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Potentials with respect to the pluricomplex Green function

by Urban Cegrell (Umeå)

Dedicated to Professor Józef Siciak

Abstract. For μ a positive measure, we estimate the pluricomplex potential of μ, $P_\mu(x) = \int_\Omega g(x,y)\,d\mu(y)$, where $g(x,y)$ is the pluricomplex Green function (relative to Ω) with pole at y.

1. Introduction. Denote by $\mathrm{PSH}(\Omega)$ the plurisubharmonic functions on Ω and by $\mathrm{PSH}^-(\Omega)$ the subclass of negative functions. A set $\Omega \subset \mathbb{C}^n$ is said to be a *hyperconvex domain* if it is open, bounded, connected and if there exists $\varphi \in \mathrm{PSH}^-(\Omega)$ such that $\{z \in \Omega;\, \varphi(z) < -c\} \subset\subset \Omega$ for all $c > 0$. For μ a positive measure on Ω we define the *pluricomplex potential* of μ (relative to Ω):

$$P_\mu(x) = \int_\Omega g(x,y)\,d\mu(y)$$

where $g(x,y)$ is the pluricomplex Green function (relative to Ω) with pole at y. We refer to [10] for facts about the pluricomplex Green function.

We let $\mathcal{E}_0$ denote the family of all bounded plurisubharmonic functions φ defined on Ω such that

$$\lim_{z\to\xi} \varphi(z) = 0 \quad \text{for every } \xi \in \partial\Omega, \quad \text{and} \quad \int_\Omega (dd^c\varphi)^n < \infty$$

where $(dd^c)^n$ is the complex Monge–Ampère operator. Let $\mathcal{E}_1$ denote the family of plurisubharmonic functions u defined on Ω such that there exists a decreasing sequence $\{u_j\}$, $u_j \in \mathcal{E}_0$, that converges pointwise to u on Ω as j tends to ∞, and

$$\sup_{j\ge 1} \int_\Omega (-u_j)(dd^c u_j)^n < \infty.$$

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If only $\sup_{j\geq 1}\int_\Omega (dd^c u_j)^n < \infty$ we say that $u \in \mathcal{F}$.

Finally, a negative plurisubharmonic function on Ω belongs to $\mathcal{E}$ if it is locally the restriction of a function in $\mathcal{F}(\Omega)$.

The complex Monge–Ampère operator is well-defined on $\mathcal{E}$.

For background and details, see [9], [10], [11], [12], [8], [5] and [6].

Throughout, we assume Ω to be a hyperconvex domain and μ to be a positive measure with $0 < \mu(\Omega) < \infty$.

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *If* $0 < \mu(\Omega) < \infty$, *then* $P_\mu(x) \in \mathcal{F}(\Omega)$ *and*

$$\int -h(dd^c P_\mu)^n \leq \Big(\int (-h)^{1/n}\, d\mu\Big)^n \leq [\mu(\Omega)]^{n-1}\int -h\, d\mu, \qquad \forall h \in \mathrm{PSH}^-(\Omega).$$

2. Proof of Theorem 1.1. The last inequality follows from the Hölder inequality.

If $\operatorname{supp}\mu \subset\subset \Omega$ then $P_\mu(x)$ is a negative plurisubharmonic function, bounded near the boundary of Ω. (It tends to zero at the boundary.) Therefore, $P_\mu \in \mathcal{E}$.

We first claim: If μ is a compactly supported measure, $P_\mu \in \mathcal{F}$ and $h \in \mathcal{E}_0$, then

$$\int -h(dd^c P_\mu)^n \leq \Big(\int (-h(w))^{1/n}\, d\mu(w)\Big)^n.$$

Following an idea of Carlehed [2] and [3], we consider, for $w_1, \ldots, w_n \in \mathbb{C}^n$,

$$\int_\Omega h(x) dd^c g(x, w_1) \wedge \cdots \wedge dd^c g(x, w_n) = \int_\Omega g(x, w_1) dd^c h(x) \wedge \cdots \wedge dd^c g(x, w_n)$$

(The equality follows from integration by parts, which is valid in $\mathcal{F}$.) By Theorem 5.5 in [6], we have

$$\begin{aligned}\int_\Omega &-h(x) dd^c g(x, w_1) \wedge \cdots \wedge dd^c g(x, w_n)\\ &\leq \Big[\int -h(x)(dd^c g(x, w_1))^n\Big]^{1/n} \times \cdots \times \Big[\int -h(x)(dd^c g(x, w_n))^n\Big]^{1/n}\\ &= (-h(w_1))^{1/n} \ldots (-h(w_n))^{1/n},\end{aligned}$$

so

$$\int_\Omega -g(x, w_1) dd^c h(x) \wedge \cdots \wedge dd^c g(x, w_n) \leq (-h(w_1))^{1/n} \ldots (-h(w_n))^{1/n}.$$

Integrating the inequality n times gives

$$\int -h(x)(dd^c P_\mu(x))^n \leq \Big(\int (-h(w))^{1/n}\, d\mu(w)\Big)^n,$$

which proves the claim.

To prove the theorem, it is thus enough to prove that $P_\mu \in \mathcal{F}$.

We can choose $P_{\mu_j} \in \mathcal{F}$ where μ_j is a sequence of finite weighted sums of Dirac measures with total mass $= \mu(\Omega)$, converging weakly to μ as $j \to \infty$. It follows from the claim that $\int (dd^c P_{\mu_j})^n \le (\mu(\Omega))^n$ and consequently $\int (dd^c P_\mu)^n \le (\mu(\Omega))^n$ so $P_\mu \in \mathcal{F}$ and the proof is complete.

Actually, using the estimate $0 \ge g(x,y) \ge \log|x-y| - \sup_{z,w\in\Omega} \log|z-w|$, one proves that P_{μ_j} tends weakly to P_μ as $j \to \infty$.

3. A general Chern–Levine–Nirenberg theorem. We now use Theorem 1.1 to generalize a theorem of Chern–Levine–Nirenberg [7]. See also [1], [9] and [4].

THEOREM 3.1. *Assume K is a compact subset of Ω. Then there is a constant d such that*

$$\int_K -h(dd^c u)^n \le d[\sup_{z\in\Omega} -u]^n \int_\Omega -h\, d\mu,$$
$$\forall u \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega),\ \forall h \in \mathrm{PSH}^-(\Omega).$$

Proof. It is no restriction to assume that K is not pluripolar, $-1 < u \in \mathcal{E}_0$ and $h \in \mathcal{E}_0$. Let $0 > -c > \sup_{z\in K} P_\mu(z)$. Then $P_\mu/c < -1$ on K so for $h \in \mathcal{E}_0$ we have

$$\begin{aligned}\int_K -h(dd^c u)^n &= \int_K -h\Big(dd^c \max\Big(u, \frac{P_\mu}{c}\Big)\Big)^n \\ &\le \frac{1}{c^n}\int_\Omega -h(dd^c P_\mu)^n \le \frac{\mu(\Omega)^{n-1}}{c^n}\int_\Omega -h\, d\mu\end{aligned}$$

where the last inequality follows from Theorem 1.1. ∎

COROLLARY 3.2. *Assume that K is a compact subset of Ω. Then there is a constant d such that*

$$\int_K (dd^c u)^n \le d[\sup_{z\in\Omega} -u]^{n-1} \int_\Omega -u\, d\mu, \qquad \forall u \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega).$$

Proof. Again, it is no restriction to assume that K is not pluripolar and that $-1 < u \in \mathcal{E}_0$. Let $0 > -c > \sup_{z\in K} P_\mu(z)$. Then $P_\mu/c < -1$ on K so

$$\begin{aligned}\int_K (dd^c u)^n &= \int_K dd^c u \wedge \Big(dd^c \max\Big(u, \frac{P_\mu}{c}\Big)\Big)^{n-1} \\ &\le \frac{1}{c^{n-1}}\int_\Omega -\frac{P_\mu}{c} dd^c u \wedge (dd^c P_\mu)^{n-1} = \frac{1}{c^n}\int_\Omega -u(dd^c P_\mu)^n \\ &\le \frac{\mu(\Omega)^{n-1}}{c^n}\int_\Omega -u\, d\mu\end{aligned}$$

where the last inequality follows from Theorem 1.1. ∎

4. Functions of finite pluricomplex energy. The class $\mathcal{E}_1$ was introduced and studied in [5]. We already know that when $\mu(\Omega) < \infty$ then $P_\mu \in \mathcal{F}$.

THEOREM 4.1. *If* $0 < \nu \leq (-u)^{(n-1)/n}(dd^c v)^n$ *where* $u, v \in \mathcal{E}_1$ *then* $P_\nu \in \mathcal{E}_1$.

Proof. Theorem 1.1 gives

$$\int -h(dd^c P_\nu)^n \leq \Big(\int (-h)^{1/n}\, d\nu\Big)^n$$

and by the Hölder inequality and Theorem 3.2 in [5] we get

$$\begin{aligned}
\int -h(dd^c P_\nu)^n &\leq \Big(\int (-h)^{1/n}(-u)^{(n-1)/n}(dd^c v)^n\Big)^n \\
&\leq \int -h(dd^c v)^n \Big(\int -u(dd^c v)^n\Big)^{n-1} \\
&\leq \Big(\int -h(dd^c h)^n\Big)^{1/(n+1)} \Big(\int -v(dd^c v)^n\Big)^{n/(n+1)} \\
&\quad \times \Big(\int -u(dd^c u)^n\Big)^{(n-1)/(n+1)} \Big(\int -v(dd^c v)^n\Big)^{n(n-1)/(n+1)}
\end{aligned}$$

so in particular

$$\int -P_\nu(dd^c P_\nu)^n \leq C\Big(\int -P_\nu(dd^c P_\nu)^n\Big)^{1/(n+1)}$$

and it follows that $P_\nu \in \mathcal{E}_1$. ■

REMARK. In the theorem, it is enough to assume that the inequality $0 \leq \nu \leq (-u)^{(n-1)/n}(dd^c v)^n$, where $u, v \in \mathcal{E}_1$, holds true in the "plurisubharmonic order" only.

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Urban Cegrell
Department of Mathematics and Mathematical Statistics
Umeå University
SE-901 87 Umeå, Sweden
E-mail: Urban.Cegrell@math.umu.se

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