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Compactness of composition operators acting on weighted Bergman–Orlicz spaces

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Abstract. We characterize compact composition operators acting on weighted Bergman–Orlicz spaces

$$\mathcal{A}_\alpha^\psi = \Big\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} \psi(|f(z)|)\, dA_\alpha(z) < \infty \Big\},$$

where $\alpha > -1$ and ψ is a strictly increasing, subadditive convex function defined on $[0, \infty)$ and satisfying $\psi(0) = 0$, the growth condition $\lim_{t\to\infty} \psi(t)/t = \infty$ and the Δ_2-condition. In fact, we prove that C_φ is compact on $\mathcal{A}_\alpha^\psi$ if and only if it is compact on the weighted Bergman space $\mathcal{A}_\alpha^2$.

1. Introduction. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the algebra of all holomorphic functions on $\mathbb{D}$. For $z, w \in \mathbb{D}$, let $\beta_z(w) = (z - w)/(1 - \bar{z}w)$ be the Möbius transformation of $\mathbb{D}$ which interchanges 0 and z. Let $dA(z) = \pi^{-1}dxdy = \pi^{-1}rdrd\theta$, where $z = x + iy$, denote the normalized area measure on $\mathbb{D}$. For each $\alpha \in (-1, \infty)$, we set

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z), \qquad z \in \mathbb{D}.$$

Then dA_α is a probability measure on $\mathbb{D}$. For $0 < p < \infty$, let $\mathcal{L}_\alpha^p$ be the weighted Lebesgue space which consists of all measurable functions f on $\mathbb{D}$ such that $\int_{\mathbb{D}} |f(z)|^p\, dA_\alpha(z) < \infty$. And $\mathcal{A}_\alpha^p = \mathcal{L}_\alpha^p \cap H(\mathbb{D})$ denotes the weighted Bergman space with the norm defined by

$$\|f\|_{\mathcal{A}_\alpha^p} = \Big(\int_{\mathbb{D}} |f(z)|^p\, dA_\alpha(z) \Big)^{1/p} < \infty.$$

The space $\mathcal{A}_\alpha^p$ is a Banach space if $1 \le p < \infty$. When $0 < p < 1$, $\mathcal{A}_\alpha^p$ is an F-space with respect to the translation invariant metric defined by $d_p^\alpha(f, g) = \|f - g\|_{\mathcal{A}_\alpha^p}^p$. The space $\mathcal{A}_0^p = \mathcal{A}^p$ is the Bergman space. Note that

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$K_z(w) = 1/(1-\overline{z}w)^2$ is the Bergman kernel and

$$k_z(w) = (1-|z|^2)/(1-\overline{z}w)^2 = (1-|z|^2)K_z(w) = -\beta_z'(w)$$

is the normalized kernel function for $\mathcal{A}^2$.

Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. By Littlewood's subordination theorem, every holomorphic self-map φ of $\mathbb{D}$ induces a bounded composition operator on the Hardy spaces and weighted Bergman spaces. These operators have gained increasing attention during the last three decades, mainly due to the fact that they provide, just as, for example, Hankel and Toeplitz operators, ways and means to link classical function theory to functional analysis and operator theory. For general background on composition operators, we refer to [CM], [S2] and references therein. In this paper, we will consider composition operators on the weighted Bergman–Orlicz spaces $\mathcal{A}^\psi_\alpha$ consisting of all holomorphic functions f on $\mathbb{D}$ which satisfy the condition

$$\int_{\mathbb{D}} \psi(|\lambda f(z)|)\, dA_\alpha(z) < \infty \tag{1.1}$$

for some $\lambda > 0$ depending upon f. Here $\psi : [0,\infty) \to [0,\infty)$ is a continuous increasing function such that $\psi(0) = 0$ and

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = \infty. \tag{1.2}$$

The space $\mathcal{A}^\psi_\alpha$ of all holomorphic functions in $\mathbb{D}$ satisfying (1.1) is an F-space. If ψ is convex in addition, then $\mathcal{A}^\psi_\alpha$ is a Banach space under the Luxemburg norm

$$N(f) = \inf\Big\{C > 0 : \int_{\mathbb{D}} \psi(|f|/C)\, dA_\alpha(z) \le 1\Big\}.$$

We say that a function ψ satisfies the Δ_2-*condition* if there exists a constant $K > 1$ such that $\psi(2t) \le K\psi(t)$ for all $t \ge 0$. If ψ is convex, then $\mathcal{A}^\psi_\alpha$ coincides with the set of all holomorphic functions f in $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} \psi(|f(z)|)\, dA_\alpha(z) < \infty. \tag{1.3}$$

Although (1.3) does not define a norm in $\mathcal{A}^\psi_\alpha$, the formula

$$d(f,g) = \int_{\mathbb{D}} \psi(|f(z)-g(z)|)\, dA_\alpha(z)$$

defines a translation-invariant metric on $\mathcal{A}^\psi_\alpha$, which turns $\mathcal{A}^\psi_\alpha$ into a complete metric space if ψ satisfies

$$\psi(x+y) \le \psi(x) + \psi(y). \tag{1.4}$$

Abusing notation, we will write

$$\|f\|_{\psi,\alpha} = \int_{\mathbb{D}} \psi(|f(z)|)\,dA_\alpha(z) \tag{1.5}$$

for $f \in \mathcal{A}^\psi_\alpha$. In this article, we will assume that $\psi : [0,\infty) \to [0,\infty)$ is a strictly increasing, convex function satisfying $\psi(0) = 0$, $\lim_{t\to\infty} \psi(t)/t = \infty$, (1.4) and the Δ_2-condition. For such a ψ, the function

$$\psi_1(t) = \int_0^t \frac{\psi(x)}{x}\,dx \quad (t \ge 0)$$

is differentiable everywhere on $[0,\infty)$. Furthermore, since ψ is a strictly increasing, convex function satisfying $\psi(0) = 0$, the function $\psi(t)/t$, $t \ge 0$, is increasing and

$$\psi(t) \ge \psi_1(t) \ge \int_{t/2}^t \frac{\psi(x)}{x}\,dx \ge \psi(t/2)$$

for all $t \ge 0$. Hence $\mathcal{A}^\psi_\alpha = \mathcal{A}^{\psi_1}_\alpha$. Similarly, we can define ψ_2 from ψ_1 as we have defined ψ_1 from ψ such that $\mathcal{A}^{\psi_1}_\alpha = \mathcal{A}^{\psi_2}_\alpha$. Hence, without loss of generality, we suppose that ψ is twice continuously differentiable, $\psi(0) = 0$ and $\psi'(0) = 0$. Recently, several authors have studied composition operators on the Hardy–Orlicz spaces (see [CK], [CKS], [LCW], [LLQR1], [Sh], [SS2]) and the Bergman–Orlicz spaces (see [LLQR2], [LC], [SS1], [St], [X]).

Suppose that φ is a holomorphic self-map of $\mathbb{D}$ with $\varphi(0) = 0$. Since $\psi(|f|)$ is subharmonic for $f \in H(\mathbb{D})$, the Littlewood subordination theorem implies that $f \circ \varphi \in \mathcal{A}^\psi_\alpha$ for $f \in \mathcal{A}^\psi_\alpha$. When $\varphi(0) \ne 0$, we consider the composition map $\Phi(z) = (\beta_{\varphi(0)} \circ \varphi)(z)$. Then $f \circ \varphi = (f \circ \Phi) \circ \beta_{\varphi(0)}$. Since $\mathcal{A}^\psi_\alpha$ is invariant under Möbius transformations, we also see that $f \circ \varphi \in \mathcal{A}^\psi_\alpha$ for $f \in \mathcal{A}^\psi_\alpha$. These facts imply that every holomorphic self-map φ of $\mathbb{D}$ induces a bounded composition operator C_φ on $\mathcal{A}^\psi_\alpha$. The goal of this paper is to characterize the compactness of C_φ in terms of properties of φ.

Throughout this paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Compactness of C_φ on $\mathcal{A}^\psi_\alpha$. Let X be a topological vector space whose topology is given by a translation invariant metric. In this paper, we say that a bounded operator T is *compact* on X if it takes every metric ball in X into a relatively compact subset in X. This type of compactness is often called *metric compactness*. When X is a Banach space, the metric compactness of T coincides with the usual notion of compactness (see [CKS]).

For $\zeta \in \partial\mathbb{D}$ and $0 < \delta < 2$, let $S(\zeta, \delta) = \{z \in \mathbb{D} : |z - \zeta| < \delta\}$. Recall that a positive Borel measure μ on $\mathbb{D}$ is an *α-Carleson measure* if

$$\sup_{\delta>0} \sup_{\zeta\in\partial\mathbb{D}} \frac{\mu(S(\zeta,\delta))}{\delta^\alpha} < \infty,$$

and μ is a *vanishing α-Carleson measure* if

$$\lim_{\delta\to 0} \sup_{\zeta\in\partial\mathbb{D}} \frac{\mu(S(\zeta,\delta))}{\delta^\alpha} = 0.$$

We proceed to present the main result of this paper.

Main Theorem 2.1. *Let φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) *C_φ is compact on $\mathcal{A}^\psi_\alpha$.*
(ii) *The pull back measure $A_\alpha \circ \varphi^{-1}$ is a vanishing $(\alpha + 2)$-Carleson measure on $\mathbb{D}$.*
(iii) *C_φ is compact on $\mathcal{A}^p_\alpha$, for every p, $0 < p < \infty$, and every $\alpha > -1$.*
(iv) *φ has an angular derivative at no point on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$.*

In order to prove Theorem 2.1, we need several lemmas.

Lemma 2.2. *Let f be a measurable function on $\mathbb{D}$. If $N(f) \le 1$, then*

$$\|f\|_{\psi,\alpha} \le N(f).$$

Proof. Since $N(f)$ is a norm on the Orlicz space $L^\psi(\mathbb{D}, dA_\alpha)$, the case $N(f) = 0$ is obvious. So we assume that $0 < N(f) \le 1$. Note that the convexity of ψ gives

$$\frac{\psi(c)}{c} \le \frac{\psi(d)}{d} \quad \text{for } 0 < c < d < \infty. \tag{2.1}$$

The assumption $N(f) \le 1$ and (2.1) show that

$$\frac{\psi(|f(z)|)}{|f(z)|} \le \frac{\psi\big(\frac{|f(z)|}{N(f)}\big)}{\frac{|f(z)|}{N(f)}},$$

and so

$$\int_{\mathbb{D}} \psi(|f(z)|)\, dA_\alpha(z) \le N(f) \int_{\mathbb{D}} \psi\left(\frac{|f(z)|}{N(f)}\right) dA_\alpha(z) \le N(f). \blacksquare$$

Now we incorporate some results from [S1]. For a holomorphic self-map φ of $\mathbb{D}$, the *Nevanlinna counting function* $N_\varphi(\cdot)$ is defined by

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}(w)} \log\frac{1}{|z|}, \qquad w \in \mathbb{D} \setminus \{\varphi(0)\},$$

where $z \in \varphi^{-1}(w)$ is repeated according to its multiplicity as a zero of $\varphi - w$. This N_φ often appears as a weight function of a measure in some change of variable formulas. In [S1], Shapiro also introduced the generalized Nevanlinna counting function $N_{\varphi,\alpha}$, defined for $\alpha > 0$ by

$$N_{\varphi,\alpha}(w) = \sum_{z\in\varphi^{-1}(w)} \left(\log\frac{1}{|z|}\right)^{\alpha}, \qquad w \in \mathbb{D}\setminus\{\varphi(0)\}.$$

He used N_φ and $N_{\varphi,\alpha}$ to study composition operators on Hardy and weighted Bergman spaces.

The counting function $N_{\varphi,\alpha}$ provides us with the following non-univalent change of variable formula (see [S1, p. 398]).

LEMMA 2.3. *If g is a positive measurable function on $\mathbb{D}$ and φ is a holomorphic map on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, then*

$$\int_{\mathbb{D}} (g\circ\varphi)(z)|\varphi'(z)|^2 \left(\log\frac{1}{|z|}\right)^{\alpha} dA(z) = \frac{2^\alpha}{\Gamma(\alpha+1)} \int_{\mathbb{D}} g(z) N_{\varphi,\alpha}(z)\, dA(z). \tag{2.2}$$

DEFINITION 2.4. We say that φ has a *finite angular derivative* at a point $\zeta \in \partial\mathbb{D}$ if there is a point $\omega \in \partial\mathbb{D}$ such that the difference quotient $(\varphi(z) - \omega)/(z-\zeta)$ has a finite limit as z tends to ζ non-tangentially.

The connection between composition operators and angular derivative is made by the following classical theorem.

THEOREM 2.5 (Julia–Carathéodory Theorem, [S2, p. 57]). *For $\zeta \in \partial\mathbb{D}$, the following are equivalent:*

(i) *φ has an angular derivative at ζ.*
(ii) *φ has a non-tangential limit of modulus 1 at ζ, and the complex derivative φ' has a finite limit at ζ. In this case, the limit of φ' is $\varphi'(\zeta)$.*
(iii) $\displaystyle\liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} = d < \infty.$

(For more information on the Julia–Carathéodory theorem and its connection with composition operators, see [MS, Section 3] or [S2, Chapter 4].)

We now recall the remarkable formula of C. S. Stanton (see [S2]) for integral means of subharmonic functions in the disk $\mathbb{D}$. If u is a positive subharmonic function on $\mathbb{D}$ and φ is a holomorphic self-map of $\mathbb{D}$, then for $0 < r < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} u(\varphi(re^{i\theta}))\, d\theta = u(\varphi(0)) + \int_{D(0,r)} N_\varphi(r,z)\, d\mu_u(z),$$

where $D(0,r) = \{z \in \mathbb{D} : |z| < r\}$ and μ_u is the Riesz measure of u, that is,

μ_u is a non-negative regular Borel measure on $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} h\, d\mu_u = \frac{1}{2}\int_{\mathbb{D}} u\Delta h\, dA$$

for all $h \in C_c^\infty(D)$, Δh denotes the Laplacian of h and $N_\varphi(r,\cdot)$ denotes the partial Nevanlinna counting function of φ defined by

$$N_\varphi(r,w) = \sum_{z\in\varphi^{-1}(w),\, |z|\le r} \log\frac{r}{|z|}.$$

For $f \in H(\mathbb{D})$, we denote the zero set of f by $Z(f) = \{z \in \mathbb{D} : f(z) = 0\}$. An easy calculation yields the following lemma.

Lemma 2.6. *Let $f \in H(\mathbb{D}) \setminus \{0\}$ and $g(z) = \psi(|f(z)|)$. Then*

$$\Delta g(z) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} g(z) = \left[\psi''(|f(z)|) + \frac{\psi'(|f(z)|)}{|f(z)|}\right]|f'(z)|^2, \qquad z \in \mathbb{D}\setminus Z(f).$$

Lemma 2.7. *Let $f \in H(\mathbb{D}) \setminus \{0\}$ and $g(z) = \psi(|f(z)|)$. Then the Riesz measure μ_g of g is given*

$$d\mu_g(z) = \frac{1}{2}\left[\psi''(|f(z)|) + \frac{\psi'(|f(z)|)}{|f(z)|}\right]|f'(z)|^2\, dA(z) \qquad \text{for } z \in \mathbb{D}\setminus Z(f).$$

Here we use the convention that the right-hand side in the above equation is defined to be zero in $Z(f)$.

Proof. Set

$$f_\psi(z) = \left[\psi''(|f(z)|) + \frac{\psi'(|f(z)|)}{|f(z)|}\right]|f'(z)|^2.$$

By Lemma 2.6, we have $\Delta g(z) = f_\psi(z)$ for $z \in \mathbb{D}\setminus Z(f)$. We need to show that

$$\int_{\mathbb{D}} g\Delta h\, dA = \int_{\mathbb{D}} h f_\psi\, dA \qquad \text{for all } h \in C_c^\infty(\mathbb{D}).$$

Since h has compact support and the zeros of f are isolated, we can assume, without loss of generality, that the support of h contains exactly one zero of f which we will take to be zero for convenience. Let $r > 0$ be such that $f(z) \ne 0$ for all $z \in \overline{D}(0,r) = \{z \in \mathbb{D} : |z| \le r\}$, $z \ne 0$. Let $D_r = \mathbb{D}\setminus \overline{D}(0,r)$ and $S_r = \partial D(0,r)$. Then, by Green's identity,

$$\int_{D_r} [h\Delta g - g\Delta h]\, dA = \int_{S_r}\left[h\frac{\partial g}{\partial \eta} - g\frac{\partial h}{\partial \eta}\right] ds.$$

We now show that the right-hand side tends to zero as $r \to 0$. Since $\partial h/\partial\eta$ and $g(re^{i\theta})$ are bounded in a neighbourhood of zero, we have

$$\lim_{r\to 0}\int_{S_r} g\frac{\partial h}{\partial \eta}\, ds = 0.$$

Note that

$$\int_{S_r} h\frac{\partial g}{\partial \eta}\,ds = -r\int_0^{2\pi} h(re^{i\theta})\frac{\partial}{\partial r}\psi(|f(re^{i\theta})|)\,d\theta,$$

and

$$\frac{\partial}{\partial r}\psi(|f(re^{i\theta})|) = \psi'(|f(re^{i\theta})|)\frac{\partial}{\partial r}(|f(re^{i\theta})|).$$

Suppose that f has a zero of mth order at 0 ($m \geq 1$). Then $f(z) = z^m q(z)$, where q is holomorphic in a neighbourhood of 0 with $q(0) \neq 0$, and so

$$\frac{\partial}{\partial r}|f(re^{i\theta})| = \frac{\partial}{\partial r}(r^m|q(re^{i\theta})|) = r^{m-1}(r|q'(re^{i\theta})| + m|q(re^{i\theta})|).$$

Suppose that $|q(z)| \leq M$ for all $z \in \overline{D}(0,r)$. Then by Cauchy's integral formula,

$$|q'(0)| \leq \frac{1}{2\pi}\int_{S_r}\frac{|q(re^{i\theta})|}{r^2}\,d\theta \leq \frac{M}{r}.$$

Thus for sufficiently small r, we have

$$\frac{\partial}{\partial r}|f(re^{i\theta})| = O(1)r^{m-1},$$

and so

$$r\left|h(re^{i\theta})\frac{\partial}{\partial r}\psi(|f(re^{i\theta})|)\right| \leq Cr^m\psi'(|f(re^{i\theta})|).$$

Since $\psi'(|f(re^{i\theta})|) \to 0$ as $r \to 0$, we see that

$$\lim_{r\to 0}\left|\int_{S_r} h\frac{\partial g}{\partial \eta}\,ds\right| = 0.$$

Finally, since h has compact support and g is bounded on the support of h, we obtain

$$\lim_{r\to 0}\int_{D_r} g\Delta h\,dA = \int_{\mathbb{D}} g\Delta h\,dA.$$

Therefore the limit

$$\lim_{r\to 0}\int_{D_r} hf_\psi\,dA$$

exists and is finite. As a consequence,

$$\int_{\mathbb{D}} g\Delta h\,dA = \int_{\mathbb{D}} hf_\psi\,dA \quad \text{for all } h \in C_c^\infty(\mathbb{D}). \ \blacksquare$$

LEMMA 2.8. *Let $f \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi f\|_{\psi,\alpha} \asymp \psi(|f(\varphi(0))|) + \int_{\mathbb{D}}\left[\psi''(|f(w)|) + \frac{\psi'(|f(w)|)}{|f(w)|}\right]|f'(w)|^2 N_\varphi^\sharp(w)\,dA(w), \tag{2.3}$$

where $N_\varphi^\sharp(w)$ *is the modified Nevanlinna counting function defined by*

$$N_\varphi^\sharp(w) = \sum_{z\in\varphi^{-1}(w)} (1-|z|^2)^{\alpha+1}\log(1/|z|).$$

Proof. Applying the Stanton formula to the subharmonic function $w \mapsto \psi(|f(w)|)$ we obtain

$$\begin{aligned}(2.4)\qquad \frac{1}{2\pi}\int_0^{2\pi}\psi(|f(\varphi(re^{i\theta}))|)\,d\theta &= \psi(|f(\varphi(0))|) + \int_{r\mathbb{D}} N_\varphi(r,w)\,d\mu_g(w)\\ &= \psi(|f(\varphi(0))|) + 2\int_{r\mathbb{D}}\Big[\psi''(|f(w)|)\\ &\quad + \frac{\psi'(|f(w)|)}{|f(w)|}\Big]|f'(w)|^2 N_\varphi(r,w)\,dA(w).\end{aligned}$$

Let

$$\Phi(|z|) = 2\int_{|z|}^{1}(1-r^2)^\alpha\log(r/|z|)r\,dr,$$

$$\Psi(|z|) = (1-|z|^2)^{\alpha+1}\log(1/|z|).$$

We claim that $\Phi(|z|)$ and $\Psi(|z|)$ are comparable with uniform constant for all $|z|>0$. Note that

$$(2.5)\qquad \Phi(|z|) = 2\int_{|z|}^{1}(1-r^2)^\alpha\log(r/|z|)r\,dr = \frac{1}{\alpha+1}\int_{|z|}^{1}(1-r^2)^{\alpha+1}\frac{dr}{r}.$$

Now if z is away from the origin, by using the elementary estimate $1-|z|^2 \asymp \log(1/|z|)$ and (2.5), we have

$$\begin{aligned}\Phi(|z|) &= \frac{1}{\alpha+1}\int_{|z|}^{1}(1-r^2)^{\alpha+1}\frac{dr}{r} \asymp \frac{1}{\alpha+1}\int_{|z|}^{1}(\log(1/r))^{\alpha+1}\frac{dr}{r}\\ &\asymp (\log(1/|z|))^{\alpha+2} \asymp (1-|z|^2)^{\alpha+1}\log(1/|z|).\end{aligned}$$

By l'Hôpital's rule,

$$\begin{aligned}\lim_{|z|\to0^+}\frac{\Phi(|z|)}{\Psi(|z|)} &= \lim_{|z|\to0^+}\frac{\Phi'(|z|)}{\Psi'(|z|)}\\ &\asymp \lim_{|z|\to0^+}\frac{1}{\frac{2(\alpha+1)^2|z|^2}{(1-|z|^2)}\log(1/|z|)+(\alpha+1)} = \frac{1}{\alpha+1}.\end{aligned}$$

Hence $\Phi(|z|) \asymp \Psi(|z|)$ for all $|z| > 0$. Thus

$$
\begin{aligned}
(2.6) \qquad 2\int_0^1 (1-r^2)^\alpha N_\varphi(r,w) r dr &= \sum_{z\in\varphi^{-1}(w)} 2\int_{|z|}^1 (1-r^2)^\alpha \log(r/|z|) r\, dr \\
&= \sum_{z\in\varphi^{-1}(w)} (1-|z|^2)^{\alpha+1} \log(1/|z|).
\end{aligned}
$$

Multiplying (2.4) by $2r(1-r^2)^\alpha$, integrating with respect to r from 0 to 1 and then applying Fubini's theorem, we get

$$
\begin{aligned}
\|C_\varphi f\|_{\psi,\alpha} \asymp \psi(|f(\varphi(0))|) &+ \int_{\mathbb{D}} \bigg[\psi''(|f(w)|) \\
&+ \frac{\psi'(|f(w)|)}{|f(w)|}\bigg] |f'(w)|^2 \Big(\int_0^1 (1-r^2)^\alpha N_\varphi(r,w) 2r\, dr\Big)\, dA(w).
\end{aligned}
$$

Thus by (2.6), we arrive at an equivalent expression for the norm of $C_\varphi f$ given in (2.3). ∎

The above lemma suggests that the Nevanlinna counting function $N_\varphi^\sharp$ is closely related to composition operators on the Bergman–Orlicz space $\mathcal{A}_\alpha^\psi$. An important special case of the previous formula is obtained by choosing φ to be the identity map:

$$
\begin{aligned}
(2.7) \qquad \|f\|_{\psi,\alpha} \asymp \psi(|f(0)|) &+ \int_{\mathbb{D}} \bigg[\psi''(|f(w)|) + \frac{\psi'(|f(w)|)}{|f(w)|}\bigg] \\
&\times |f'(w)|^2 (1-|w|)^{\alpha+1} \log(1/|w|)\, dA(w).
\end{aligned}
$$

The next criteria for the compactness of C_φ on $\mathcal{A}_\alpha^p$ were proved by MacCluer and Shapiro [MS] and are useful in the proof of the main result of this paper.

Lemma 2.9. *Let $1 \le p < \infty$, $\alpha \in (-1,\infty)$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following conditions are equivalent:*

(i) *C_φ is compact on $\mathcal{A}_\alpha^p$.*
(ii) *φ has an angular derivative at no point of $\partial\mathbb{D}$.*
(iii) *The pull back measure $A_\alpha \circ \varphi^{-1}$ is a vanishing $(\alpha+2)$-Carleson measure on $\mathbb{D}$.*

The following lemma characterizes the compactness of C_φ on $\mathcal{A}_\alpha^\psi$ in terms of sequential convergence. It can be proved along similar lines to Proposition 3.11 in [CM].

Lemma 2.10. *Let φ be a holomorphic self-map of $\mathbb{D}$. Then C_φ is compact on $\mathcal{A}_\alpha^\psi$ if and only if for every sequence $\{f_n\}$ which is norm bounded and converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|f_n \circ \varphi\|_{\psi,\alpha} \to 0$.*

Proof of Main Theorem. In view of Lemma 2.9, we need only show that (i)⇒(ii) and (iv)⇒(i).

(iv)⇒(i). First note that we can assume $\varphi(0) = 0$. For $a = \varphi(0) \neq 0$, we may consider the composition map $\phi = \beta_a \circ \varphi$. Fix a sequence $\{f_n\}$ in A^ψ_α which is bounded by a finite constant M and converges to zero uniformly on compact subsets of $\mathbb{D}$. It is enough to show that $\|f_n \circ \varphi\|_{\psi,\alpha} \to 0$. By (2.3), there is a constant $C > 0$ such that

$$\|C_\varphi f_n\|_{\psi,\alpha} \leq C\Bigg[\psi(|f_n(\varphi(0))|) + \int_{\mathbb{D}} \left\{\psi''(|f_n(w)|) + \frac{\psi'(|f_n(w)|)}{|f_n(w)|}\right\}|f_n'(w)|^2 N^\sharp_\varphi(w)\, dA(w)\Bigg].$$

Since $f_n(\varphi(0)) \to 0$ as $n \to \infty$, it follows that $\psi(|f_n(\varphi(0))|) \to 0$ as $n \to \infty$. Thus it remains to show that

$$\lim_{n\to\infty} \int_{\mathbb{D}} \left\{\psi''(|f_n(w)|) + \frac{\psi'(|f_n(w)|)}{|f_n(w)|}\right\}|f_n'(w)|^2 N^\sharp_\varphi(w)\, dA(w) = 0. \tag{2.8}$$

Let $\varepsilon > 0$ be given. By the Julia–Carathéodory theorem, the statement that φ does not have a finite angular derivative at any point on $\partial\mathbb{D}$ is equivalent to the condition

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

Thus we can choose $0 < r < 1$ such that

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} < \varepsilon \tag{2.9}$$

for $r < |z| < 1$. Furthermore there exists $0 < \delta < 1$ such that if $|w| > 1-\delta$ and $\varphi(z) = w$, then $|z| > r$. We now split the integral in (2.8) as follows:

$$\int_{\mathbb{D}} \left\{\psi''(|f_n(w)|) + \frac{\psi'(|f_n(w)|)}{|f_n(w)|}\right\}|f_n'(w)|^2 N^\sharp_\varphi(w)\, dA(w) = \int_{|w|\leq 1-\delta} + \int_{|w|>1-\delta} = I_1 + I_2.$$

Since

$$\lim_{t\to 0} \psi''(t) = \psi'(0) \quad \text{and} \quad \lim_{t\to 0} \frac{\psi'(t)}{t} = \psi''(0),$$

and f_n' as well as f_n tend to zero uniformly on $|w| \leq 1-\delta$ as $n \to \infty$, we have

$$\left\{\psi''(|f_n(w)|) + \frac{\psi'(|f_n(w)|)}{|f_n(w)|}\right\}|f_n'(w)|^2 < C\varepsilon.$$

Since $2+\varphi \in A^\psi_\alpha$, by (1.4) and (2.3) we have

$$\psi(2)+\psi(1) \geq \|2+\varphi\|_{\psi,\alpha} \geq C\left(\psi''(1)+\frac{\psi'(1)}{3}\right) \int\limits_{|w|\leq 1-\delta} N^\sharp_\varphi(w)\,dA(w).$$

Consequently, I_1 tends to zero as $n \to \infty$. Finally, we show that I_2 is bounded by a constant times $\varepsilon^{\alpha+1}$. By using (2.9) we have

$$1-|z|^2 \leq \varepsilon(1-|\varphi(z)|^2) = \varepsilon(1-|w|^2)$$

if $|w| > 1-\delta$ and $w=\varphi(z)$. Therefore,

$$N^\sharp_\varphi(w) \leq \varepsilon^{\alpha+1}(1-|w|^2)^{\alpha+1} N_\varphi(w) \leq \varepsilon^{\alpha+1}(1-|w|^2)^{\alpha+1}\log(1/|w|)$$

for $|w| > 1-\delta$. Here the last inequality is the Littlewood inequality for the Nevanlinna counting function N_φ. Combining this with (2.7), we have

$$\begin{aligned} I_2 &\leq C\varepsilon^{\alpha+1} \int\limits_{\mathbb{D}} \left\{\psi''(|f_n(w)|) + \frac{\psi'(|f_n(w)|)}{|f_n(w)|}\right\} |f_n'(w)|^2 \\ &\qquad\qquad \times (1-|w|^2)^{\alpha+1}\log(1/|w|)\,dA(w) \\ &\leq C\varepsilon^{\alpha+1}\|f_n\|_{\psi,\alpha} \leq CM\varepsilon^{\alpha+1}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\|f_n \circ \varphi\|_{\psi,\alpha} \to 0$, which establishes the compactness of C_φ on A^ψ_α by Lemma 2.10.

To prove (i)⇒(ii), assume that C_φ is compact on A^ψ_α and

$$(A_\alpha \circ \varphi^{-1})(S(\zeta,\delta)) \neq o(\delta^{\alpha+2})$$

as $\delta \to 0$ uniformly in $\zeta \in \partial\mathbb{D}$. Then there are sequences $\{\zeta_j\} \subset \partial\mathbb{D}$, $\{\delta_j\} \subset (0,1)$ with $\delta_j \to 0$ as $j \to \infty$ and $\varepsilon_0 > 0$ such that

$$(A_\alpha \circ \varphi^{-1})(S(\zeta_j,\delta_j)) \geq \varepsilon_0 \delta_j^{\alpha+2}$$

for each positive integer j. Put $a_j = (1-\delta_j)\zeta_j$ and

$$f_j(z) := \frac{\delta_j^{\alpha+2}}{2^{\alpha+2}}\psi^{-1}(1/\delta_j^{\alpha+2})\{k_{a_j}(z)\}^{\alpha+2}.$$

Since

$$N(\{k_{a_j}\}^{\alpha+2}) \leq \frac{2^{\alpha+2}}{\delta_j^{\alpha+2}} \cdot \frac{1}{\psi^{-1}(1/\delta_j^{\alpha+2})},$$

we see that $N(f_j) \leq 1$. So Lemma 2.2 shows that $\|f_j\|_{\psi,\alpha} \leq 1$ for any j. The assumption (1.2) on ψ implies $x \geq \psi^{-1}(x)$ for sufficiently large x. Thus $\{f_j\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.10 we deduce that $\|f_j \circ \varphi\|_{\psi,\alpha} \to 0$ as $j \to \infty$.

On the other hand, if $z \in S(\zeta_j,\delta_j)$, then $|1-\overline{a_j}z|^{2(\alpha+2)} < (2\delta_j)^{\alpha+2}$, and so

$$|f_j(z)| \geq \frac{1}{8^{\alpha+2}} \cdot \psi^{-1}(1/\delta_j^{\alpha+2})$$

for each $z \in S(\zeta_j, \delta_j)$. Thus

$$\begin{aligned}
(2.10)\qquad \|f_j \circ \varphi\|_{\psi,\alpha} &\geq \int\limits_{S(\zeta_j,\delta_j)} \psi(|f_j(z)|)\, d(A_\alpha \circ \varphi^{-1})(z) \\
&\geq \psi\biggl(\frac{1}{8^{\alpha+2}} \cdot \psi^{-1}(1/\delta_j^{\alpha+2})\biggr) \cdot (A_\alpha \circ \varphi^{-1})(S(\zeta_j, \delta_j)) \\
&\geq \psi\biggl(\frac{1}{8^{\alpha+2}} \cdot \psi^{-1}(1/\delta_j^{\alpha+2})\biggr) \cdot \varepsilon_0 \delta_j^{\alpha+2}
\end{aligned}$$

for each j. Since $x \geq \psi^{-1}(x)$ for sufficiently large x, we see that $\psi(x) \geq x$ for sufficiently large x. By the Δ_2-condition for ψ, there exists an absolute constant $C > 0$ such that $\psi(x) \leq C\psi(x/8^{\alpha+2})$ for sufficiently large x. Hence

$$(2.11)\qquad \psi\biggl(\frac{1}{8^{\alpha+2}} \cdot \psi^{-1}(1/\delta_j^{\alpha+2})\biggr) \geq \frac{1}{C}\psi(\psi^{-1}(1/\delta_j^{\alpha+2})) = \frac{1}{C\delta_j^{\alpha+2}}$$

for sufficiently large j. It follows from (2.10) and (2.11) that

$$\|f_j \circ \varphi\|_{\psi,\alpha} \geq \frac{1}{C\delta_j^{\alpha+2}} \cdot \varepsilon_0 \delta_j^{\alpha+2} = \frac{\varepsilon_0}{C}$$

for sufficiently large j. This contradicts $\|f_j \circ \varphi\|_{\psi,\alpha} \to 0$ as $j \to \infty$. Hence the compactness of C_φ on A_α^ψ implies that $(A_\alpha \circ \varphi^{-1})(S(\zeta, \delta)) = o(\delta^{\alpha+2})$ as $\delta \to 0$ uniformly in $\zeta \in \partial\mathbb{D}$. This completes the proof.

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