \int\limits_0^1 y^{j+1}(-\ln y)\ln(1-y)\ln(-\ln y)\,dy \\
&\quad - \frac{2k^2}{\beta}\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j \frac{1}{(j+2)^2}\,(\psi(2) - \ln(j+2)) \\
&= \frac{2k^3}{\beta}\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j \sum_{i=1}^{\infty}\int\limits_0^1 y^{i+j+1}(-\ln y)\ln(-\ln y)\,dy \\
&\quad - \frac{2k^2}{\beta}\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j \frac{1}{(j+2)^2}\,(1 - \gamma - \ln(j+2)) \\
&= \frac{2k^3}{\beta}\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j \sum_{i=1}^{\infty}\frac{1}{i(i+j+2)^2}\,(1 - \gamma - \ln(i+j+2)) \\
&\quad - \frac{2k^2}{\beta}\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j \frac{1}{(j+2)^2}\,(1 - \gamma - \ln(j+2))
\end{aligned}
$$

$$= -\frac{2k^2}{\beta}\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\frac{1}{(j+2)^2}\Bigg[1-\gamma-\ln(j+2)$$
$$-k(j+2)^2\sum_{i=1}^{\infty}\frac{1}{i(i+j+2)^2}\left(1-\gamma-\ln(i+j+2)\right)\Bigg].$$

Hence

$$\boldsymbol{d}_k = \frac{2k^2}{\beta}\left[\begin{array}{c} \frac{1}{k-1}\left(\frac{\pi^2}{6}-T_k^*-\frac{1}{k-1}(T_k-1)\right) \\ -\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\frac{1}{(j+2)^2}\big[1-\gamma-\ln(j+2) \\ -k(j+2)^2\sum_{i=1}^{\infty}\frac{1}{i(i+j+2)^2}(1-\gamma-\ln(i+j+2))\big] \end{array}\right].$$

Using the fact that

$$\mathcal{I}^{-1} = \frac{6\beta^2}{\pi^2}\begin{bmatrix}(1-\gamma)^2+\pi^2/6 & 1-\gamma \\ 1-\gamma & 1\end{bmatrix},$$

we obtain

$$\mathcal{K}_k = \frac{24k^4}{\pi^2}\left[\left((1-\gamma)^2+\frac{\pi^2}{6}\right)a_k^2+2(1-\gamma)a_kb_k+b_k^2\right],$$

where

$$a_k = \frac{1}{k-1}\left(\frac{\pi^2}{6}-T_k^*-\frac{1}{k-1}(T_k-1)\right),$$
$$b_k = -\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\frac{1}{(j+2)^2}\Bigg[1-\gamma-\ln(j+2)$$
$$-k(j+2)^2\sum_{i=1}^{\infty}\frac{1}{i(i+j+2)^2}\left(1-\gamma-\ln(i+j+2)\right)\Bigg].$$

Now we consider $k=1$. Then as above

$$d_1(\alpha) = -\frac{2}{\beta}\left[\int_0^1\frac{\ln y}{y}\,(1-y)\ln(1-y)\,dy+\int_0^1\frac{(1-y)}{y}\ln(1-y)\,dy\right]$$
$$= -\frac{2}{\beta}\left[\sum_{n=1}^{\infty}\frac{1}{n^3}-\left(2-\frac{\pi^2}{6}\right)-\frac{\pi^2}{6}+1\right] = \frac{2}{\beta}\,[1-\zeta(3)],$$

and similarly

$$d_1(\beta) = -\frac{2}{\beta}\Bigg[\int_0^1\frac{\ln(1-y)}{1-y}\,y(-\ln y)\ln(-\ln y)\,dy$$
$$+\int_0^1\frac{y}{1-y}\,(-\ln y)\ln(-\ln y)\,dy\Bigg]$$

$$
\begin{aligned}
&= -\frac{2}{\beta}\Big[-\sum_{n=1}^{\infty} T_n \int_0^1 y^{n+1}(-\ln y)\ln(-\ln y)\,dy \\
&\quad + \sum_{n=0}^{\infty}\int_0^1 y^{n+1}(-\ln y)\ln(-\ln y)\Big] \\
&= -\frac{2}{\beta}\Big[-\sum_{n=1}^{\infty} T_n \frac{1}{(n+2)^2}(1-\gamma-\ln(n+2)) \\
&\quad + \sum_{n=0}^{\infty}\frac{1}{(n+2)^2}(1-\gamma-\ln(n+2))\Big] \\
&= -\frac{2}{\beta}\Big[-(1-\gamma)\sum_{n=1}^{\infty} T_n \frac{1}{(n+2)^2} + \sum_{n=1}^{\infty} T_n \frac{\ln(n+2)}{(n+2)^2} \\
&\quad + (1-\gamma)\sum_{n=0}^{\infty}\frac{1}{(n+2)^2} - \sum_{n=1}^{\infty}\frac{\ln(n+2)}{(n+2)^2} - \frac{1}{4}\ln 2\Big] \\
&= -\frac{2}{\beta}\Big[(1-\gamma)\Big(\frac{\pi^2}{6} - 1 - \sum_{n=1}^{\infty} T_n \frac{1}{(n+2)^2}\Big) \\
&\quad + \sum_{n=2}^{\infty}(T_n - 1)\frac{\ln(n+2)}{(n+2)^2} - \frac{1}{4}\ln 2\Big] \\
&= \frac{2}{\beta}\Big[\frac{1}{4}\ln 2 - \sum_{n=2}^{\infty}(T_n - 1)\frac{\ln(n+2)}{(n+2)^2} \\
&\quad - (1-\gamma)\Big(\frac{\pi^2}{6} - 1 - \sum_{n=1}^{\infty} T_n \frac{1}{(n+2)^2}\Big)\Big].
\end{aligned}
$$

It follows that

$$
\mathcal{K}_1 = \frac{24}{\pi^2}\bigg(\Big((1-\gamma)^2 + \frac{\pi^2}{6}\Big)(1-\zeta(3))^2 + 2(1-\gamma)(1-\zeta(3))d_0 + d_0^2\bigg),
$$

where

$$
d_0 = \frac{1}{4}\ln 2 - \sum_{n=2}^{\infty}(T_n - 1)\frac{\ln(n+2)}{(n+2)^2} - (1-\gamma)\Big(\frac{\pi^2}{6} - 1 - \sum_{n=1}^{\infty} T_n \frac{1}{(n+2)^2}\Big).
$$

Thus we have

PROPOSITION 9. *A goodness-of-fit test for $F \in \mathrm{EV}(\alpha, \beta)$ is given by*

$$
\widehat{T}_{kN} = \frac{k^5}{N(8k - \mathcal{K}_k)}\bigg(\sum_{i=1}^{N} \ln^2\Big(1 - \exp\Big(-\exp\Big(-\frac{U_{ki} - \widehat{\alpha}_n}{\widehat{\beta}_n}\Big)\Big)\Big)\bigg)
$$

$$+\frac{2}{k}\sum_{i=1}^{N}\ln\left(1-\exp\left(-\exp\left(-\frac{U_{ki}-\widehat{\alpha}_n}{\widehat{\beta}_n}\right)\right)\right)\Bigg)^2 \xrightarrow{D} \chi^2(1),$$

for $k \geq 2$, *and for* $k = 1$,

$$\widehat{T}_{1N} = \frac{1}{N(8-\mathcal{K}_1)}\left(\sum_{i=1}^{N}\ln^2\left((1-\exp\left(-\frac{X_i-\widehat{\alpha}_n}{\widehat{\beta}_n}\right)\right)\right.$$
$$\left. + 2\sum_{i=1}^{N}\ln\left(1-\exp\left(-\exp\left(-\frac{X_i-\widehat{\alpha}_n}{\widehat{\beta}_n}\right)\right)\right)\right)^2 \xrightarrow{D} \chi^2(1),$$

where $\widehat{\alpha}_n$ *and* $\widehat{\beta}_n$ *are obtained by solving numerically the equations*

$$\frac{\partial L_n}{\partial\alpha} = \frac{\partial L_n}{\partial\beta} = 0$$

for

$$L_n(\alpha,\beta) = -n\ln\beta - \sum_{i=1}^{n}\exp\left(-\frac{x_i-\alpha}{\beta}\right) - \frac{n}{\beta}\,(\overline{x}_n-\alpha).$$

Numerical evaluation of $\mathcal{K}_k$

k	1	2	3	4	5
$\mathcal{K}_k$	3.47977	4.09061	4.48721	4.85553	5.24759

9°. Let $F \in \mathrm{U}(\alpha,\beta)$, i.e. $F(x) = \frac{x-\alpha}{\beta-\alpha}$, $\alpha \leq x \leq \beta$. Then by (2.5) we can prove

Proposition 10. *A goodness-of-fit test for* $F \in \mathrm{U}(\alpha,\beta)$ *is given by*

$$\widehat{T}_{kN} = \frac{k^4}{8N}\left(\sum_{i=1}^{N}\ln^2\frac{\widehat{\beta}_n-U_{ki}}{\widehat{\beta}_n-\widehat{\alpha}_n} + \frac{2}{k}\sum_{i=1}^{N}\ln\frac{\widehat{\beta}_n-U_{ki}}{\widehat{\beta}_n-\widehat{\alpha}_n}\right)^2 \xrightarrow{D} \chi^2(1),$$

where

$$\widehat{\alpha}_n = \min(X_1,\ldots,X_n), \qquad \widehat{\beta}_n = \max(X_1,\ldots,X_n).$$

This follows because $\sqrt{n}(\widehat{\alpha}_n-\alpha)$ and $\sqrt{n}(\widehat{\beta}_n-\beta)$ both converge to 0 in probability.

3. Comparisons with other tests. We intend to publish extensive simulations of the above tests in a further paper. Here we give some results for the 5% Normal tests in 7°, but only for $k = 1$, and $k = 2$, and only when the sample size is $n = 20$. The critical values were obtained using 100,000 samples, and the powers using 25,000 samples. The alternatives used were chosen from the paper [2]. With the numbering used there, they are:

	Symmetric		Skew
1	X, where $Y = \frac{1}{2}\ln(\frac{X}{1-X})$ and $Y \sim N(0,1)$	37	$X \sim$ Beta(3,2)
7	$X \sim$ Beta(2,2)	38	$X \sim$ Beta(2,1)
14	$X \sim$ t(10)	44	$X \sim$ Weibull(1,2)
32	$X \sim$ t(1)	50	$X \sim \chi^2(4)$
		56	$X \sim$ Weibull(1, $\frac{1}{2}$)

The following table shows the powers of $\widehat{T}_1$ and $\widehat{T}_2$, and also the powers of various other tests; these are quoted from [2], but there they were obtained using only 200 samples. The tests are:

K^2: the Bowman & Shenton test
R: the rectangle test
W: the Shapiro–Wilk test
Y: the D'Agostino tests

Alt	T_1	T_2	K^2	R	W	Y
1	2.6	9.7	38	40	48	8
7	1.1	6.6	14	4	2	7
14	10.6	4.2	13	10	13	10
32	74.7	1.4	79	80	91	92
37	4.3	5.2	4	5	8	6
38	21.4	3.7	12	11	29	6
44	14.2	6.6	14	14	19	7
50	42.4	6.7	40	39	46	24
56	94.9	5.0	96	96	100	96

$\widehat{T}_2$ performs very poorly here, with the power often less than the level. But $\widehat{T}_1$ compares favourably with the others for some alternatives, although W is clearly the best.

Acknowledgements. The authors are grateful to the referee for a number of useful suggestions and comments.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards Appl. Math. Ser. 55, 1964.

[2] K. O. Bowman, R. B. D'Agostino and E. S. Pearson, *Tests for departure from normality: Comparison of powers*, Biometrika 64 (1977), 231–246.

[3] W. Dziubdziela and B. Kopociński, *Limiting properties of the k-th record values*, Zastos. Mat. 5 (1976), 187–190.

[4] Z. Grudzień and D. Szynal, *On the expected values of k-th record values and associated characterizations of distributions*, in: Proc. 4th Pannonian Symp. on Math. Statist. (Bad Tatzmannsdorf, 1985), 119–127.
[5] —, —, *Characterization of continuous distributions via moments of record values*, J. Appl. Statist. Sci. 9 (2000), 93–104.
[6] A. M. Hossain and W. J. Zimmer, *Comparisons of methods of estimation for a Pareto distribution of the first kind*, Comm. Statist. Theory Methods 29 (2000), 859–878.
[7] S. A. Klugman, H. H. Panjer and G. E. Willmot, *Loss Models*, *From Data to Decisions*, Wiley, 1998.
[8] G. D. Lin, *Characterizations of continuous distributions via expected values of two functions of order statistics*, Sankhyā Ser. A 52 (1990), 84–90.
[9] K. W. Morris and D. Szynal, *Goodness-of-fit tests using characterizations of continuous distributions*, Appl. Math. (Warsaw) 28 (2001), 151–168.
[10] —, —, *Goodness-of-fit tests based on characterizations in terms of moments of order statistics*, ibid. 29 (2002), 251–283.
[11] D. A. Pierce, *The asymptotic effect of substituting estimators for parameters in certain types of statistics*, Ann. Statist. 10 (1982), 475–478.
[12] I. M. Ryzhik and I. S. Gradshteĭn, *Tables of Integrals, Sums and Products*, PWN, Warszawa, 1964 (in Polish).

Department of Applied Mathematics
University of Adelaide
North Tce, Adelaide
South Australia, 5001
E-mail: kmorris@stats.adelaide.edu.au

Institute of Mathematics
Maria Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
E-mail: szynal@golem.umcs.lublin.pl

Received on 4.3.2002;
revised version on 2.12.2002 (1615)