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On lifting of connections to Weil bundles

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Abstract. We prove that the problem of finding all $\mathcal{M}f_m$-natural operators $B : Q \rightsquigarrow QT^A$ lifting classical linear connections ∇ on m-manifolds M to classical linear connections $B_M(\nabla)$ on the Weil bundle $T^A M$ corresponding to a p-dimensional (over $\mathbb{R}$) Weil algebra A is equivalent to the one of finding all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ transforming classical linear connections ∇ on m-manifolds M into base-preserving fibred maps $C_M(\nabla) : T^1_{p-1}M = \bigoplus_M^{p-1} TM \to T^*M \otimes T^*M \otimes TM$.

1. Introduction. Let A be a p-dimensional (over $\mathbb{R}$) Weil algebra and $T^A M$ be the Weil bundle of infinitesimal near A-points of a manifold M (see [We], [GMP], [KMS]). For $A = \mathbf{D} = J^1_0(\mathbb{R}, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$, we get the tangent bundle $TM = J^1_0(\mathbb{R}, M)$. For $A = \mathbf{D}^1_{p-1} = J^1_0(\mathbb{R}^{p-1}, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}^{p-1}$, we get the bundle $T^1_{p-1}M = J^1_0(\mathbb{R}^{p-1}, M)$ of $(p-1)^1$-velocities of M. We have the usual vector bundle identifications (isomorphisms) $T^1_{p-1}M = TM \otimes \mathbb{R}^{p-1} = \bigoplus_M^{p-1} TM = TM \times_M \cdots \times_M TM$ ($p-1$-fold product).

In [Mo], A. Morimoto constructed canonically a classical linear connection ∇^A on $T^A M$ from a classical linear connection ∇ on M. One can observe that the construction ∇^A is affine in ∇. In [D], J. Dębecki described all affine (in ∇) canonical constructions of torsion free classical linear connections $B(\nabla)$ on $T^A M$ from torsion free classical linear connections ∇ on M.

The problem of finding all (not necessarily affine in ∇) canonical constructions of classical linear connections $B(\nabla)$ on $T^A M$ from classical linear connections ∇ on M is still open. In the present note we prove that this problem is equivalent to the one of finding all canonical constructions of base-preserving fibred maps $C(\nabla) : T^1_{p-1}M \to T^*M \otimes T^*M \otimes TM$ from classical linear connections ∇ on M.

The canonical constructions of base-preserving fibred maps $C(\nabla)$ as above are not base-extending (see Remark 4.2) and therefore they are more

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convenient than those of classical linear connections $B(\nabla)$ on T^AM. So, it seems that our result can simplify the problem of finding all canonical constructions of classical linear connections $B(\nabla)$ on T^AM from classical linear connections ∇ on M.

From now on, the category of m-dimensional manifolds and their embeddings will be denoted by $\mathcal{M}f_m$. All canonical constructions $B(\nabla)$ or $C(\nabla)$ (as above) will always be identified with the corresponding $\mathcal{M}f_m$-natural operators $B : Q \rightsquigarrow QT^A$ or $C : Q \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ in the sense of [KMS]. For the reader's convenience the definitions of such $\mathcal{M}f_m$-natural operators are recalled in Section 2. Some examples of such operators are presented in Section 3. The main result is Theorem 4.1 in Section 4. Its proof is given in Section 5.

All manifolds and maps are assumed to be smooth (of class C^∞). All manifolds are assumed to be Hausdorff, finite-dimensional and without boundaries.

2. Basic definitions. We will only use the following partial definitions of natural operators. (The general concept of natural operators can be found in [KMS].)

Definition 2.1. Let T^A be the Weil functor corresponding to a Weil algebra A. An *$\mathcal{M}f_m$-natural operator* $B : Q \rightsquigarrow QT^A$ transforming classical linear connections ∇ on m-manifolds M into classical linear connections $B_M(\nabla)$ on T^AM is an $\mathcal{M}f_m$-invariant system $B = \{B_M\}_{M\in \mathrm{obj}(\mathcal{M}f_m)}$ of regular operators (functions)

$$B_M : \underline{Q}(M) \to \underline{Q}(T^AM)$$

for any m-manifold M, where $\underline{Q}(M)$ is the set of all classical linear connections on M. The *$\mathcal{M}f_m$-invariance* of B means that if $\nabla_1 \in \underline{Q}(M_1)$ and $\nabla_2 \in \underline{Q}(M_2)$ are φ-related by an embedding $\varphi : M_1 \to M_2$ between m-manifolds (i.e. φ is (∇_1, ∇_2)-affine), then $B_{M_1}(\nabla_1)$ and $B_{M_2}(\nabla_2)$ are $T^A\varphi$-related. The regularity of B means that B_M transforms smoothly parametrized families of connections into smoothly parametrized ones. We say that B is *affine* if $B_M : \underline{Q}(M) \to \underline{Q}(T^AM)$ is affine for any m-manifold M.

Definition 2.2. An *$\mathcal{M}f_m$-natural operator* $C : Q \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ transforming classical linear connections ∇ on m-manifolds M into base-preserving fibred maps $C_M(\nabla) : T^1_{p-1}M \to T^*M \otimes T^*M \otimes TM$ is an $\mathcal{M}f_m$-invariant system $C = \{C_M\}_{M\in \mathrm{obj}(\mathcal{M}f_m)}$ of regular operators

$$C_M : \underline{Q}(M) \to C^\infty_M(T^1_{p-1}M, T^*M \otimes T^*M \otimes TM)$$

for any m-manifold M, where $\underline{Q}(M)$ is as in Definition 2.1 and $C^\infty_M(T^1_{p-1}M, T^*M \otimes T^*M \otimes TM)$ is the set of all base-preserving fibred maps $T^1_{p-1}M \to$

$T^*M \otimes T^*M \otimes TM$. The $\mathcal{M}f_m$-invariance of C means that if $\nabla_1 \in \underline{Q}(M_1)$ and $\nabla_2 \in \underline{Q}(M_2)$ are φ-related by an embedding $\varphi : M_1 \to M_2$ between m-manifolds, then the fibred maps $C_{M_1}(\nabla_1) : T^1_{p-1}M_1 \to T^*M_1 \otimes T^*M_1 \otimes TM_1$ and $C_{M_2}(\nabla_2) : T^1_{p-1}M_2 \to T^*M_2 \otimes T^*M_2 \otimes TM_2$ are also φ-related (i.e. $C_{M_2}(\nabla_2) \circ T^1_{p-1}\varphi = (T^*\varphi \otimes T^*\varphi \otimes T\varphi) \circ C_{M_1}(\nabla_1)$). The regularity means almost the same as in Definition 2.1.

3. Some examples

Example 3.1. An example of an affine $\mathcal{M}f_m$-natural operator $B : Q \rightsquigarrow QT^A$ is the $\mathcal{M}f_m$-natural operator $B^A : Q \rightsquigarrow QT^A$ transforming classical linear connections ∇ on m-manifolds M into the complete (Morimoto) lifting ∇^A of ∇ to T^AM (see [Mo]).

Example 3.2. Many $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ can be obtained as follows. Let ∇ be a classical linear connection on an m-manifold M. Let τ_∇ be a tensor field of type $\bigotimes^k T^* \otimes T$ canonically depending on ∇, e.g. the curvature tensor $\mathcal{R}_\nabla$ of ∇, the torsion tensor $\mathcal{T}_\nabla$ of ∇, the higher order covariant derivatives $\nabla^l\mathcal{R}_\nabla$ of the curvature tensor of ∇, etc. (The full description of such tensor fields τ_∇ depending of torsion free classical linear connections ∇ can be found in Section 33.4 of [KMS].) Let $a : T^1_{p-1}M \to T^1_{k-2}M$ be a base-preserving vector bundle map, e.g. the system $(\mathrm{pr}_{i_1}, \dots, \mathrm{pr}_{i_{k-2}})$ of usual fibred projections $T^1_{p-1}M \to TM$. We put $C_M(\nabla) := \tau_\nabla(a, -, -) : T^1_{p-1}M \to T^*M \otimes T^*M \otimes TM$ (i.e. $C_M(\nabla)(v_1, \dots, v_{p-1})(w_1, w_2) = \tau_\nabla(a(v_1, \dots, v_{p-1}), w_1, w_2)$, $v_1, \dots, v_{p-1}, w_1, w_2 \in T_xM$, $x \in M$).

Remark 3.3. The complete description of all affine $\mathcal{M}f_m$-natural operators $B : Q_\tau \rightsquigarrow Q_\tau T^A$ transforming torsion free classical linear connections ∇ on m-manifolds M into torsion free classical linear connections $B_M(\nabla)$ on T^AM can be found in [D]. A full description of all $\mathcal{M}f_m$-natural operators $B : Q \rightsquigarrow QT^A$ is unknown. A full description of all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ is also unknown.

4. The main theorem. The main result of this note is the following theorem which shows that the problems of finding all $\mathcal{M}f_m$-natural operators $B : Q \rightsquigarrow QT^A$ and of finding all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ are equivalent.

Theorem 4.1. *Let A be a p-dimensional* (*over* $\mathbb{R}$) *Weil algebra. The $\mathcal{M}f_m$-natural operators $B : Q \rightsquigarrow QT^A$ transforming classical linear connections ∇ on m-manifolds M into classical linear connections $B_M(\nabla)$ on T^AM are in bijection with the p^3-tuples $C = (C_i)_{i=1,\dots,p^3}$ of $\mathcal{M}f_m$-natural operators $C_i : Q \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ transforming classical linear con-*

nections ∇ *on* m*-manifolds* M *into base-preserving fibred maps* $(C_i)_M(\nabla) : T^1_{p-1}M \to T^*M \otimes T^*M \otimes TM$.

REMARK 4.2. Natural operators $C : Q \rightsquigarrow (T^1_{p-1}, T^*\otimes T^*\otimes T)$ can be interpreted (in obvious way) as natural operators sending tuples $(\nabla, X_1, \ldots, X_{p-1})$ of classical linear connections ∇ on M and vector fields $X_1, \ldots, X_{p-1}$ on M into tensor fields $C_M(\nabla, X_1, \ldots, X_{p-1})$ of type $T^*\otimes T^*\otimes T$ on M of order 0 in $X_1, \ldots, X_{p-1}$. Natural operators $C : Q \rightsquigarrow (T^1_{p-1}, T^*\otimes T^*\otimes T)$ so interpreted are not base-extending and therefore they are more convenient than natural operators $B : Q \rightsquigarrow QT^A$ (which are base-extending). So, it seems that Theorem 4.1 can simplify the problem of finding all natural operators $B : Q \rightsquigarrow QT^A$.

5. Proof of the main theorem. First we deduce the following lemma.

LEMMA 5.1. *Let* $A = \mathbb{R} \oplus N_A$ *be a* p*-dimensional Weil algebra with the maximal nilpotent ideal* N_A*. For any classical linear connection* ∇ *on an* m*-manifold* M *we have a base-preserving fibred diffeomorphism* $I^A_\nabla : T^AM \to TM \otimes \mathbb{R}^{p-1}$ *canonically depending on* ∇.

Proof. We fix a linear isomorphism $N_A \cong \mathbb{R}^{p-1}$. By the theory of Weil functors we have the standard $\mathrm{Gl}(m)$-invariant identification $T^A_0\mathbb{R}^m \cong \mathbb{R}^m \otimes N_A$ (the restriction of the $\mathrm{Gl}(V)$-invariant one $T^AV \cong V \otimes A$ for $V = \mathbb{R}^m$). So, we have the $\mathrm{Gl}(m)$-invariant identification

$$I^A : T^A_0\mathbb{R}^m \to T_0\mathbb{R}^m \otimes \mathbb{R}^{p-1}$$

defined by $T^A_0\mathbb{R}^m \cong \mathbb{R}^m \otimes N_A \cong T_0\mathbb{R}^m \otimes N_A \cong T_0\mathbb{R}^m \otimes \mathbb{R}^{p-1}$, where the first $\cong$ is as above, the second $\cong$ is the tensor product of the standard $\mathrm{Gl}(m)$-invariant identification $\mathbb{R}^m \cong T_0\mathbb{R}^m$ and id_{N_A} and the last $\cong$ is the tensor product of $N_A \cong \mathbb{R}^{p-1}$ and $\mathrm{id}_{T_0\mathbb{R}^m}$.

Now, we define a base-preserving fibred diffeomorphism $I^A_\nabla : T^AM \to TM \otimes \mathbb{R}^{p-1}$ by

$$I^A_\nabla(v) := (T_0\varphi^{-1} \otimes \mathrm{id}_{\mathbb{R}^{p-1}})(I^A(T^A_x\varphi(v))), \qquad v \in T^A_xM,\ x \in M,$$

where φ is a ∇-normal coordinate system on M with centre x.

If φ_1 is another ∇-normal coordinate system on M with centre x, then $\varphi_1 = B \circ \varphi$ for some $B \in \mathrm{Gl}(m)$. Then using the $\mathrm{Gl}(m)$-invariance of I^A we easily verify that

$$(T_0\varphi_1 \otimes \mathrm{id}_{\mathbb{R}^{p-1}})(I^A(T^A_x\varphi_1(v))) = (T_0\varphi \otimes \mathrm{id}_{\mathbb{R}^{p-1}})(I^A(T^A_x\varphi(v))).$$

So, the definition of $I^A_\nabla(v)$ is correct. ∎

Using Lemma 5.1 we can prove Theorem 4.1 as follows.

Proof of Theorem 4.1. Let ∇ be a classical linear connection on an m-manifold M. Then for any point $v \in T_xM \otimes \mathbb{R}^{p-1}$, $x \in M$, we have the standard linear isomorphisms

$$\begin{aligned} T_v(TM \otimes \mathbb{R}^{p-1}) &\cong H_v^\nabla \oplus V_v(TM \otimes \mathbb{R}^{p-1}) \cong T_xM \oplus (T_xM \otimes \mathbb{R}^{p-1}) \\ &= T_xM \oplus \cdots \oplus T_xM \ (p\text{-fold direct sum}), \end{aligned}$$

canonically depending on ∇, where $H_v^\nabla \subset T_v(TM \otimes \mathbb{R}^{p-1})$ is the ∇-horizontal subspace and $V_v(TM \otimes \mathbb{R}^{p-1}) \subset T_v(TM \otimes \mathbb{R}^{p-1})$ is the vertical subspace. So, we have the linear identification (linear isomorphism)

$$\begin{aligned} &T_v^*(TM \otimes \mathbb{R}^{p-1}) \otimes T_v^*(TM \otimes \mathbb{R}^{p-1}) \otimes T_v(TM \otimes \mathbb{R}^{p-1}) \\ &\qquad \cong (T_x^*M \otimes T_x^*M \otimes T_xM) \oplus \cdots \oplus (T_x^*M \otimes T_x^*M \otimes T_xM) \end{aligned}$$

(p^3-fold direct sum) canonically depending on ∇. Consequently, any $\mathcal{M}f_m$-natural operator $B : Q \rightsquigarrow QT^A$ can be identified with the system $C = (C_i)_{i=1,\dots,p^3}$ of $\mathcal{M}f_m$-natural operators $C_i : Q \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ defined by

$$((C_i)_M(\nabla)(v))_{i=1,\dots,p^3} = (T^*_{\tilde v} I^A_\nabla \otimes T^*_{\tilde v} I^A_\nabla \otimes T_{\tilde v} I^A_\nabla)(\delta_M(\nabla)(\tilde v))$$

modulo the last identification, $v \in T_xM \otimes \mathbb{R}^{p-1}$, $x \in M$, $\tilde v = (I^A_\nabla)^{-1}(v) \in T^A_xM$, where $\delta_M(\nabla) : T^AM \to T^*T^AM \otimes T^*T^AM \otimes TT^AM$ is the tensor field of type $T^* \otimes T^* \otimes T$ on T^AM with $B_M(\nabla) = \nabla^A + \delta_M(\nabla)$. The proof of Theorem 4.1 is complete. ∎

6. On lifting torsion free connections to Weil bundles. Quite similarly to Theorem 4.1 one can prove

THEOREM 6.1. *Let A be a p-dimensional (over $\mathbb{R}$) Weil algebra. The $\mathcal{M}f_m$-natural operators $B : Q_\tau \rightsquigarrow Q_\tau T^A$ transforming torsion free classical linear connections ∇ on m-manifolds M into torsion free classical linear connections $B_M(\nabla)$ on T^AM are in bijection with the pairs (C, D) of pC^p_2-tuples $C = (C_i)_{i=1,\dots,pC^p_2}$ of $\mathcal{M}f_m$-natural operators $C_i : Q_\tau \rightsquigarrow (T^1_{p-1}, T^* \otimes T^* \otimes T)$ transforming torsion free classical linear connections ∇ on m-manifolds M into base-preserving fibred maps $(C_i)_M(\nabla) : T^1_{p-1}M \to T^*M \otimes T^*M \otimes TM$, and p^2-tuples $D = (D_j)_{j=1,\dots,p^2}$ of $\mathcal{M}f_m$-natural operators $D_j : Q_\tau \rightsquigarrow (T^1_{p-1}, (T^* \odot T^*) \otimes T)$ transforming torsion free classical linear connections ∇ on m-manifolds M into base-preserving fibred maps $(D_j)_M(\nabla) : T^1_{p-1}M \to (T^*M \odot T^*M) \otimes TM$, where $C^p_q = p!/(q!(p-q)!)$ is the Newton symbol.*

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