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Commutators of weighted Hardy operators on Herz-type spaces

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Abstract. A sufficient condition for boundedness on Herz-type spaces of the commutator generated by a Lipschitz function and a weighted Hardy operator is obtained.

1. Introduction. Let f be a non-negative integrable function on $\mathbb{R}^+$. The classical Hardy operator is defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt,$$

and the celebrated Hardy integral inequality [9] is

$$\|Hf\|_p \le \frac{p}{p-1}\|f\|_p$$

when $p > 1$. The Hardy integral inequality has drawn considerable attention (see [1], [8] and [13]).

In 1984, Carton-Lebrun and Fosset [3] defined a weighted Hardy operator U_ψ as follows:

$$U_\psi f(x) = \int_0^1 f(tx)\psi(t)\,dt, \qquad x \in \mathbb{R}^n,$$

for a fixed function $\psi : [0,1] \to [0,\infty)$. In fact, U_ψ is related to the Hardy–Littlewood maximal operator in harmonic analysis. If $\psi \equiv 1$ and $n = 1$, then U_ψ is the classical Hardy operator. Carton-Lebrun and Fosset [3] found that U_ψ is bounded on $\mathrm{BMO}(\mathbb{R}^n)$ under certain conditions on ψ. And Xiao [14] proved that U_ψ is bounded on $L^p(\mathbb{R}^n)$ iff $\int_0^1 t^{-n/p}\psi(t)\,dt < \infty$.

On the other hand, the study of commutators of operators attracted much attention. A well known result of Coifman, Rochberg and Weiss states

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that the commutator U^b of a Calderón–Zygmund singular integral operator U is bounded on L^p $(1 < p < \infty)$ if and only if $b \in \mathrm{BMO}$. Further results are obtained in [12], [2], [10], [4], [6], [15] etc. In this paper we study the commutator of the weighted Hardy operator, U_ψ^b.

Definition 1.1. Let b a measurable, locally integrable function and $\psi : [0,1] \to [0,\infty)$ be a measurable function. Define the *commutator* of the weighted Hardy operator, U_ψ^b, as

$$U_\psi^b f = bU_\psi f - U_\psi bf.$$

In [5], Fu, Liu and Lu established a sufficient and necessary condition on the weight function ψ to ensure the L^p $(1 < p < \infty)$ boundedness of U_ψ^b when $b \in \mathrm{BMO}$. And Fu [4] obtained the $L^p \to L^q$ boundedness of the commutator of the classical Hardy operator when $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$. Can one generalize the boundedness to the commutators of the weighted Hardy operator when $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$? We will find a sufficient condition on the weight function ψ to get the boundedness on Herz-type spaces.

We first recall some definitions. Let q' be the conjugate exponent of q, that is, $1/q+1/q' = 1$. C will often be used to denote a constant, but it may vary from line to line. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_E is the characteristic function of the set E.

Definition 1.2. Let $0 < \beta < 1$. The *Lipschitz space* $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} = \sup_{x,h\in\mathbb{R}^n,\, h\neq 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.$$

Definition 1.3 ([11]). Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q < \infty$. The *homogeneous Herz space* $\dot{K}_q^{\alpha,p}$ is defined by $\dot{K}_q^{\alpha,p} = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\}$, where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \Big\}^{1/p}.$$

It is obvious that $\dot{K}_q^{0,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and for $\alpha \in \mathbb{R}$, $\dot{K}_q^{\alpha,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n, |x|^{\alpha q})$, the Lebesgue space $L^q(\mathbb{R}^n)$ with power weight $|x|^{\alpha q}$. Therefore, the Herz spaces are natural generalizations of the Lebesgue spaces with power weights.

Definition 1.4 ([7]). Let $0 < q < \infty$, $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $\lambda \ge 0$. The *homogeneous Morrey–Herz space* $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p}$$

with the usual modifications when $p=\infty$.

Comparing the homogeneous Morrey–Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ with the homogeneous Herz spaces $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$, we see that $M\dot{K}^{\alpha,0}_{p,q}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$.

Now, we state our main result.

THEOREM 1.1. *Let $\psi:[0,1]\to[0,\infty)$ be a measurable function, $0<\beta<1$, $b\in \mathrm{Lip}_\beta(\mathbb{R}^n)$, $0<q_2\le q_1<\infty$. If $\int_0^1 t^{-(\alpha+\beta+n/q_2-\lambda)}\psi(t)\,dt<\infty$, then U^b_ψ is bounded from $M\dot{K}^{\alpha+\beta+n/q_2-n/q_1,\lambda}_{p,q_1}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$.*

REMARK 1.2. If $\alpha+\beta+n/q_2-n/q_1=\lambda=0$, $q_1=p$ and $q_2=q$, then the conclusion of Theorem 1.1 is as follows: If $\int_0^1 t^{-n/p}\psi(t)\,dt<\infty$, then U^b_ψ is bounded from $L^p(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$.

2. Proof of main theorem. In this section, we will prove our main result. Before giving the proof, we state some lemmas we need.

LEMMA 2.1 ([12]). *For any x, $y\in\mathbb{R}^n$, if $f\in\mathrm{Lip}_\beta(\mathbb{R}^n)$, $0<\beta<1$, then $|f(x)-f(y)|\le|x-y|^\beta\|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)}$. For any cube $I\subset\mathbb{R}^n$,*

$$\sup_{x\in I}|f(x)-f_I|\le C|I|^{\beta/n}\|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)},\quad \textit{where}\quad f_I=\frac{1}{|I|}\int f.$$

The following lemma is easily proved. We omit the proof here.

LEMMA 2.2. *Let $b\in\mathrm{Lip}_\beta(\mathbb{R}^n)$, and define*

$$b_k=\frac{1}{|B_k|}\int_{B_k} b(x)\,dx\quad \textit{and}\quad b_{tk}=\frac{1}{|tB_k|}\int_{tB_k} b(x)\,dx$$

for $k\in\mathbb{Z}$ and $t\in(0,1)$. Then

$$|b_{tk}-b_k|\le C|B_k|^{\beta/n}\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)}.$$

Proof of Theorem 1.1. Observe that

$$\begin{aligned}\|(U^b_\psi f)\chi_k\|_{L^{q_2}(\mathbb{R}^n)} &= \Big\{\int_{C_k}\Big|\int_0^1 f(tx)(b(x)-b(tx))\psi(t)\,dt\Big|^{q_2}dx\Big\}^{1/q_2}\\ &\le \int_0^1\Big\{\int_{C_k}|f(tx)(b(x)-b_k)|^{q_2}\,dx\Big\}^{1/q_2}\psi(t)\,dt\end{aligned}$$

$$+\int_0^1 \Big\{ \int_{C_k} |f(tx)(b_{tk}-b_k)|^{q_2}\,dx \Big\}^{1/q_2} \psi(t)\,dt$$
$$+\int_0^1 \Big\{ \int_{C_k} |f(tx)(b(tx)-b_{tk})|^{q_2}\,dx \Big\}^{1/q_2} \psi(t)\,dt$$
$$= I_1 + I_2 + I_3.$$

Let $r = 1/q_2 - 1/q_1$. Since $t \in (0,1)$, there exists a non-negative number m such that $2^{-m-1} < t \le 2^{-m}$. By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} I_1 &\le \int_0^1 \Big(\int_{C_k} |f(tx)|^{q_1}\,dx \Big)^{1/q_1} \Big(\int_{C_k} |b(x)-b_k|^{1/r}\,dx \Big)^{r} \psi(t)\,dt \\ &\le C|B_k|^{\beta/n+r} \|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \int_0^1 \Big\{ \int_{C_k} |f(tx)|^{q_1}\,dx \Big\}^{1/q_1} \psi(t)\,dt \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} 2^{kn(\beta/n+r)} \int_0^1 \sum_{i=0}^{1} \|f\chi_{k-m-i}\|_{q_1} t^{-n/q_1} \psi(t)\,dt. \end{aligned} \tag{2.1}$$

Similarly, since $0 < \beta < 1$ and $0 < t^\beta < 1$, we can deduce that

$$I_3 \le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} 2^{kn(\beta/n+r)} \int_0^1 \sum_{i=0}^{1} \|f\chi_{k-m-i}\|_{q_1} t^{-n/q_1} \psi(t)\,dt, \tag{2.2}$$

Finally, by Lemma 2.2 and $1 < q_2 \le q_1$, we obtain

$$\begin{aligned} I_2 &\le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} |B_k|^{\beta/n} \int_0^1 \Big\{ \int_{C_k} |f(tx)|^{q_2}\,dx \Big\}^{1/q_2} \psi(t)\,dt \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} |B_k|^{\beta/n+r} \int_0^1 \Big\{ \int_{C_k} |f(tx)|^{q_1}\,dx \Big\}^{1/q_1} \psi(t)\,dt \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} 2^{kn(\beta/n+r)} \int_0^1 \sum_{i=0}^{1} \|f\chi_{k-m-i}\|_{q_1} t^{-n/q_1} \psi(t)\,dt. \end{aligned} \tag{2.3}$$

Combining of (2.1)–(2.3), we have

$$\|(U_\psi^b f)\chi_k\|_{q_2} \le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} 2^{-kn(\beta/n+r)} \int_0^1 \sum_{i=0}^{1} \|f\chi_{k-m-i}\|_{q_1} t^{-n/q_1} \psi(t)\,dt.$$

Denote $s = \alpha + \beta + n/q_2 - n/q_1$. We will consider the following two cases.

CASE 1: $p > 1$. Then

$$\begin{aligned}
&\|U_\psi^b f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}} \\
&\quad \le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \int_0^1 \Big(\sum_{k=-\infty}^{k_0} 2^{ksp} \|f\chi_{k-m-1}\|_{q_1}^p t^{-np/q_1} \psi(t)^p\Big)^{1/p} dt \\
&\qquad + C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \int_0^1 \Big(\sum_{k=-\infty}^{k_0} 2^{ksp} \|f\chi_{k-m}\|_{q_1}^p t^{-np/q_1} \psi(t)^p\Big)^{1/p} dt \\
&\quad \equiv A_1 + A_2,
\end{aligned}$$

where

$$\begin{aligned}
A_1 \le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} &\int_0^1 \sup_{k_0\in\mathbb{Z}} 2^{-(k_0-m-1)\lambda} \\
&\cdot \Big(\sum_{k=-\infty}^{k_0-m-1} 2^{ksp} \|f\chi_k\|_{q_1}^p 2^{(m+1)(s-\lambda)p} t^{-np/q_1} \psi(t)^p\Big)^{1/p} dt,
\end{aligned}$$

and

$$\begin{aligned}
A_2 \le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} &\int_0^1 \sup_{k_0\in\mathbb{Z}} 2^{-(k_0-m)\lambda} \\
&\cdot \Big(\sum_{k=-\infty}^{k_0-m} 2^{ksp} \|f\chi_k\|_{q_1}^p 2^{m(s-\lambda)p} t^{-np/q_1} \psi(t)^p\Big)^{1/p} dt,
\end{aligned}$$

and so

$$\begin{aligned}
\|U_\psi^b f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}} &\le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{s,\lambda}_{p,q_1}} \int_0^1 \{2^{(m+1)s} + 2^{ms}\} t^{-(n/q_1-\lambda)} \psi(t)\, dt \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{s,\lambda}_{p,q_1}} \int_0^1 t^{-(\alpha+\beta+n/q_2-\lambda)} \psi(t)\, dt.
\end{aligned}$$

CASE 2: $0 < p < 1$. Then

$$\begin{aligned}
&\|U_\psi^b f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}} \\
&\quad \le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Big\{\sum_{k=-\infty}^{k_0} 2^{ksp} \Big[\int_0^1 \|f\chi_{k-m-1}\|_{q_1} t^{-n/q_1} \psi(t)\, dt\Big]^p\Big\}^{1/p} \\
&\qquad + C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Big\{\sum_{k=-\infty}^{k_0} 2^{ksp} \Big[\int_0^1 \|f\chi_{k-m}\|_{q_1} t^{-n/q_1} \psi(t)\, dt\Big]^p\Big\}^{1/p} \\
&\quad \equiv B_1 + B_2,
\end{aligned}$$

where

$$B_1 \le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \sup_{k_0\in\mathbb{Z}} \Big\{ \sum_{k=-\infty}^{k_0} 2^{ksp} \Big[\int_0^1 2^{-(k-m-1)s}$$

$$\cdot \Big(\sum_{i=-\infty}^{k-m-1} 2^{isp} \|f\chi_i\|_{q_1}^p \Big)^{1/p} t^{-n/q_1} \psi(t)\, dt \Big]^p \Big\}^{1/p}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{s,\lambda}_{p,q_1}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}$$

$$\cdot \Big\{ \sum_{k=-\infty}^{k_0} 2^{k\lambda p} \Big[\int_0^1 2^{(m+1)(s-\lambda)} t^{-n/q_1} \psi(t)\, dt \Big]^p \Big\}^{1/p}$$

and

$$B_2 \le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{s,\lambda}_{p,q_1}}$$

$$\cdot \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Big\{ \sum_{k=-\infty}^{k_0} 2^{k\lambda p} \Big[\int_0^1 2^{m(s-\lambda)} t^{-n/q_1} \psi(t)\, dt \Big]^p \Big\}^{1/p},$$

and so

$$\|U_\psi^b f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta} \|f\|_{M\dot{K}^{s,\lambda}_{p,q_1}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Big\{ \sum_{k=-\infty}^{k_0} 2^{k\lambda p} \Big[\int_0^1 t^{-(\alpha+\beta+n/q_2-\lambda)} \psi(t)\, dt \Big]^p \Big\}^{1/p}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{s,\lambda}_{p,q_1}} \int_0^1 t^{-(\alpha+\beta+n/q_2-\lambda)} \psi(t)\, dt.$$

Thus the proof of Theorem 1.1 is finished. ■

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