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A counterexample to the Γ-interpolation conjecture

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Abstract. Agler, Lykova and Young introduced a sequence C_ν, where $\nu \geq 0$, of necessary conditions for the solvability of the finite interpolation problem for analytic functions from the open unit disc $\mathbb{D}$ into the symmetrized bidisc Γ. They conjectured that condition C_{n-2} is necessary and sufficient for the solvability of an n-point interpolation problem. The aim of this article is to give a counterexample to that conjecture.

1. Introduction. In this paper we will denote by $\mathbb{D}$ the open unit disc, by Δ its closure, and by $\mathbb{T}$ the unit circle. We also denote by $\mathcal{S}$ the *Schur class*, i.e. the set of holomorphic functions $f : \mathbb{D} \to \Delta$. For λ_1 and λ_2 in $\mathbb{D}$, we denote

$$[\lambda_1, \lambda_2] := \frac{\lambda_2 - \lambda_1}{1 - \overline{\lambda}_2 \lambda_1} \quad \text{and} \quad \rho(\lambda_1, \lambda_2) := |[\lambda_1, \lambda_2]|.$$

The function ρ is called the *pseudo-hyperbolic distance*.

We denote by M_m the set of all $m \times m$ complex matrices.

For $W \in M_m$, we write $\sigma(W)$ for the spectrum of W. If $\sigma(W) = \{w_1, \ldots, w_k\}$, and if m_j is the multiplicity of w_j as a root of the minimal polynomial of W, we shall call the product

$$b_1(w) := \prod_{j=1}^{k} \left(\frac{w - w_j}{1 - w\overline{w}_j} \right)^{m_j}$$

the *minimal Blaschke product* of W.

The *spectral unit ball* Ω_m is the set of all matrices $W \in M_m$ whose spectral radius is less than 1. In this paper, Ω will refer to Ω_2.

Given n distinct points $\lambda_1, \ldots, \lambda_n$ in the open unit disc and n points $W_1, \ldots, W_n$ in the spectral unit ball Ω_m, the spectral Nevanlinna–Pick problem with data

$$\lambda_j \mapsto W_j, \quad j = 1, \ldots, n,$$

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consists in finding necessary and sufficient conditions for the existence of an analytic map $F : \mathbb{D} \to \Omega_m$ such that $F(\lambda_j) = W_j$ for $j = 1, \ldots, n$.

A method for determining if such a function F exists, with the additional condition $\sup_{z\in\mathbb{D}} r(F(z)) < 1$, where $r(W)$ denotes the spectral radius of the matrix W, was obtained by Bercovici, Foias and Tannenbaum [7]. But this method provides criteria which are in practice hard to implement and therefore some other approaches to solve the problem have been made by several authors.

One of these approaches is to consider, as was done by Agler and Young, an interpolation problem involving the elementary symmetric functions of the eigenvalues of the matrices concerned. With this new approach, the study of the 2×2 spectral Nevanlinna–Pick problem led them to the introduction of the symmetrized bidisc G which is defined as:

$$G := \{(s, p) = (z + w, zw) : z, w \in \mathbb{D}\}.$$

We will denote by

$$\Gamma := \{(z + w, zw) : z, w \in \Delta\},$$

the closure of G.

There is a close relationship between the interpolation with target data in Ω and the interpolation problem with data in G, as stated in the following theorem:

THEOREM 1.1 ([3, Theorem 1.1]). *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ be distinct and let $W_1, \ldots, W_n \in \Omega$. Suppose that either all or none of $W_1, \ldots, W_n$ are scalar matrices. The following statements are equivalent:*

(1) *there exists an analytic function $F : \mathbb{D} \to \Omega$ such that*

$$F(\lambda_j) = W_j, \qquad j = 1, \ldots, n;$$

(2) *there exists an analytic function $f : \mathbb{D} \to G$ such that*

$$f(\lambda_j) = (\mathrm{tr}(W_j), \det(W_j)), \qquad j = 1, \ldots, n.$$

The study of the hyperbolic geometry of the symmetrized bidisc allows us to give a full answer to the 2×2 spectral Nevanlinna–Pick problem with two interpolating points (see [4], [9] and [11]).

Similarly, Nikolov, Pflug and Thomas have shown in [10] that the interpolation problem in Ω_3 can be reduced to an interpolating problem on the symmetrized three-disc.

For the general case, although the solution is not known, some necessary conditions for the solvability of the spectral Nevanlinna–Pick problem have been given: see for example [5] and [8].

From now on we consider the case where $m = 2$. In a recent paper Agler, Lykova, and Young [2] introduced the class of n-extremal holomorphic maps for the interpolation problem into the symmetrized bidisc. Furthermore, they introduced a sequence of necessary conditions of increasing strength C_ν for the solvability of a given interpolation problem into Γ. In the same paper they conjectured that C_{n-2} is necessary and sufficient for a problem with n interpolating points to be solvable.

Condition C_0 is sufficient for $n = 2$ and even for some special cases with $n \geq 2$ (see [2, Theorem 4.4]). Some examples where C_1 is sufficient when $n = 3$ are given in [1]. Our aim is to give a counterexample to the Γ-interpolation conjecture.

2. The Γ-interpolation conjecture. Let $\lambda_1, \ldots, \lambda_n$ be n distinct points in $\mathbb{D}$, let $(s_1, p_1), \ldots, (s_n, p_n) \in \Gamma$, and let $\nu \geq 0$.

The n-Γ-interpolation problem with data (λ, z) where $\lambda = (\lambda_1, \ldots, \lambda_n)$, $z = (z_1, \ldots, z_n)$ and $z_j = (s_j, p_j)$, consists in finding, if possible, an analytic function $h : \mathbb{D} \to \Gamma$ such that $h(\lambda_j) = (s_j, p_j)$ for $j = 1, \ldots, n$, and to give a criterion that guarantees that such a function exists.

The $C_\nu(\lambda, z)$ condition, introduced in [2], is the following: for every Blaschke product v of degree at most ν, the classical Nevanlinna–Pick data

$$\lambda_j \mapsto \Phi(v(\lambda_j), s_j, p_j), \qquad j = 1, \ldots, n, \tag{2.1}$$

are solvable, where

$$\Phi(z, s, p) := \frac{2zp - s}{2 - zs}.$$

These conditions are necessary for the Γ-interpolation problem.

Theorem 2.1 ([2, Theorem 4.3]). *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $z_j \in G$ for $j = 1, \ldots, n$. If there exists an analytic function $h : \mathbb{D} \to \Gamma$ such that $h(\lambda_j) = z_j$ for $j = 1, \ldots, n$, then, for any function v in the Schur class $\mathcal{S}$, the Nevanlinna–Pick data* (2.1) *are solvable. In particular, the condition $C_\nu(\lambda, z)$ holds for every non-negative integer ν.*

The Γ-interpolation conjecture is stated in [2] as follows: condition C_{n-2} is necessary and sufficient for the solvability of the n-Γ-interpolation problem.

We are going to give shortly a counterexample to the 3-Γ-interpolation conjecture. For this, we need to recall a few results from [5]. Let $F : \mathbb{D} \to \Omega$ be holomorphic, fix $z_0 \in \mathbb{D}$, and denote by b_1 the minimal Blaschke product of $F(z_0)$. It is shown in [5, Theorem 1.3] that $\sigma(b_1(F(z))/[z, z_0])$ is a subset of $\overline{\mathbb{D}}$ for all $z \in \mathbb{D}$, and that if it intersects $\mathbb{D}$ for some $z \in \mathbb{D}$, then it does for all $z \in \mathbb{D}$. In this case we have the following estimate:

$$\Delta_\rho\left(\sigma\left(\frac{b_1(F(z_1))}{[z_1, z_0]}\right) \cap \mathbb{D}, \sigma\left(\frac{b_1(F(z_2))}{[z_2, z_0]}\right) \cap \mathbb{D}\right) \leq \rho(z_1, z_2)^{1/2} \tag{2.2}$$

for any $z_1, z_2 \in \mathbb{D}\backslash\{z_0\}$, where

$$\varDelta_\rho(K_1, K_2) := \max\big(\max_{z\in K_1}\min_{w\in K_2}\rho(z,w), \max_{z\in K_2}\min_{w\in K_1}\rho(z,w)\big)$$

is the Hausdorff distance on compact sets corresponding to ρ (see [5, Corollary 3.1]). By Schur's algorithm, condition C_1 can be translated into a set of inequalities to be satisfied by the interpolation data. By comparing this to the necessary condition provided by (2.2), we found the following example of a 3-$\varGamma$-interpolation problem for which C_1 holds, but which is not solvable.

EXAMPLE 2.2. *Let*

$$\lambda_0 = 0, \quad \lambda_1 = -0.12 + 0.5i \quad \textit{and} \quad \lambda_2 = -0.874,$$

and let

$$\alpha = -0.32 + 0.15i, \qquad \beta = 0.5 + 0.77i, \qquad \gamma = -0.38;$$

set $s = \beta + \gamma$ and $p = \beta\gamma$. Then the $\varGamma$-interpolation problem

$$\begin{cases} 0 = \lambda_0 \mapsto (0,0), \\ \quad\;\; \lambda_1 \mapsto (-2\alpha, \alpha^2), \\ \quad\;\; \lambda_2 \mapsto (s,p) \end{cases} \tag{2.3}$$

satisfies C_1, whereas it is not solvable.

Proof. C_1 holds for the $\varGamma$-interpolating data (2.3) if and only if for all $z \in \mathbb{D}$ the Nevanlinna–Pick interpolating data

$$\begin{cases} \lambda_0 \mapsto 0, \\ \lambda_1 \mapsto \alpha, \\ \lambda_2 \mapsto \varPhi(z,s,p) = \dfrac{2zp - s}{2 - zs} \end{cases} \tag{2.4}$$

are solvable.

First note that $|\alpha| < |\lambda_1|$. Recall that the Möbius transformation $T(z) = (az+b)/(cz+d)$ maps the unit disc into the disc with center $C = T(-\overline{c}/\overline{d})$ and radius $R = |ad-bc|/(|d|^2-|c|^2)$. Applying this to our Möbius transformation $z \mapsto \varPhi(z,s,p)$, we find that

$$\sup_{|z|=1} |\varPhi(z,s,p)| = |C| + R = 0.8479 < 0.874 = |\lambda_2|.$$

In particular, we infer that, for all $z \in \mathbb{D}$, we have $\varPhi(z,s,p)/\lambda_2 \in \mathbb{D}$. Thus applying Schur reduction and the maximum modulus principle we deduce that (2.4) is solvable for all $z \in \mathbb{D}$ if and only if

$$\sup_{|z|=1} \rho\bigg(\frac{\alpha}{\lambda_1}, \frac{\varPhi(z,s,p)}{\lambda_2}\bigg) \le \rho(\lambda_1, \lambda_2).$$

The supremum on the left is $\sup_{|z|=1}|T(z)|$, where

$$T(z) := \left[\frac{\Phi(z,s,p)}{\lambda_2}, \frac{\alpha}{\lambda_1}\right] = \frac{az+b}{cz+d}$$

with $a = (-s\alpha\lambda_2 - 2p\lambda_1)\overline{\lambda}_1/\lambda_1$, $b = (2\lambda_2\alpha + s\lambda_1)\overline{\lambda}_1/\lambda_1$, $c = -s\lambda_2\overline{\lambda}_1 - 2\overline{\alpha}p$ and $d = 2\overline{\lambda}_1\lambda_2 + \overline{\alpha}s$. Again $\sup_{|z|=1}|T(z)|$ for the latter Möbius transformation is given by

$$\sup_{|z|=1}|T(z)| = |C| + R = 0.8792.$$

This implies that

$$\sup_{|z|=1}\rho\left(\frac{\alpha}{\lambda_1}, \frac{\Phi(z,s,p)}{\lambda_2}\right) = 0.8792 < 0.90 < \rho(\lambda_1,\lambda_2).$$

Therefore C_1 holds for the data (2.3).

It remains to show that the Γ-interpolating problem (2.3) is not solvable. By Theorem 1.1, problem (2.3) is equivalent to the following spectral Nevanlinna–Pick problem:

$$\tag{2.5} \begin{cases} \lambda_0 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =: W_0, \\ \lambda_1 \mapsto \begin{pmatrix} -\alpha & 1 \\ 0 & -\alpha \end{pmatrix} =: W_1, \\ \lambda_2 \mapsto \begin{pmatrix} \beta & 1 \\ 0 & \gamma \end{pmatrix} =: W_2. \end{cases}$$

Note that a necessary and sufficient condition for the solvability of (2.5) is obtained in [6] by Bercovici. This condition implies a search over four parameters for the problem (2.5). But for our purpose it is enough to show that the necessary condition given by (2.2) is not satisfied.

The minimal polynomial of W_0 is $b_1(w) = w^2$. Suppose (2.5) is solvable. We observe that $\sigma(W_j^2/-\lambda_j) \subseteq \mathbb{D}$ for $j = 1,2$, and therefore we must have, by (2.2), the inequality

$$\Delta_\rho\left(\sigma\left(\frac{W_1^2}{-\lambda_1}\right), \sigma\left(\frac{W_2^2}{-\lambda_2}\right)\right) \le \rho(\lambda_1,\lambda_2)^{1/2}.$$

But direct calculations give

$$\Delta_\rho\left(\sigma\left(\frac{W_1^2}{-\lambda_1}\right), \sigma\left(\frac{W_2^2}{-\lambda_2}\right)\right) = \Delta_\rho\left(\left\{\frac{\alpha^2}{\lambda_1}\right\}, \left\{\frac{\beta^2}{\lambda_2}, \frac{\gamma^2}{\lambda_2}\right\}\right) = \rho\left(\frac{\alpha^2}{\lambda_1}, \frac{\beta^2}{\lambda_2}\right)$$
$$> 0.9678 > 0.9531 > \rho(\lambda_1,\lambda_2)^{1/2}. \ \blacksquare$$

3. Concluding remarks

Remark 3.1. The example we give shows that none of the C_n conditions is sufficient for the solvability of the problem (2.3).

Remark 3.2. As the numbers in Example 2.2 might suggest, this example was not easy to find, as the region where the inequalities were incompatible was very narrow. This could be an indication that condition C_1 is not far from a sufficient condition. It would be interesting to use the necessary and sufficient criterion of Bercovici to try to identify the extreme counterexamples.

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