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Analytic functions in the unit disc sharing values in a sector

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Abstract. We deal with the uniqueness of analytic functions in the unit disc sharing four distinct values and obtain two theorems improving a previous result given by Mao and Liu (2009).

1. Introduction. We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}$ $(= \mathbb{C} \cup \{\infty\})$ for the extended complex plane, $\mathbb{D} = \{z : |z| < 1\}$ for the unit disc, and X $(\subseteq \mathbb{C})$ for an angular domain. We will study the uniqueness of analytic functions and adopt the standard notation of the Nevanlinna theory of meromorphic functions as explained in [4, 18].

For $a \in \widehat{\mathbb{C}}$, we say that meromorphic functions f and g share the value a CM (resp. IM) in X (or $\mathbb{D}$) if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities (resp. ignoring multiplicities) in X (or $\mathbb{D}$). In addition, we write $f = a \rightleftharpoons g = a$ in X (or $\mathbb{D}$) to mean that f and g share the value a CM in X (or $\mathbb{D}$), $f = a \Leftrightarrow g = a$ in X (or $\mathbb{D}$) to mean that f and g share a IM in X (or $\mathbb{D}$), and $f = a \Rightarrow g = a$ in X (or $\mathbb{D}$) to mean that $f = a$ implies $g = a$ in X (or $\mathbb{D}$).

R. Nevanlinna (see [10]) proved the following well-known theorem.

THEOREM 1.1 (see [10]). *If f and g are nonconstant meromorphic functions that share five distinct values it in $\mathbb{C}$, then $f(z) \equiv g(z)$.*

After his theorem, the uniqueness theory of meromorphic functions sharing values in the whole complex plane attracted many researchers (see [18]). In [21], Zheng studied the uniqueness problem under the condition that five values are shared in some angular domain in $\mathbb{C}$. There are many results on uniqueness with shared values in the complex plane and in angular domains (see [2, 7–9, 14–17, 20–22]). J. H. Zheng [22], T. B. Cao and H. X. Yi [2], and J. F. Xu and H. X. Yi [17] continued to investigate the uniqueness of meromorphic functions sharing five values and four values in an angular

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domain. W. C. Lin, S. Mori and K. Tohge [7] and W. C. Lin, S. Mori and H. X. Yi [8] investigated the uniqueness of meromorphic and entire functions sharing sets in an angular domain. Some important results were obtained by applying Nevanlinna's theory on angular domains (see [4, 21, 22]).

In 2009, Zhang [20] found a relationship between two characteristic functions and applied it to study the uniqueness of meromorphic functions in an angular domain. He proved the following theorems:

THEOREM 1.2 (see [20]). *Let f, g be meromorphic functions of finite order in $\mathbb{C}$, $a_j \in \widehat{\mathbb{C}}$ $(j = 1, \ldots, 5)$ be five distinct values, and let $\Delta_\delta = \{z : |\arg z - \theta_0| \le \delta\}$ $(0 < \delta < \pi)$ be an angular domain satisfying*

$$\limsup_{\varepsilon\to 0^+} \limsup_{r\to+\infty} \frac{\log T(r, \Delta_{\delta-\varepsilon}, f)}{\log r} > \omega, \tag{1.1}$$

where $\omega = \pi/2\delta$ and $T(r, \Delta_{\delta-\varepsilon}, f)$ denotes the Ahlfors characteristic function of f in $\Delta_{\delta-\varepsilon}$. If f and g share a_j $(j = 1, \ldots, 5)$ IM in Δ_δ, then $f \equiv g$.

THEOREM 1.3 (see [20]). *Let f, g be meromorphic functions of finite order in $\mathbb{C}$, $a_j \in \widehat{\mathbb{C}}$ $(j = 1, 2, 3, 4)$ be four distinct values, and let $\Delta_\delta = \{z : |\arg z - \theta_0| \le \delta\}$ $(0 < \delta < \pi)$ be an angular domain satisfying* (1.1). *If f and g share a_j $(j = 1, 2, 3, 4)$ CM in Δ_δ, then $f(z)$ is a linear fractional transformation of $g(z)$.*

It is also an interesting topic to investigate the uniqueness of meromorphic functions in $\mathbb{D}$ (see [3, 9, 12]). To state some uniqueness theorems for meromorphic functions in $\mathbb{D}$, we need the following basic notations and definitions.

DEFINITION 1.1 (see [6]). A meromorphic function f in $\mathbb{D}$ is called *admissible* if

$$\limsup_{r\to 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = \infty,$$

and *non-admissible* if

$$\limsup_{r\to 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} < \infty.$$

Let $f(z)$ be a meromorphic function in $\mathbb{D}$ and let $\Delta(\theta_0, \delta) = \{z : |z| < 1\} \cap \{z : |\arg z - \theta_0| < \delta\}$, where $0 \le \theta_0 \le 2\pi$, $0 < \delta < \pi$. We use $n(r, \Delta(\theta_0, \delta), f(z) = a)$ to denote the number of zeros of $f(z) - a$ in $\Delta(\theta_0, \delta) \cap \{z : |z| < r\}$ counting multiplicities.

THEOREM 1.4 (see [12]). *If admissible functions f, g share five distinct values, then $f \equiv g$.*

THEOREM 1.5 (see [9]). *Let f, g be meromorphic functions in $\mathbb{D}$, $a_j \in \widehat{\mathbb{C}}$ $(j = 1, \dots, 5)$ be five distinct values, and $\Delta(\theta_0, \delta)$ $(0 < \delta < \pi)$ be an angular domain such that for some $a \in \widehat{\mathbb{C}}$,*

$$\limsup_{r\to 1^-} \frac{\log n(r, \Delta(\theta_0, \delta/2), f(z) = a)}{\log \frac{1}{1-r}} = \tau > 1. \tag{1.2}$$

If f and g share a_j $(j = 1, \dots, 5)$ IM in $\Delta(\theta_0, \delta)$, then $f(z) \equiv g(z)$.

REMARK 1.1. Let f be a meromorphic function of finite order in the unit disc. If for arbitrarily small $\varepsilon > 0$, we have

$$\limsup_{r\to 1^-} \frac{\log n(r, \Delta(\theta_0, \varepsilon), f(z) = a)}{\log \frac{1}{1-r}} =: \tau$$

for all but at most two $a \in \widehat{\mathbb{C}}$, then $e^{i\theta_0}$ is called a *Borel point of order* τ of $f(z)$. In [13], G. Valiron proved that every meromorphic function of finite order ρ in the unit disc must have at least one Borel point of order $\rho + 1$.

In this paper, we will investigate the uniqueness of analytic functions in the unit disc $\mathbb{D}$ sharing four distinct values. Relaxing the assumptions of Theorem 1.5, we obtain the following results.

THEOREM 1.6. *Let f, g be analytic functions in $\mathbb{D}$, $a_j \in \mathbb{C}$ $(j = 1, 2, 3, 4)$ be four distinct values, and $\Delta(\theta_0, \delta)$ $(0 < \delta < \pi)$ be an angular domain satisfying* (1.2)*. If f and g share a_1, a_2 CM in $\Delta(\theta_0, \delta)$, and $f = a_3 \Rightarrow g = a_3$ and $f = a_4 \Rightarrow g = a_4$ in $\Delta(\theta_0, \delta)$, then $f(z) \equiv g(z)$.*

THEOREM 1.7. *Under the assumptions of Theorem* 1.6 *with CM replaced by IM, we have either $f(z) \equiv g(z)$ or*

$$f \equiv \frac{a_3 g - a_1 a_2}{g - a_4},$$

and $a_1 + a_2 = a_3 + a_4$ and a_3, a_4 are exceptional values of f and g in $\Delta(\theta_0, \delta)$, respectively.

2. Some lemmas

LEMMA 2.1 (see [4]). *Let f be an admissible function in $\mathbb{D}$, q a positive integer and $a_1, \dots, a_q$ pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r, f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f),$$

where $E \subset (0, 1)$ is a possible exceptional set with $\int_E \frac{dr}{1-r} < \infty$, and the term $\overline{N}\big(r, \frac{1}{f-a_j}\big)$ is replaced by $\overline{N}(r, f)$ when some a_j is ∞. We use $S(r, f)$

to denote any quantity satisfying

$$S(r,f) = O\left\{\log\frac{1}{1-r}\right\} + O\{\log^+ T(r,f)\}$$

as $r \to 1^-$ possibly outside a set E such that $\int_E \frac{dr}{1-r} < \infty$. If the order of f is finite, the remainder $S(r,f)$ is $O\big(\log\frac{1}{1-r}\big)$ without any exceptional set.

Lemma 2.2 (see [5]). *Let $f(z)$ be meromorphic in $\mathbb{D}$ and k be a positive integer. Then*

$$m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = S(r,f).$$

If $f(z)$ is of finite order, then

$$m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = O\left\{\log\frac{1}{1-r}\right\} \quad (r \to 1^-).$$

Lemma 2.3 (see [1, 5]). *Let $h_1(r)$ and $h_2(r)$ be increasing, real valued functions on $[0,1)$ such that $h_1(r) \le h_2(r)$ possibly outside an exceptional set $E \subset [0,1)$ for which $\int_E \frac{dr}{1-r} < \infty$. Then there exists a constant $b \in (0,1)$ such that if $s(r) = 1 - b(1-r)$, then $h_1(r) \le h_2(r)$ for all $r \in (0,1)$.*

Lemma 2.4. *Let f, g be distinct analytic functions in $\mathbb{D}$, $a_j \in \mathbb{C}$ $(j = 1,2,3,4)$ be distinct. If f is admissible, and $f = a_j \Rightarrow g = a_j$ in $\mathbb{D}$ for $j = 1,2,3,4$, then g is also admissible.*

Proof. By the assumption of this lemma and applying Lemma 2.1, we get

$$\begin{aligned}3T(r,f) &\le \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f) \le \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{g-a_j}\right) + S(r,f)\\ &\le 4T(r,g) + S(r,f).\end{aligned}$$

Therefore

$$T(r,f) \le 4T(r,g) + O\left\{\log\frac{1}{1-r}\right\}$$

as $r \to 1^-$ possibly outside a set E such that $\int_E \frac{dr}{1-r} < \infty$. Then g is admissible by Lemma 2.3. ∎

Lemma 2.5. *Suppose that f is an admissible meromorphic function in $\mathbb{D}$. Let $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p$ $(a_0 \ne 0)$ be a polynomial of f with degree p, where the coefficients a_j $(j = 0,1,\ldots,p)$ are constants, and let b_j $(j = 1,\ldots,q)$ be q $(q \ge p+1)$ distinct finite complex numbers. Then*

$$m\left(r, \frac{P(f)\cdot f'}{(f-b_1)\cdots(f-b_q)}\right) = S(r,f).$$

Proof. Use the same argument as in Lemma 4.3 of [19]. ∎

LEMMA 2.6. *Let f, g be distinct analytic functions in $\mathbb{D}$. Suppose that f and g share a_1, a_2 IM in $\mathbb{D}$, and $f = a_3 \Rightarrow g = a_3$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{D}$, and $a_j \in \mathbb{C}$ $(j = 1, 2, 3, 4)$ are four distinct finite complex numbers. If f is an admissible function in $\mathbb{D}$, then g is also admissible, and*

(i) $T(r, g) = 2T(r, f) + S(r)$;
(ii) $T(r, f - g) = 3T(r, f) + S(r)$;
(iii) $T(r, f) = \overline{N}\big(r, \frac{1}{f-a_3}\big) + \overline{N}\big(r, \frac{1}{f-a_4}\big) + S(r)$;
(iv) $T(r, f) = \overline{N}\big(r, \frac{1}{f-a_j}\big) + S(r)$, $j = 1, 2$;
(v) $T(r, g) = \overline{N}\big(r, \frac{1}{g-a_j}\big) + S(r)$, $j = 3, 4$;
(vi) $T(r, f') = T(r, f) + S(r)$, $T(r, g') = T(r, g) + S(r)$,

where $S(r) := S(r, f) = S(r, g)$.

Proof. By the assumption of this lemma, and by Lemma 2.1, we have $T(r, f) \leq 3T(r, g) + S(r, f)$ and $T(r, g) \leq 3T(r, f) + S(r, g)$. From [12], we get $S(r, f) = S(r, g)$.

Let

$$\eta := \frac{f'g'(f - g)}{(f - a_3)(f - a_4)(g - a_1)(g - a_2)}. \tag{2.1}$$

From the assumptions of this lemma, η is analytic in $\mathbb{D}$ and $\eta \not\equiv 0$ unless $f \equiv g$. By Lemma 2.3, we have $m(r, \eta) = S(r, f) + S(r, g) = S(r)$. Thus, $S(r, \eta) = S(r)$.

Since f, g are nonconstant analytic functions in $\mathbb{D}$, and share a_1, a_2 IM in $\mathbb{D}$, and $f = a_3 \Rightarrow g = a_3$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{D}$, again by Lemma 2.1 we have

$$3T(r, f) \leq \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f) \tag{2.2}$$

$$\leq \overline{N}\left(r, \frac{1}{f - g}\right) + S(r, f) = T(r, f - g) + S(r, f) \tag{2.3}$$

$$\leq T(r, f) + T(r, g) + S(r), \tag{2.4}$$

and

$$T(r, g) \leq \overline{N}\left(r, \frac{1}{g - a_1}\right) + \overline{N}\left(r, \frac{1}{g - a_2}\right) + S(r, g) \tag{2.5}$$

$$= \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r) \tag{2.6}$$

$$\leq 2T(r, f) + S(r). \tag{2.7}$$

From (2.4) and (2.7), we get (i); from (2.3), (2.4) and (i), we get (ii); and from (2.2), (2.4), (2.6), (2.7) and (i), we get (iii). Then, we can easily deduce that (iv) and (v) hold from (2.2)–(2.7) and (i)–(iii). Now, we prove (vi). First, we can rewrite (2.1) as

$$f = f'\frac{g'}{\eta(g-a_1)(g-a_2)} + \frac{f'g'(a_3f + a_4f - a_3a_4 - fg)}{\eta(f-a_3)(f-a_4)(g-a_1)(g-a_2)}. \tag{2.8}$$

From (2.8) and Lemma 2.5, we can get $m(r,f) \le m(r,f') + S(r,f)$. Since f is analytic in $\mathbb{D}$, we have $T(r,f') = T(r,f) + S(r,f)$. Similarly, $T(r,g') = T(r,g) + S(r,g)$. ■

LEMMA 2.7. *Suppose f,g are analytic in $\mathbb{D}$. Assume f and g share a_1,a_2 CM in $\mathbb{D}$, and $f = a_3 \Rightarrow g = a_3$ in $\mathbb{D}$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{D}$, and $a_j \in \mathbb{C}$ $(j = 1,2,3,4)$ are four distinct finite complex numbers. If f is admissible, then $f \equiv g$.*

Proof. Suppose $f \not\equiv g$. By the assumption of this lemma, we infer that g is admissible and the conclusions (i)–(vi) of Lemma 2.6 hold. Set

$$\psi_1 := \frac{f'(f-a_3)}{(f-a_1)(f-a_2)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)},$$
$$\psi_2 := \frac{f'(f-a_4)}{(f-a_1)(f-a_2)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)}.$$

By Lemma 2.5, we get

$$m(r,\psi_i) = S(r,f) + S(r,g) = S(r), \qquad i = 1,2. \tag{2.9}$$

Moreover, $N(r,\psi_i) = O(1)$ $(i = 1,2)$. In fact, the poles of ψ_i in $\mathbb{D}$ can only occur at the zeros of $f - a_j$ and $g - a_j$ $(i,j = 1,2)$ in $\mathbb{D}$. Since f,g share a_1,a_2 CM in $\mathbb{D}$, we see that if $z_0 \in \mathbb{D}$ is a zero of $f - a_j$ with multiplicity m (≥ 1), then it is a zero of $g - a_j$ $(j = 1,2)$ with multiplicity m. Suppose that

$$f - a_j = (z-z_0)^m \alpha_j(z), \qquad g - a_j = (z-z_0)^m \beta_j(z),$$

where $\alpha_j(z), \beta_j(z)$ are analytic functions in $\mathbb{D}$ and $\alpha_j(z_0) \ne 0$, $\beta_j(z_0) \ne 0$, $(j = 1,2)$. By a simple calculation, we have

$$\psi_i(z_0) = K\left(\frac{\alpha_j'(z_0)}{\alpha_j(z_0)} - \frac{\beta_j'(z_0)}{\beta_j(z_0)}\right) \qquad (i,j = 1,2),$$

where K is a constant. Therefore, ψ_i $(i = 1,2)$ are analytic in $\mathbb{D}$. Thus, from (2.9), we get $T(r,\psi_i) = S(r)$ $(i = 1,2)$.

If $\psi_i \not\equiv 0, i = 1,2$, then

$$\overline{N}\left(r, \frac{1}{f-a_3}\right) \le N\left(r, \frac{1}{\psi_1}\right) \le T(r,\psi_1) + S(r,f) = S(r), \tag{2.10}$$

$$\overline{N}\left(r, \frac{1}{f-a_4}\right) \le N\left(r, \frac{1}{\psi_2}\right) \le T(r,\psi_2) + S(r,f) = S(r). \tag{2.11}$$

From (2.10), (2.11) and Lemma 2.6(iv), we have $T(r,f) \le S(r)$. Thus, since f, g are admissible functions, that is, f and g are of unbounded characteristic, and from the definition of $S(r)$, we get a contradiction.

Assume that one of ψ_1 and ψ_2 is identically zero, say $\psi_1 \equiv 0$. Then

$$\overline{N}_{(2}\Big(r, \frac{1}{g-a_4}\Big) = \overline{N}_{(2}\Big(r, \frac{1}{f-a_4}\Big), \tag{2.12}$$

where $\overline{N}_{(2}\big(r, \frac{1}{f-a}\big)$ is the counting function of the distinct zeros of $f-a$ in $\mathbb{D}$ with multiplicity $q \ge 2$.

From (2.1), we see that $g(z_1) = a_4$ implies $f(z_1) = a_4$ for $z_1 \in \mathbb{D}$ satisfying $\eta(z_1) \ne 0$. Since $T(r,\eta) = S(r)$, we have

$$\overline{N}_{1)}\Big(r, \frac{1}{g-a_4}\Big) = \overline{N}_{1)}\Big(r, \frac{1}{f-a_4}\Big) + S(r), \tag{2.13}$$

where $\overline{N}_{1)}\big(r, \frac{1}{f-a}\big)$ is the counting function of the distinct simple zeros of $f-a$ in $\mathbb{D}$.

From (2.12) and (2.13), we get

$$\overline{N}\Big(r, \frac{1}{g-a_4}\Big) = \overline{N}\Big(r, \frac{1}{f-a_4}\Big) + S(r). \tag{2.14}$$

Similarly, when $\psi_2 \equiv 0$, we get

$$\overline{N}\Big(r, \frac{1}{g-a_3}\Big) = \overline{N}\Big(r, \frac{1}{f-a_3}\Big) + S(r). \tag{2.15}$$

From (2.14), (2.15) and Lemma 2.6(i), (v), we get

$$2T(r,f) = \overline{N}\Big(r, \frac{1}{f-a_3}\Big) + S(r),$$

or

$$2T(r,f) = \overline{N}\Big(r, \frac{1}{f-a_4}\Big) + S(r).$$

Since f, g are admissible functions in the unit disc, we get a contradiction again. ∎

Lemma 2.8. *Suppose f, g are analytic in $\mathbb{D}$. Assume f and g share two distinct values a_1, a_2 IM in $\mathbb{D}$, and $f = a_3 \Rightarrow g = a_3$ and $g = a_4 \Rightarrow f = a_4$ in $\mathbb{D}$. If f is admissible, then so is g; moreover, either $f(z) \equiv g(z)$ or*

$$f \equiv \frac{a_3 g - a_1 a_2}{g - a_4},$$

and $a_1 + a_2 = a_3 + a_4$, and a_3, a_4 are Picard exceptional values of f and g in $\mathbb{D}$, respectively.

Proof. Suppose that $f \not\equiv g$. By Lemma 2.1 and f is admissible, we have

$$\begin{aligned}
&2T(r,f) + \overline{N}\left(r, \frac{1}{g-a_4}\right)\\
&\quad \le \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + \overline{N}\left(r, \frac{1}{f-a_3}\right)\\
&\qquad + \overline{N}\left(r, \frac{1}{g-a_4}\right) + S(r,f)\\
&\quad \le \overline{N}\left(r, \frac{1}{f-g}\right) + S(r,f) \le T(r,f) + T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Therefore,

$$T(r,f) + \overline{N}\left(r, \frac{1}{g-a_4}\right) \le T(r,g) + S(r,f) + S(r,g). \tag{2.16}$$

Similarly,

$$T(r,g) + \overline{N}\left(r, \frac{1}{f-a_3}\right) \le T(r,f) + S(r,g) + S(r,f). \tag{2.17}$$

From (2.16) and (2.17), we see that $T(r,f) = T(r,g) + S(r,f) + S(r,g)$, and

$$\overline{N}\left(r, \frac{1}{f-a_3}\right) = S(r,f) + S(r,g), \qquad \overline{N}\left(r, \frac{1}{g-a_4}\right) = S(r,f) + S(r,g), \tag{2.18}$$

Thus, from [12], (2.16), (2.17) and the definition of $S(r)$, we deduce that g is admissible when f is.

From (2.16)–(2.18), we also get

$$2T(r,f) = \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + S(r). \tag{2.19}$$

From (2.19), we can see that "almost all" zeros of $f - a_i$ $(i = 1, 2)$ in $\mathbb{D}$ are simple. Similarly, "almost all" zeros of $g - a_i$ $(i = 1, 2)$ in $\mathbb{D}$ are simple. Let

$$\begin{aligned}
\varphi_1 &:= \frac{(a_1-a_3)f'(f-a_2)}{(f-a_1)(f-a_3)} - \frac{(a_1-a_4)g'(g-a_2)}{(g-a_1)(g-a_4)},\\
\varphi_2 &:= \frac{(a_2-a_3)f'(f-a_1)}{(f-a_2)(f-a_3)} - \frac{(a_2-a_4)g'(g-a_1)}{(g-a_2)(g-a_4)}.
\end{aligned}$$

By Lemma 2.5, $m(r,\varphi_i) = S(r)$ $(i = 1, 2)$. Since f, g share a_1, a_2 IM in $\mathbb{D}$ and from (2.18), we have $N(r,\varphi_i) = S(r)$ $(i = 1, 2)$. Therefore, $T(r,\varphi_i) = S(r)$ $(i = 1, 2)$.

If $\varphi_1 \not\equiv 0$, then $\overline{N}(r, 1/(f-a_2)) \le \overline{N}(r, 1/\varphi_1) = S(r)$. Thus, from (2.19), we get a contradiction easily. Similarly, when $\varphi_2 \not\equiv 0$, we get a contradiction, too. Hence, φ_1, φ_2 are identically equal to 0. Then $\frac{\varphi_1-\varphi_2}{a_1-a_2} \equiv 0$, i.e.,

$$\frac{f'}{f-a_3} - \frac{g'}{g-a_4} - \frac{f'}{f-a_1} + \frac{g'}{g-a_1} - \frac{f'}{f-a_2} + \frac{g'}{g-a_2} \equiv 0,$$

which implies that

$$\frac{f-a_3}{g-a_4} \cdot \frac{(g-a_1)(g-a_2)}{(f-a_1)(f-a_2)} \equiv c, \tag{2.20}$$

where c is a nonzero constant. Rewrite (2.20) as

$$g^2 - \left(a_1 + a_2 - \frac{c\gamma(f)}{f-a_3}\right)g + a_1a_2 + \frac{ca_4\gamma(f)}{f-a_3} \equiv 0, \tag{2.21}$$

where $\gamma(f) := (f-a_1)(f-a_2)$. The discriminant of (2.21) is

$$\varDelta(f) = \left(a_1 + a_2 - \frac{c\gamma(f)}{f-a_3}\right)^2 - 4\left(a_1a_2 + \frac{ca_4\gamma(f)}{f-a_3}\right) = \frac{Q(f)}{(f-a_3)^2},$$

where

$$Q(z) := ((a_1+a_2)(z-a_3) - c\gamma(z))^2 - 4a_1a_2(z-a_3)^2 - 4ca_4\gamma(z)(z-a_3)$$

is a polynomial of degree 4 in z. If a is a zero of $Q(z)$ in $\mathbb{D}$, obviously $a \neq a_3$. Then from (2.21), $f(z) = a$ implies that

$$g(z) = \frac{1}{2}\left(a_1 + a_2 - \frac{c\gamma(a)}{a-a_3}\right) =: b. \tag{2.22}$$

Set

$$\phi_1 := \frac{f'g'(f-g)}{(f-a_1)(g-a_2)(f-a_3)(g-a_4)},$$
$$\phi_2 := \frac{f'g'(f-g)}{(f-a_2)(g-a_1)(f-a_3)(g-a_4)},$$
$$\phi := \frac{\phi_2}{\phi_1} = \frac{(f-a_1)(g-a_2)}{(f-a_2)(g-a_1)}.$$

By Lemma 2.5, $m(r,\phi_i) = S(r)$ $(i = 1,2)$, and by a simple calculation, $N(r,\phi_i) = S(r), (i = 1,2)$. Then $T(r,\phi_i) = S(r)$ $(i = 1,2)$, and thus $T(r,\phi) = S(r)$.

Assume that f is not a Möbius transformation of g. Then ϕ is a nonconstant function. Since

$$Q(a_1) = ((a_1+a_2)(a_1-a_3))^2 - 4a_1a_2(a_1-a_3)^2 = (a_1-a_3)^2(a_1-a_2)^2 \neq 0,$$
$$Q(a_2) = ((a_1+a_3)(a_2-a_3))^2 - 4a_1a_2(a_2-a_3)^2 = (a_2-a_3)^2(a_1-a_2)^2 \neq 0,$$

from $a \neq a_i$ $(i = 1,2)$ and (2.20), we get

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leq \overline{N}\left(r, \frac{1}{\phi-\xi}\right) \leq T(r,\phi) = S(r), \tag{2.23}$$

where $\xi = \frac{(a-a_1)(b-a_2)}{(a-a_2)(b-a_1)}$. Since f is analytic in $\mathbb{D}$, by Lemma 2.1 and (2.18) we get

$$T(r,f) \leq \overline{N}\left(r, \frac{1}{f-a_3}\right) + \overline{N}\left(r, \frac{1}{f-a}\right) + S(r) = S(r).$$

Since f, g are admissible, we get a contradiction. Therefore f is a Möbius transformation of g. Since f, g are analytic functions in $\mathbb{D}$, by a simple calculation we easily get $a_1 + a_2 = a_3 + a_4$ and

$$f \equiv \frac{a_3 g - a_1 a_2}{g - a_4};$$

furthermore, a_3, a_4 are Picard exceptional values of f and g in $\mathbb{D}$, respectively. ■

LEMMA 2.9 (see [9]). *Set*

$$u = u(z) = \frac{z^{\pi/\delta} + 2z^{\pi/2\delta} - 1}{z^{\pi/\delta} - 2z^{\pi/2\delta} - 1},$$

where $0 < \delta < \pi$. *Then* u *maps conformally* $\{z : |\arg z| < \delta,\ |z| < 1\}$ *onto the unit disc* $\{u : |u| < 1\}$.

3. Proofs of the main results

3.1. Proof of Theorem 1.6. Without loss of generality, we may assume $\theta_0 = 0$. Set

$$u = u(z) = \frac{z^{\pi/\delta} + 2z^{\pi/2\delta} - 1}{z^{\pi/\delta} - 2z^{\pi/2\delta} - 1}. \tag{3.1}$$

Let $z = z(u)$ denote its inverse function. By Lemma 2.9 we know that u maps conformally $\Delta(0,\delta)$ onto the unit disc $\mathbb{D}' := \{u : |u| < 1\}$.

Using the same argument as in [10, Theorem 1.2], we find that $f(z(u)$ and $g(z(u))$ are meromorphic functions in $\mathbb{D}'$, and $f(z(u))$ is admissible in $\mathbb{D}'$. For the convenience of the reader, we repeat the argument.

Set $z_0 = pe^{i\vartheta} \in \Delta(0,\delta)$. By (3.1) we get

$$\begin{aligned} 1 - |u(z_0)| &= 1 - \sqrt{\frac{A^2+B^2}{C^2+D^2}} = \frac{C^2 + D^2 - A^2 - B^2}{C^2 + D^2 + \sqrt{(A^2+B^2)(C^2+D^2)}} \\ &= \frac{8p^{\pi/2\delta}\left(1 - p^{\pi/\delta}\right)\cos\frac{\pi\vartheta}{2\delta}}{C^2 + D^2 + \sqrt{(A^2+B^2)(C^2+D^2)}}, \end{aligned} \tag{3.2}$$

where

$$A = p^{\pi/\delta}\cos\frac{\pi\vartheta}{\delta} + 2p^{\pi/2\delta}\cos\frac{\pi\vartheta}{2\delta} - 1, \quad B = p^{\pi/\delta}\sin\frac{\pi\vartheta}{\delta} + 2p^{\pi/2\delta}\sin\frac{\pi\vartheta}{2\delta},$$

$$C = p^{\pi/\delta}\cos\frac{\pi\vartheta}{\delta} - 2p^{\pi/2\delta}\cos\frac{\pi\vartheta}{2\delta} - 1, \quad D = p^{\pi/\delta}\sin\frac{\pi\vartheta}{\delta} - 2p^{\pi/2\delta}\sin\frac{\pi\vartheta}{2\delta}.$$

Since

$$C^2 + D^2 = p^{2\pi/\delta} + 2p^{\pi/\delta} + 1 + 4p^{2\pi/\delta}(1 - p^{\pi/\delta}) \cos \frac{\pi\vartheta}{2\delta} + 2p^{\pi/\delta}\left(1 - \cos \frac{\pi\vartheta}{\delta}\right),$$

we get

$$1 \le C^2+D^2 \le C^2+D^2+\sqrt{(A^2 + B^2)(C^2 + D^2)} \le 2(C^2+D^2) \le 20. \tag{3.3}$$

Since $\lim_{p\to 1^-} \frac{1-p^{\pi/\delta}}{1-p} = \pi/\delta$, there exists $b \in ((1/2)^{2\delta/\pi}, 1)$ such that for all p satisfying $b < p < 1$, we have

$$\frac{1}{2} < p^{\pi/2\delta} < 1, \qquad \frac{\pi}{2\delta}(1 - p) < 1 - p^{\pi/\delta} < \frac{3\pi}{2\delta}(1 - p). \tag{3.4}$$

Therefore, from (3.2)–(3.4), we get

$$\min\{1 - |u(pe^{i\vartheta})| : b < p < r,\ |\vartheta| < \delta/2\} > \frac{\pi}{20\delta}(1 - r) \tag{3.5}$$

for all $r \in (b, 1)$.

We now prove that $f(z(u))$ is admissible in $\mathbb{D}' = \{u : |u| < 1\}$. From (1.2), there exists a sequence $\{r_n\}$ of positive numbers such that $r_n \to 1$ as $n \to \infty$ and

$$n(r_n, \Delta(0, \delta/2), f(z) = a) > \left(\frac{1}{1 - r_n}\right)^{\tau_1} \tag{3.6}$$

for sufficiently large n and $\tau > \tau_1 > 1$. Then from (3.6) and Theorem 1.3.2 in [6, pp. 16–17], we have

$$\limsup_{t\to 1} \frac{T(t, f(z(u)))}{\log \frac{1}{1-t}} \ge \limsup_{t'_n\to 1} \frac{T(t'_n, f(z(u)))}{\log \frac{1}{1-t'_n}} \ge \infty. \tag{3.7}$$

Since $f(z(u))$ is a meromorphic function in $\mathbb{D}'$, from (3.7) we see that $f(z(u))$ is admissible in $\mathbb{D}'$.

From the assumption of Theorem 1.6, we infer that $f(z(u))$ and $g(z(u))$ share the two distinct values a_1, a_2 CM in $\mathbb{D}'$, and $f = a_3 \Rightarrow g = a_3$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{D}'$. Then by Lemmas 2.4 and 2.7, we get $f(z(u)) \equiv g(z(u))$.

This completes the proof of Theorem 1.6.

3.2. Proof of Theorem 1.7. We deduce that $f(z(u))$ is admissible in $\mathbb{D}'$, as in Theorem 1.6. Then $f(z(u))$ and $g(z(u))$ share the two distinct values a_1, a_2 IM in $\mathbb{D}'$, and $f = a_3 \Rightarrow g = a_3$ and $g = a_4 \Rightarrow f = a_4$ in $\mathbb{D}'$. Thus, by Lemma 2.8, we get the conclusion of Theorem 1.7.

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