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On para-Kähler–Norden structures on the tangent bundles

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Abstract. The main purpose of this article is to investigate the paraholomorphy property of the Sasaki and Cheeger–Gromoll metrics by using compatible paracomplex stuctures on the tangent bundle.

1. Introduction. Let M be an n-dimensional Riemannian manifold with metric g. We denote by $\Im_q^p(M)$ the set of all tensor fields of type (p,q) on M. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

An *almost paracomplex manifold* is an almost product manifold (M,φ), $\varphi^2 = \mathrm{id}$, $\varphi \neq \pm\mathrm{id}$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure φ, we obtain the following set of affinors on M_{2k}: $\{\mathrm{id},\varphi\}$, $\varphi^2 = \mathrm{id}$, which is an isomorphic representation of the algebra of order 2 over the field $\mathbb{R}$ of real numbers, which is called the *algebra of paracomplex* (or *double*) *numbers* and is denoted by $R(j) = \{a_0 + a_1 j \mid j^2 = 1,\ j \neq \pm 1;\ a_0, a_1 \in \mathbb{R}\}$. Obviously, it is associative, commutative and unital, i.e., it admits principal unit 1. The canonical base of this algebra $\{1, j\}$. The structure constants of this algebra are $C_{11}^1 = C_{12}^2 = C_{21}^2 = C_{22}^1 = 1$, all the others being zero, with respect to the canonical base $\{e_1, e_2\} = \{1, j\}$ of $R(j)$, i.e. $e_i e_j = C_{ij}^k e_k$.

Consider $R(j)$ endowed with the usual topology of $\mathbb{R}^2$ and a domain U of $R(j)$. Let

$$X = x^1 + jx^2$$

be a variable in $R(j)$, where x^i are real coordinates of a point of U for $i = 1, 2$. Using two real-valued functions $f^i(x^1, x^2)$, $i = 1, 2$, we introduce a

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paracomplex function

$$F = f^1 + jf^2$$

of variable X. It is said to be *paraholomorphic* if

$$dF = F'(X)dX$$

for the differentials $dX = dx^1 + jdx^2$, $dF = df^1 + jdf^2$ and the derivative $F'(X)$. The paraholomorphy of the function $F = f^1 + jf^2$ in the variable $X = x^1 + jx^2$ is equivalent to the fact that the Jacobian matrix $D = (\partial_k f^i)$ commutes with the matrix

$$\begin{pmatrix} C_{21}^1 & C_{22}^1 \\ C_{21}^2 & C_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see [30, p. 87]). It follows that F is paraholomorphic if and only if f^1 and f^2 satisfy the para-Cauchy–Riemann equations

$$\frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2}, \qquad \frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}.$$

The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor N_φ. On the other hand, in order that an almost paracomplex structure be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection such that $\nabla\varphi = 0$. A *paracomplex manifold* is an almost paracomplex manifold (M_{2k}, φ) such that the G-structure defined by the affinor field φ is integrable. We can give another, equivalent definition of paracomplex manifold in terms of local homeomorphisms in the space $R^k(j) = \{(X^1, \ldots, X^k) \mid X^i \in R(j), i = 1, \ldots, k\}$ and paraholomorphic changes of charts in a way similar to [8] (for more details see [30]), i.e. a manifold M_{2k} with an integrable paracomplex structure φ is a real realization of the paraholomorphic manifold $\mathrm{X}_k(R(j))$ over the algebra $R(j)$.

1.1. Para-Norden metric. Let M_{2k} be an almost paracomplex manifold with the structure φ. A Riemannian metric g is a *para-Norden metric* (*B-metric*) if

$$g(\varphi X, \varphi Y) = g(X, Y)$$

or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_{2k})$. If (M_{2k}, φ) is an almost paracomplex manifold with a para-Norden metric g, we say that (M_{2k}, φ, g) is an *almost paracomplex Norden manifold* [12, 24, 22, 29]. If φ is integrable, we say that (M_{2k}, φ, g) is a *paracomplex Norden manifold.*

1.2. Paraholomorphic (or almost paraholomorphic) tensor fields. Let $\overset{*}{t}$ be a paracomplex tensor field on $\mathrm{X}_k(R(j))$. The real model of such a tensor field is a tensor field on M_{2k} of the same order that is independent of whether its vector or covector argument is subject to the action of the affinor structure φ. Such tensor fields are said to be *pure* with respect to φ. They were studied by many authors (see, e.g., [12, 18, 21, 24, 22, 30, 31]). In particular, being applied to a $(0, q)$-tensor field ω, the purity means that for any $X_1, \dots, X_q \in \Im^1_0(M_{2k})$, the following conditions hold:

$$\omega(\varphi X_1, X_2, \dots, X_q) = \omega(X_1, \varphi X_2, \dots, X_q) = \dots = \omega(X_1, X_2, \dots, \varphi X_q).$$

Consider the operator

$$\varPhi_\varphi : \Im^0_q(M_{2k}) \to \Im^0_{q+1}(M_{2k})$$

associated with φ and applied to the pure tensor field ω by (see [31])

$$\begin{aligned}(\varPhi_\varphi \omega)(X, Y_1, Y_2, \dots, Y_q) &= (\varphi X)(\omega(Y_1, Y_2, \dots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \dots, Y_q)) \\ &\quad + \omega((L_{Y_1}\varphi)X, Y_2, \dots, Y_q) + \dots + \omega(Y_1, Y_2, \dots, (L_{Y_q}\varphi)X),\end{aligned}$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a paracomplex structure on M_{2k} and the tensor field $\varPhi_\varphi \omega$ vanishes, the paracomplex tensor field $\overset{*}{\omega}$ on $\mathrm{X}_k(R(j))$ is said to be *paraholomorphic* [18]. Thus a paraholomorphic tensor field $\overset{*}{\omega}$ on $\mathrm{X}_k(R(j))$ is realized on M_{2k} in the form of a pure tensor field ω such that

$$(\varPhi_\varphi \omega)(X, Y_1, \dots, Y_q) = 0$$

for any $X, Y_1, \dots, Y_q \in \Im^1_0(M_{2k})$. Therefore such a tensor field ω on M_{2k} is also called paraholomorphic.

1.3. Paraholomorphic Norden (or para-Kähler–Norden) metrics. If (M_{2k}, φ, g) is an almost paracomplex Norden manifold with $\varPhi_\varphi g = 0$, we say that (M_{2k}, φ, g) is an *almost paraholomorphic Norden manifold.* If φ is integrable, we say that (M_{2k}, φ, g) is a *paraholomorphic Norden manifold.* If $\nabla \varphi = 0$, where ∇ is the Levi-Civita connection of g, then we say that (M_{2k}, φ, g) is a *para-Kähler–Norden manifold.*

In some respects, paraholomorphic Norden manifolds are similar to para-Kähler manifolds. The following theorem is an analogue of the known result that an almost para-Hermitian manifold is para-Kähler if and only if the almost paracomplex structure is parallel with respect to the Levi-Civita connection.

THEOREM 1.1 ([22, 24]; for a complex version see [12]). *For an almost paracomplex manifold with a para-Norden metric g, the condition $\Phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g.*

A para-Kähler–Norden manifold can be defined as a triple (M_{2k}, φ, g) which consists of a manifold M_{2n} endowed with an almost paracomplex structure φ and a Riemannian metric g such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be Nordenian. Therefore, there exists a one-to-one correspondence between para-Kähler–Norden manifolds and Norden manifolds with a paraholomorphic metric.

2. Lifts to tangent bundles. Let TM be the tangent bundle over an n-dimensional manifold M, and π the natural projection $\pi : TM \to M$. Let the manifold M be covered by a system of coordinate neighborhoods (U, x^i), where (x^i), $i = 1, \ldots, n$, is a local coordinate system in U. Let (y^i) be the Cartesian coordinates in each tangent space T_pM at $p \in M$ with respect to the natural base $\{\frac{\partial}{\partial x^i}|_p\}$, p being an arbitrary point in U whose coordinates are (x^i). Then we can introduce local coordinates (x^i, y^i) in the open set $\pi^{-1}(U) \subset TM$. We call them the *induced coordinates*. The projection π is represented by $(x^i, y^i) \mapsto (x^i)$. The indices $I, J, \ldots$ run from 1 to $2n$, the indices $\bar{i}, \bar{j}, \ldots$ run from $n+1$ to $2n$. Summation over repeated indices is always assumed.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the horizontal lift ${}^HX$ and the vertical lift ${}^VX$ of X are given, in the induced coordinates, by

$$ {}^VX = X^i \partial_{\bar{i}}, \tag{2.1} $$

$$ {}^HX = X^i \partial_i - y^j \Gamma^i_{jk} X^k \partial_{\bar{i}}, \tag{2.2} $$

where Γ^i_{jk} are the coefficients of the Levi-Civita connection ∇ of g (for more details, see [32]).

In particular, we have the vertical spray ${}^Vu$ and the horizontal spray ${}^Hu$ on TM defined by

$$ {}^Vu = y^i\, {}^V(\partial_i) = y^i \partial_{\bar{i}}, \qquad {}^Hu = y^i\, {}^H(\partial_i) = y^i \delta_i, \tag{2.3} $$

where $\delta_i = \partial_i - y^j \Gamma^s_{ji} \partial_{\bar{s}}$. ${}^Vu$ is also called the *canonical* or *Liouville vector field* on TM.

Now, let r be the norm of a vector $u \in TM$. Then, for any smooth function $f : \mathbb{R} \to \mathbb{R}$, we have

$$ {}^HX(f(r^2)) = 0, \tag{2.4} $$

$$ {}^VX(f(r^2)) = 2f'(r^2) g(X, u) \tag{2.5} $$

and in particular,

$$ {}^{H}X(r^2) = 0, \tag{2.6} $$

$$ {}^{V}X(r^2) = 2g(X, u). \tag{2.7} $$

Let X, Y and Z be any vector fields on M. Then (see [3])

$$ {}^{H}X(g(Y,u)) = g((\nabla_X Y), u), \tag{2.8} $$

$$ {}^{V}X(g(Y,u)) = g(X,Y), \tag{2.9} $$

$$ {}^{H}X({}^{V}(g(Y,Z))) = X(g(Y,Z)), \tag{2.10} $$

$$ {}^{V}X({}^{V}(g(Y,Z))) = 0. \tag{2.11} $$

Explicit expressions for the Lie bracket $[\,,]$ of the tangent bundle TM are given by Dombrowski [9]. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$ \begin{cases} [{}^{H}X, {}^{H}Y] = {}^{H}[X,Y] - {}^{V}(R(X,Y)u), \\ [{}^{H}X, {}^{V}Y] = {}^{V}(\nabla_X Y), \\ [{}^{V}X, {}^{V}Y] = 0, \end{cases} \tag{2.12} $$

for all vector fields X and Y on M, where R is the Riemannian curvature of g defined by

$$ R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. $$

3. Almost paracomplex structures with para-Norden metrics on tangent bundles. Let (M,g) be a Riemannian manifold. A Remannian metric $\tilde{g}$ on the tangent bundle TM of M is said to be *natural* with respect to g on M if

$$ \tilde{g}({}^{H}X, {}^{H}Y) = g(X,Y), \quad \tilde{g}({}^{H}X, {}^{V}Y) = 0 $$

for all vector fields $X, Y \in \Im_0^1(M)$. A natural metric $\tilde{g}$ is constructed in such a way that the vertical and horizontal subbundles are orthogonal and the bundle map $\pi : (TM, \tilde{g}) \to (M, g)$ is a Riemannian submersion. All the preceding metrics belong to the wide class of so-called *g-natural metrics* on the tangent bundle, initially classified by Kowalski and Sekizawa [15] and fully characterized by Abbassi and Sarih [1]–[3] (see also [13] for other presentations of the basic result from [15] and for more details about the concept of naturality).

3.1. The well-known example of a g-natural metric is the Sasaki metric ${}^{S}g$ introduced in [26]. Its construction is based on a natural splitting of the tangent bundle TTM of TM into its vertical and horizontal subbundles by means of the Levi-Civita connection ∇ on (M,g). The Sasaki metric is defined by

$$(3.1)\qquad {}^Sg({}^HX,{}^HY) = {}^V(g(X,Y)),$$

$$(3.2)\qquad {}^Sg({}^VX,{}^HY) = {}^Sg({}^HX,{}^VY) = 0,$$

$$(3.3)\qquad {}^Sg({}^VX,{}^VY) = {}^V(g(X,Y))$$

for all $X, Y \in \Im_0^1(M)$(see [32, pp. 155–175]. The Sasaki metric has been extensively studied by several authors, including Kowalski [14], Musso and Tricerri [20], and Aso [4]. Kowalski [14] calculated the Levi-Civita connection ${}^S\nabla$ of the Sasaki metric on TM and its Riemannian curvature tensor ${}^SR$. With this in hand Kowalski [14], Aso [4], and Musso and Tricerri [20] derived interesting connections between the geometric properties of (M, g) and $(TM, {}^Sg)$.

Now, define an almost paracomplex structure J_S on TM by

$$(3.4)\qquad J_S({}^HX) = {}^VX, \qquad J_S({}^VX) = {}^HX,$$

for all $X, Y \in \Im_0^1(M)$ [8]. We put

$$A(\tilde{X},\tilde{Y}) = {}^Sg(J_S\tilde{X},\tilde{Y}) - {}^Sg(\tilde{X},J_S\tilde{Y})$$

for any $\tilde{X},\tilde{Y} \in \Im_0^1(TM)$. For all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ${}^VX$, ${}^VY$ or ${}^HX$, ${}^HY$, from (3.1)–(3.4), we have $A(\tilde{X},\tilde{Y}) = 0$, i.e. ${}^Sg$ is pure with respect to J_S. Hence we have the following theorem:

THEOREM 3.1. *Let (M, g) be a Riemannian manifold and let TM be its tangent bundle equipped with the Sasaki metric ${}^Sg$ and the paracomplex structure J_S defined by* (3.4). *Then the triple $(TM, J_S, {}^Sg)$ is an almost paracomplex Norden manifold.*

Having determined both the Sasaki metric ${}^Sg$ and the almost paracomplex structure J_S and by using the fact that ${}^VX^V(g(Y,Z)) = 0$ and ${}^HX^V(g(Y,Z)) = {}^V(Xg(Y,Z))$ we calculate

$$\begin{aligned}(\varPhi_{J_S}{}^Sg)(\tilde{X},\tilde{Y},\tilde{Z}) &= (J_S\tilde{X})({}^Sg(\tilde{Y},\tilde{Z})) - \tilde{X}(g(J_S\tilde{Y},\tilde{Z}))\\ &\quad + {}^Sg((L_{\tilde{Y}}J_S)\tilde{X},\tilde{Z}) + {}^Sg(\tilde{Y},(L_{\tilde{Z}}J_S)\tilde{X})\end{aligned}$$

for all $\tilde{X},\tilde{Y},\tilde{Z} \in \Im_0^1(TM)$. Then we get

$$\begin{aligned}
&(\varPhi_{J_S}{}^Sg)({}^VX,{}^VY,{}^HZ) = {}^Sg({}^H(R(u,Y)X),{}^HZ),\\
&(\varPhi_{J_S}{}^Sg)({}^VX,{}^VY,{}^VZ) = 0,\\
&(\varPhi_{J_S}{}^Sg)({}^VX,{}^HY,{}^VZ) = {}^Sg({}^V(R(X,Y)u),{}^VZ),\\
&(\varPhi_{J_S}{}^Sg)({}^VX,{}^HY,{}^HZ) = 0,\\
&(\varPhi_{J_S}{}^Sg)({}^HX,{}^VY,{}^HZ) = 0,\\
&(\varPhi_{J_S}{}^Sg)({}^HX,{}^VY,{}^VZ) = 0,\\
&(\varPhi_{J_S}{}^Sg)({}^HX,{}^HY,{}^HZ) = {}^Sg({}^H(R(Y,X)u - R(u,Y)X),{}^HZ),\\
&(\varPhi_{J_S}{}^Sg)({}^HX,{}^HY,{}^VZ) = 0.
\end{aligned}$$

Therefore, from Theorem 1.1 we have

THEOREM 3.2. *Let (M, g) be a Riemannian manifold and let TM be its tangent bundle equipped with the Sasaki metric ${}^{S}g$ and the paracomplex structure J_S defined by* (3.4)*. The triple $(TM, J_S, {}^{S}g)$ is a para-Kähler–Norden manifold if and only if M is locally flat.*

3.2. Another well-known g-natural Riemannian metric g_{CG} was considered by Muso and Tricerri [20] who, inspired by the paper [7] of Cheeger and Gromoll, called it the *Cheeger–Gromoll metric*. The metric was defined by Cheeger and Gromoll; yet, it was Musso and Tricerri who wrote down its expression, constructed it in a more "comprehensible" way, and gave it the name. The Levi-Civita connection of g_{CG} and its Riemannian curvature tensor were calculated by Sekizawa [27] (for more details see [10, 11]). In [19], Munteanu considered a Cheeger–Gromoll type metric on TM, as well as a compatible complex structure. By direct computations, he obtained some conditions under which TM is almost Kählerian, locally conformal Kählerian or Kählerian. The geometry of Cheeger–Gromoll metric is well known and has been intensively studied (see [5, 6, 10, 11, 16, 17, 19, 23, 25]). A similar metric in theoretical physics has been obtained by Tamm (the 1958 Nobel Laureate in Physics, see [28]).

Let (M, g) be a Riemannian manifold and denote by r the norm of a vector $u = (u^i)$, i.e. $r^2 = g_{ji}u^j u^i$. The *Cheeger–Gromoll metric* g_{CG} on the tangent bundle TM is given by

$$g_{CG}({}^{H}X, {}^{H}Y) = {}^{V}(g(X, Y)), \tag{3.5}$$

$$g_{CG}({}^{H}X, {}^{V}Y) = 0, \tag{3.6}$$

$$g_{CG}({}^{V}X, {}^{V}Y) = \frac{1}{\alpha}[\,{}^{V}(g(X, Y)) + g(X, u)g(Y, u)] \tag{3.7}$$

for all vector fields $X, Y \in \Im^1_0(M)$, where ${}^{V}(g(X, Y)) = (g(X, Y)) \circ \pi$ and $\alpha = 1 + r^2$.

THEOREM 3.3 ([10, 11]). *Let (M, g) be a Riemannian manifold and equip its tangent bundle TM with the Cheeger–Gromoll metric g_{CG}. Then the corresponding Levi-Civita connection ${}^{CG}\nabla$ satisfies the following:*

$$ {}^{CG}\nabla_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) - \frac{1}{2}\,{}^{V}(R(X, Y)u),$$

$$ {}^{CG}\nabla_{{}^{H}X}{}^{V}Y = \frac{1}{2\alpha}\,{}^{H}(R(u, Y)X) + {}^{V}(\nabla_X Y),$$

$$ {}^{CG}\nabla_{{}^{V}X}{}^{H}Y = \frac{1}{2\alpha}\,{}^{H}(R(u, X)Y),$$

$$
\begin{aligned}
{}^{CG}\nabla_{{}^V X}{}^V Y = &-\frac{1}{\alpha}(g_{CG}({}^V X, {}^V u)\,{}^V Y + g_{CG}({}^V Y, {}^V u)\,{}^V X \\
&+ \frac{1+\alpha}{\alpha} g_{CG}({}^V X, {}^V Y)\,{}^V u - \frac{1}{\alpha} g_{CG}({}^V X, {}^V u) g({}^V Y, {}^V u)\,{}^V u
\end{aligned}
$$

for any $X, Y \in \Im_0^1(M)$, *where* R *and* ${}^V u$ *denote respectively the curvature tensor of* ∇ *and the canonical vector field on* TM.

We define another almost paracomplex structure J_{CG} on TM by the formulas

$$
\begin{cases}
J_{CG}({}^H X) = \sqrt{\alpha}\ {}^V X - \dfrac{1}{1+\sqrt{\alpha}} g(X, u)\,{}^V u, \\
J_{CG}({}^V X) = \dfrac{1}{\sqrt{\alpha}}\ {}^H X + \dfrac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} g(X, u)\,{}^H u.
\end{cases}
\tag{3.8}
$$

Note that $J_{CG}{}^V u = {}^H u$ and $J_{CG}{}^H u = {}^V u$. It is easily seen that $J_{CG}^2 = I$. In fact, by (3.8) we have

$$
\begin{aligned}
J_{CG}^2({}^H X) &= J_{CG}(J_{CG}{}^H X) = J_{CG}\left(\sqrt{\alpha}\ {}^V X - \frac{1}{1+\sqrt{\alpha}} g(X, u)\,{}^V u\right) \\
&= \sqrt{\alpha}\, J_{CG}{}^V X - \frac{1}{1+\sqrt{\alpha}} g(X, u) J_{CG}\,{}^V u \\
&= \sqrt{\alpha}\left(\frac{1}{\sqrt{\alpha}}\ {}^H X + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} g(X, u)\,{}^H u\right) - \frac{1}{1+\sqrt{\alpha}} g(X, u)\,{}^V u \\
&= {}^H X, \\
J_{CG}^2({}^V X) &= J_{CG}(J_{CG}{}^V X) = J_{CG}\left(\frac{1}{\sqrt{\alpha}}\ {}^H X + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} g(X, u)\,{}^H u\right) \\
&= \frac{1}{\sqrt{\alpha}}\, J_{CG}{}^H X + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} g(X, u) J_{CG}\,{}^H u \\
&= \frac{1}{\sqrt{\alpha}}\left(\sqrt{\alpha}\ {}^V X - \frac{1}{1+\sqrt{\alpha}} g(X, u)\,{}^V u\right) + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} g(X, u)\,{}^H u \\
&= {}^V X
\end{aligned}
$$

for any $X \in \Im_0^1(M)$, which implies $J_{CG}^2 = I$.

Theorem 3.4. *Let* (M, g) *be a Riemannian manifold and let* TM *be its tangent bundle equipped with the Cheeger–Gromoll metric* g_{CG} *and the almost paracomplex structure* J_{CG} *defined by* (3.8). *Then the triple* (TM, J_{CG}, g_{CG}) *is an almost paracomplex Norden manifold.*

Proof. We put

$$
A(\tilde{X}, \tilde{Y}) = g_{CG}(J_{CG}\tilde{X}, \tilde{Y}) - g_{CG}(\tilde{X}, J_{CG}\tilde{Y})
$$

for any $\tilde{X}, \tilde{Y} \in \Im_0^1(TM)$. From (3.5)–(3.8), we get

$$\begin{aligned}
A({}^VX, {}^VY) &= g_{CG}(J_{CG}\,{}^VX, {}^VY) - g_{CG}({}^VX, J_{CG}\,{}^VY) \\
&= g_{CG}\left(\frac{1}{\sqrt{\alpha}}\,{}^HX + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(X,u)\,{}^Hu, {}^VY\right) \\
&\quad - g_{CG}\left({}^VX, \frac{1}{\sqrt{\alpha}}\,{}^HY + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)\,{}^Hu\right) \\
&= \frac{1}{\sqrt{\alpha}}\,g_{CG}({}^HX, {}^VY) + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(X,u)g_{CG}({}^Hu, {}^VY) \\
&\quad - \frac{1}{\sqrt{\alpha}}\,g_{CG}({}^VX, {}^HY) + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)g_{CG}({}^VX, {}^Hu) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
A({}^VX, {}^HY) &= g_{CG}(J_{CG}{}^VX, {}^HY) - g_{CG}({}^VX, J_{CG}{}^HY) \\
&= g_{CG}\left(\frac{1}{\sqrt{\alpha}}\,{}^HX + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(X,u)\,{}^Hu, {}^HY\right) \\
&\quad - g_{CG}({}^VX, \sqrt{\alpha}\,{}^VY - \frac{1}{1+\sqrt{\alpha}}g(Y,u)\,{}^Vu) \\
&= \frac{1}{\sqrt{\alpha}}\,g_{CG}({}^HX, {}^HY) + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(X,u)g_{CG}({}^Hu, {}^HY) \\
&\quad - \sqrt{\alpha}\,g_{CG}({}^VX, {}^VY) + \frac{1}{1+\sqrt{\alpha}}g(Y,u)g_{CG}({}^VX, {}^Vu) \\
&= \frac{1}{\sqrt{\alpha}}\,{}^V(g(X,Y)) + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(X,u)g(Y,u) \\
&\quad - \frac{1}{\sqrt{\alpha}}\,{}^V(g(X,Y)) - \frac{1}{\sqrt{\alpha}}g(X,u)g(Y,u) + \frac{1}{1+\sqrt{\alpha}}g(Y,u)g(X,u) \\
&= \left(\frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} - \frac{1}{\sqrt{\alpha}} + \frac{1}{1+\sqrt{\alpha}}\right)g(X,u)g(Y,u) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
A({}^HX, {}^VY) &= g_{CG}(J_{CG}{}^HX, {}^VY) - g_{CG}({}^HX, J_{CG}{}^VY) \\
&= g_{CG}\left(\sqrt{\alpha}\,{}^VX - \frac{1}{1+\sqrt{\alpha}}g(X,u)\,{}^Vu, {}^VY\right) \\
&\quad - g_{CG}({}^HX, \frac{1}{\sqrt{\alpha}}\,{}^HY + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)\,{}^Hu) \\
&= \sqrt{\alpha}\,g_{CG}({}^VX, {}^VY) - \frac{1}{1+\sqrt{\alpha}}g(X,u)g_{CG}({}^Vu, {}^VY) \\
&\quad - \frac{1}{\sqrt{\alpha}}\,g_{CG}({}^HX, {}^HY) - \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)g_{CG}({}^HX, {}^Hu)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\alpha}}\,{}^{V}(g(X,Y)) + \frac{1}{\sqrt{\alpha}} g(X,u)g(Y,u) - \frac{1}{\sqrt{\alpha}}\,{}^{V}(g(X,Y)) \\
&\quad - \frac{1}{1+\sqrt{\alpha}}\, g(X,u)g(Y,u) - \frac{1}{\sqrt{\alpha}\,(1+\sqrt{\alpha})} g(X,u)g(Y,u) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
A({}^{H}X, {}^{H}Y) &= g_{CG}(J_{CG}{}^{H}X, {}^{H}Y) - g_{CG}({}^{H}X, J_{CG}{}^{H}Y) \\
&= g_{CG}\Big(\sqrt{\alpha}\,{}^{V}X - \frac{1}{1+\sqrt{\alpha}} g(X,u)\,{}^{V}u, {}^{H}Y\Big) \\
&\quad - g_{CG}\Big({}^{H}X, \sqrt{\alpha}\,{}^{V}Y - \frac{1}{1+\sqrt{\alpha}} g(Y,u)\,{}^{V}u\Big) \\
&= \sqrt{\alpha}\, g_{CG}({}^{V}X, {}^{H}Y) - \frac{1}{1+\sqrt{\alpha}} g(X,u) g_{CG}({}^{V}u, {}^{H}Y) \\
&\quad - \sqrt{\alpha} g_{CG}({}^{H}X, {}^{V}Y) - \frac{1}{1+\sqrt{\alpha}}\, g(Y,u) g_{CG}({}^{H}X, {}^{V}u) \\
&= 0,
\end{aligned}$$

i.e. g_{CG} is pure with respect to J_{CG}. Thus Theorem 3.4 is proved. ∎

We now consider the covariant derivative of J_{CG}. Let us begin with the following lemma which will be used later on.

Lemma 3.5. *Let ${}^{CG}\nabla$ be the Levi-Civita connection of the Cheeger–Gromoll metric g_{CG} and ${}^{V}u$ and ${}^{H}u$ be the vertical spray and horizontal spray on TM, respectively. Then*

$$\begin{aligned}
{}^{CG}\nabla_{{}^{H}X}{}^{V}u &= 0, \\
{}^{CG}\nabla_{{}^{H}X}{}^{H}u &= \frac{1}{2}\,{}^{V}(R(u,X)u), \\
{}^{CG}\nabla_{{}^{V}X}{}^{V}u &= \frac{1}{\alpha}({}^{V}X + g(X,u){}^{V}u) \\
{}^{CG}\nabla_{{}^{V}X}{}^{H}u &= \frac{1}{2\alpha}\,{}^{H}(R(u,X)u).
\end{aligned}$$

Proof. The equalities follow directly from the definition of the vertical and horizontal spray and Theorem 3.3. ∎

Also note that the definition of the Cheeger–Gromoll metric leads to

$$g_{CG}({}^{V}X, {}^{V}u) = \frac{1}{\alpha}(g(X,u) + g(X,u)g(u,u)) = g(X,u). \tag{3.9}$$

Using (2.4)–(2.11), (3.8), (3.9), Theorem 3.3 and Lemma 3.5, by direct computation we obtain the following identities:

(i) $$({}^{CG}\nabla_{{}^HX}\, J_{CG})({}^HY) = {}^{CG}\nabla_{{}^HX}(J_{CG}\,{}^HY) - J_{CG}({}^{CG}\nabla_{{}^HX}\,{}^HY)$$

$$= {}^{CG}\nabla_{{}^HX}\left(\sqrt{\alpha}\,{}^VY - \frac{1}{1+\sqrt{\alpha}}g(Y,u)\,{}^Vu\right)$$

$$- J_{CG}\left({}^H(\nabla_XY) - \frac{1}{2}\,{}^V(R(X,Y)u)\right)$$

$$= {}^HX(\sqrt{\alpha})\,{}^VY + \sqrt{\alpha}\,{}^{CG}\nabla_{{}^HX}\,{}^VY - {}^HX\left(\frac{1}{1+\sqrt{\alpha}}g(Y,u)\right){}^Vu$$

$$- \frac{1}{1+\sqrt{\alpha}}g(Y,u)\,{}^{CG}\nabla_{{}^HX}\,{}^Vu - J_{CG}({}^H(\nabla_XY)) + \frac{1}{2}J_{CG}({}^V(R(X,Y)u))$$

$$= \frac{1}{2\sqrt{\alpha}}\,{}^H(R(u,Y)X) + \sqrt{\alpha}\,{}^V(\nabla_XY)) - \frac{1}{1+\sqrt{\alpha}}g(\nabla_XY,u)\,{}^Vu$$

$$- \sqrt{\alpha}\,{}^V(\nabla_XY) + \frac{1}{1+\sqrt{\alpha}}g(\nabla_XY,u)\,{}^Vu$$

$$+ \frac{1}{2\sqrt{\alpha}}\,{}^H(R(X,Y)u) + \frac{1}{2\sqrt{\alpha}(1+\sqrt{\alpha})}g(R(X,Y)u,u)\,{}^Hu$$

$$= \frac{1}{2\sqrt{\alpha}}\,{}^H(R(u,Y)X + R(X,Y)u);$$

(ii) $$({}^{CG}\nabla_{{}^HX}\, J_{CG})({}^VY) = {}^{CG}\nabla_{{}^HX}(J_{CG}\,{}^VY) - J_{CG}({}^{CG}\nabla_{{}^HX}\,{}^VY)$$

$$= {}^{CG}\nabla_{{}^HX}\left(\frac{1}{\sqrt{\alpha}}\,{}^HY + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)\,{}^Hu\right)$$

$$- J_{CG}\left(\frac{1}{2\alpha}\,{}^H(R(u,Y)X) + {}^V(\nabla_XY)\right)$$

$$= {}^HX\left(\frac{1}{\sqrt{\alpha}}\right){}^HY + \frac{1}{\sqrt{\alpha}}\,{}^{CG}\nabla_{{}^HX}\,{}^HY + {}^HX\left(\frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha}}g(Y,u)\right){}^Hu$$

$$+ \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)\,{}^{CG}\nabla_{{}^HX}\,{}^Hu - \frac{1}{2\alpha}J_{CG}({}^H(R(u,Y)X))$$

$$- J_{CG}({}^V(\nabla_XY))$$

$$= \frac{1}{\sqrt{\alpha}}\,{}^H(\nabla_XY) - \frac{1}{2\sqrt{\alpha}}\,{}^V(R(X,Y)u) + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(\nabla_XY,u)^Hu$$

$$- \frac{1}{2\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)^V(R(X,u)u) - \frac{1}{2\sqrt{\alpha}}\,{}^V(R(u,Y)X)$$

$$+ \frac{1}{2\alpha(1+\sqrt{\alpha})}g((R(u,Y)X),u)\,{}^Vu - \frac{1}{\sqrt{\alpha}}\,{}^H(\nabla_XY)$$

$$- \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(\nabla_XY,u)\,{}^Hu$$

$$= \frac{-1}{2\sqrt{\alpha}}\,{}^{V}(R(X,Y)u + R(u,Y)X) - \frac{1}{2\sqrt{\alpha}(1+\sqrt{\alpha})} g(Y,u)^{V}(R(X,u)u)$$
$$+ \frac{1}{2\alpha(1+\sqrt{\alpha})} g((R(u,Y)X),u)\,{}^{V}u;$$

(iii) $$({}^{CG}\nabla_{{}^{V}X}\, J_{CG})({}^{H}Y) = {}^{CG}\nabla_{{}^{V}X}(J_{CG}{}^{H}Y) - J_{CG}({}^{CG}\nabla_{{}^{V}X}{}^{H}Y)$$
$$= {}^{CG}\nabla_{{}^{V}X}\left(\sqrt{\alpha}\,{}^{V}Y - \frac{1}{1+\sqrt{\alpha}} g(Y,u)\,{}^{V}u\right) - J_{CG}\left(\frac{1}{2\alpha}\,{}^{H}(R(u,X)Y)\right)$$
$$= {}^{V}X(\sqrt{\alpha})\,{}^{V}Y + \sqrt{\alpha}\,{}^{CG}\nabla_{{}^{V}X}{}^{V}Y - {}^{V}X\left(\frac{1}{1+\sqrt{\alpha}} g(Y,u)\right){}^{V}u$$
$$- \frac{1}{1+\sqrt{\alpha}} g(Y,u)\,{}^{CG}\nabla_{{}^{V}X}{}^{V}u - \frac{1}{2\sqrt{\alpha}}\,{}^{V}(R(u,X)Y)$$
$$+ \frac{1}{2\alpha(1+\sqrt{\alpha})} g(R(u,X)Y,u)\,{}^{V}u$$
$$= \frac{1}{\sqrt{\alpha}}\, g(X,u)\,{}^{V}Y + \sqrt{\alpha}\bigg(-\frac{1}{\alpha} g(X,u)\,{}^{V}Y - \frac{1}{\alpha} g(Y,u)\,{}^{V}X$$
$$+ \frac{1+\alpha}{\alpha^2}\,{}^{V}(g(X,Y))\,{}^{V}u + \frac{1}{\alpha^2} g(X,u)g(Y,u)\,{}^{V}u\bigg)$$
$$+ \frac{1}{2\sqrt{\alpha}(1+\sqrt{\alpha})^2} g(X,u)g(Y,u)\,{}^{V}u - \frac{1}{\sqrt{\alpha}}\,{}^{V}(g(X,Y))\,{}^{V}u$$
$$- \frac{1}{\alpha(1+\sqrt{\alpha})} g(Y,u)({}^{V}X + g(X,u)\,{}^{V}u) - \frac{1}{2\sqrt{\alpha}}\,{}^{V}(R(u,X)Y)$$
$$+ \frac{1}{2\alpha(1+\sqrt{\alpha})} g(R(u,X)Y,u)\,{}^{V}u$$
$$= -\left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\alpha(1+\sqrt{\alpha})}\right) g(Y,u)\,{}^{V}X$$
$$+ \left(\frac{\sqrt{\alpha}(1+\alpha)}{\alpha^2} - \frac{1}{1+\sqrt{\alpha}}\right){}^{V}(g(X,Y))\,{}^{V}u$$
$$+ \left(\frac{\sqrt{\alpha}}{\alpha^2} + \frac{1}{2\sqrt{\alpha}(1+\sqrt{\alpha})^2} - \frac{1}{\alpha(1+\sqrt{\alpha})}\right) g(X,u)g(Y,u)\,{}^{V}u$$
$$- \frac{1}{2\sqrt{\alpha}}\,{}^{V}(R(u,X)Y) + \frac{1}{2\alpha(1+\sqrt{\alpha})} g(R(u,X)Y,u)\,{}^{V}u;$$

(iv) $$({}^{CG}\nabla_{{}^{V}X}\, J_{CG})({}^{V}Y) = {}^{CG}\nabla_{{}^{V}X}(J_{CG}{}^{V}Y) - J_{CG}({}^{CG}\nabla_{{}^{V}X}{}^{V}Y)$$
$$= {}^{CG}\nabla_{{}^{V}X}\left(\frac{1}{\sqrt{\alpha}}\,{}^{H}Y + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} g(Y,u)\,{}^{H}u\right)$$

$$
\begin{aligned}
&- J_{CG}\Big(-\frac{1}{\alpha}g(X,u)\,{}^VY - \frac{1}{\alpha}g(Y,u)\,{}^VX + \frac{1+\alpha}{\alpha^2}\,{}^V(g(X,Y))\,{}^Vu \\
&+ \frac{1}{\alpha^2}g(X,u)g(Y,u)\,{}^Vu\Big) \\
= {}&{}^VX\Big(\frac{1}{\sqrt{\alpha}}\Big){}^HY + \frac{1}{\sqrt{\alpha}}{}^{CG}\nabla_{{}^VX}{}^HY + {}^VX\Big(\frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)\Big){}^Hu \\
&- \frac{1+\alpha}{\alpha^2}\,{}^V(g(X,Y))J_{CG}{}^Vu - \frac{1}{\alpha^2}g(X,u)J_{CG}{}^Vu \\
= {}&- \frac{1}{\alpha\sqrt{\alpha}}g(X,u)\,{}^HY + \frac{1}{2\alpha\sqrt{\alpha}}\,{}^H(R(u,X)Y) \\
&+ \frac{-(1+2\sqrt{\alpha})}{\alpha\sqrt{\alpha}(1+\sqrt{\alpha})^2}g(X,u)g(Y,u)\,{}^Hu + \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}\,{}^V(g(X,Y)){}^Hu \\
&+ \frac{1}{2\alpha\sqrt{\alpha}(1+\sqrt{\alpha})}g(Y,u)\,{}^H(R(u,X)u) + \frac{1}{\alpha\sqrt{\alpha}}g(X,u)\,{}^HY \\
&+ \frac{2}{\alpha\sqrt{\alpha}(1+\sqrt{\alpha})}g(X,u)g(Y,u)\,{}^Hu + \frac{1}{\alpha\sqrt{\alpha}}g(Y,u)\,{}^HX \\
&- \frac{1+\alpha}{\alpha^2}\,{}^V(g(X,Y))\,{}^Hu - \frac{1}{\alpha^2}g(X,u)\,{}^Hu \\
= {}&\frac{1}{\alpha\sqrt{\alpha}}g(Y,u)\,{}^HX - \frac{1}{\alpha^2}g(X,u)\,{}^Hu + \frac{1}{2\alpha\sqrt{\alpha}}\Big({}^H(R(u,X)Y) \\
&+ \frac{1}{1+\sqrt{\alpha}}g(Y,u)\,{}^H(R(u,X)u)\Big) + \Big(\frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} - \frac{1+\alpha}{\alpha^2}\Big){}^V(g(X,Y))\,{}^Hu \\
&+ \frac{1}{\alpha\sqrt{\alpha}(1+\sqrt{\alpha})^2}g(X,u)g(Y,u)\,{}^Hu.
\end{aligned}
$$

Hence, using Theorem 1.1 we deduce:

THEOREM 3.6. *Let (M,g) be a Riemannian manifold and let TM be its tangent bundle equipped with the Cheeger–Gromoll metric g_{CG} and the paracomplex structure J_{CG} defined by* (3.8). *Then the triple (TM, J_{CG}, g_{CG}) is never a para-Kähler–Norden manifold.*

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