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On a time-dependent subdifferential evolution inclusion with Carathéodory perturbation

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Abstract. On an infinite-dimensional Hilbert space, we establish the existence of solutions for some evolution problems associated with time-dependent subdifferential operators whose perturbations are Carathéodory single-valued maps.

1. Introduction. We discuss the existence of solutions of an evolution inclusion governed by subdifferential operators of the form

$$(\mathcal{P}_{f(\cdot,\cdot)})\quad \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + f(t, x(t)) & \text{a.e. } t \in I, \\ x(T_0) = x_0 \in \operatorname{dom}\varphi(T_0, \cdot), \end{cases}$$

on an interval $I := [T_0, T]$, where for each $t \in I$, $\partial\varphi(t, \cdot)$ denotes the subdifferential of a time-dependent proper lower semicontinuous (lsc) convex function $\varphi(t, \cdot)$ defined on a Hilbert space H into $\mathbb{R} \cup \{\infty\}$, and $\operatorname{dom}\varphi(t, \cdot)$ is the effective domain of the function $\varphi(t, \cdot)$. The perturbation $f : I \times H \to H$ is a Carathéodory single-valued mapping that satisfies the natural growth condition

$$\|f(t, x)\| \le \beta(t)(1 + \|x\|) \quad \forall (t, x) \in I \times H,$$

with a non-negative function $\beta(\cdot)$ in $L^2_{\mathbb{R}}(I)$.

The existence and uniqueness of solutions for the unperturbed problem

$$(\mathcal{P})\quad \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) & \text{a.e. } t \in I, \\ x(T_0) = x_0 \in \operatorname{dom}\varphi(T_0, \cdot), \end{cases}$$

and the perturbed one

$$(\mathcal{P}_{h(\cdot)})\quad \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + h(t), \\ x(T_0) = x_0 \in \operatorname{dom}\varphi(T_0, \cdot), \end{cases}$$

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with $h \in L^2_H(I)$, were proved in the early work of Peralba [11, 12] under an assumption expressed in terms of the conjugate function $\varphi^*(t,\cdot)$ of the convex function $\varphi(t,\cdot)$: there exist a non-negative Lipschitz function $k : H \to \mathbb{R}$ and an absolutely continuous function $a : I \to \mathbb{R}$ with $\dot{a} \in L^2_{\mathbb{R}}(I)$ such that for all $x \in H$ and $s,t \in I$,

$$\varphi^*(t,x) \le \varphi^*(s,x) + k(x)|a(t) - a(s)|.$$

Note that several results are also known under various conditions expressed in terms of φ or the Fenchel conjugate function $\varphi^*(t,\cdot)$ or the Yosida approximation of $\partial\varphi(t,\cdot)$; we refer to [3], [10], and [13].

In Saïdi–Thibault–Yarou [13], we consider a single-valued perturbation $f(\cdot,\cdot)$ to $(\mathcal{P})$ with a Lipschitz property with respect to the second variable; we get existence and uniqueness of an absolutely continuous solution of $(\mathcal{P}_{f(\cdot,\cdot)})$, and apply this result to a Bolza type optimal control problem.

A particular case of $(\mathcal{P}_{f(\cdot,\cdot)})$, called the sweeping process, with $\varphi(t,\cdot)$ taken as the indicator function of a closed convex or uniformly r-prox-regular moving set $C(t)$, has been studied by Castaing–Salvadori–Thibault [7] in the finite-dimensional setting, and by Edmond–Thibault [9] in the infinite-dimensional setting. Related results can be found in [10] for the class of primal lower nice functions $\varphi : H \to \mathbb{R} \cup \{\infty\}$.

The case of a multi-valued perturbation

$$(\mathcal{P}_{F(\cdot,\cdot)}) \qquad \begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + F(t,x(t)) & \text{a.e. } t \in I, \\ x(T_0) = x_0 \in \operatorname{dom}\varphi(T_0,\cdot), \end{cases}$$

has been studied in [3], [5], and [6] for a convex compact valued perturbation $F(\cdot,\cdot)$ under a compactness assumption on the sublevel sets of $\varphi(t,\cdot)$. In our recent paper [14], this condition on $\varphi(t,\cdot)$ is replaced by a compactness condition imposed only on F.

In this paper, just as in [13], we adopt a discretization approach. We weaken the assumption on $f(\cdot,\cdot)$, namely the perturbation is assumed to be only Carathéodory, and we prove an existence result for $(\mathcal{P}_{f(\cdot,\cdot)})$ under the hypothesis that $\varphi(t,\cdot)$ is inf-ball-compact. This condition permits us to obtain the convergence of the approximate sequence via Ascoli's theorem. The content of this paper is as follows. In Section 2, we give some preliminaries. In Section 3, we recall some results obtained in [11, 13] for $(\mathcal{P})$ and $(\mathcal{P}_{h(\cdot)})$. In Section 4, we establish the main existence theorem for the problem $(\mathcal{P}_{f(\cdot,\cdot)})$ under consideration.

2. Notation and preliminaries. Throughout the paper, $I := [T_0, T]$ is an interval of $\mathbb{R}$, and H is a real Hilbert space whose inner product is denoted by $\langle\cdot,\cdot\rangle$, and the associated norm by $\|\cdot\|$. Let φ be a lsc convex function from H into $\mathbb{R} \cup \{\infty\}$ which is proper in the sense that its effective domain defined by

$$\operatorname{dom}\varphi = \{x \in H : \varphi(x) < \infty\}$$

is non-empty. As usual, its *Fenchel conjugate* is defined by

$$\varphi^*(v) := \sup_{x \in H}[\langle v, x\rangle - \varphi(x)].$$

It is often useful to regularize φ via its *Moreau envelope*

$$\varphi_\lambda(x) := \inf_{y \in H}\left[\varphi(y) + \frac{1}{2\lambda}\|x - y\|^2\right]$$

for $\lambda > 0$. The family $(\varphi_\lambda)_\lambda$ increases when $\lambda \downarrow 0$ to a proper lsc convex function φ, and hence it epi-converges to φ (see [1]). This entails, in particular, that

$$\varphi(x) \le \liminf_{\lambda \downarrow 0} \varphi_\lambda(x_\lambda) \tag{2.1}$$

for any net $(x_\lambda)_\lambda$ in H converging to x.

The Moreau envelope φ_λ is also known to have a Lipschitz continuous derivative $\nabla\varphi_\lambda$.

The subdifferential $\partial\varphi(x)$ of φ at $x \in \operatorname{dom}\varphi$ is defined by

$$\partial\varphi(x) = \{v \in H : \varphi(y) \ge \langle v, y - x\rangle + \varphi(x)\ \forall y \in \operatorname{dom}\varphi\},$$

and its effective domain is $\operatorname{Dom}\partial\varphi = \{x \in H : \partial\varphi(x) \ne \emptyset\}$. It is well known that if φ is a proper lsc convex function, then $\partial\varphi$ is a maximal monotone operator (see [4]).

The function φ is said to be *inf-ball-compact* if for every $r > 0$, the set $\{x \in H : \varphi(x) \le r\}$ is ball-compact, i.e., its intersection with any closed ball in H is compact.

A map $f : I \times H \to H$ is said to be *Carathéodory* if $f(t, \cdot)$ is continuous for a.e. $t \in I$, and $f(\cdot, x)$ is measurable for each $x \in H$.

For more details concerning the properties of maximal monotone operators in Hilbert spaces, we refer to [2, 4]. The basic facts of convex analysis and measurable multifunctions can be found in [8].

3. Single-valued time-dependent perturbation. We recall here the existence theorems obtained in [11, 12], along with some results of [13].

THEOREM 3.1. *Let $\varphi : I \times H \to \mathbb{R}_+ \cup \{\infty\}$ be such that*

(H_1) *for each $t \in I$, the function $x \mapsto \varphi(t, x)$ is proper, lsc, and convex;*

(H_2) *there exist a non-negative ρ-Lipschitz function $k : H \to \mathbb{R}_+$ and an absolutely continuous function $a : I \to \mathbb{R}$, with a non-negative derivative $\dot a \in L^2_{\mathbb{R}}(I)$, such that*

$$\varphi^*(t, x) \le \varphi^*(s, x) + k(x)|a(t) - a(s)| \tag{3.1}$$

for every $(t,s,x) \in I \times I \times H$, where $\varphi^*(t,\cdot)$ is the conjugate function of $\varphi(t,\cdot)$ (recalled above).

Then for any fixed $x_0 \in \operatorname{dom}\varphi(T_0,\cdot)$, the problem $(\mathcal{P})$ has a unique absolutely continuous solution $x(\cdot)$ on $[T_0,T]$ which satisfies

$$|\varphi(t_2,x(t_2)) - \varphi(t_1,x(t_1))| \le \int_{t_1}^{t_2} \big[(k(0)+\rho\|\dot{x}(t)\|)\dot{a}(t) + \|\dot{x}(t)\|^2\big]\,dt \tag{3.2}$$

for $T_0 \le t_1 \le t_2 \le T$. Moreover, $x(t) \in \operatorname{dom}\varphi(t,\cdot)$ for all $t \in I$, and $t \mapsto \varphi(t,x(t))$ is absolutely continuous on I.

Remark 3.2. The requirement $\dot{a}(\cdot) \ge 0$ in (H_2) can be omitted. Indeed, it is enough to replace $a(\cdot)$ by the (absolutely continuous) function $t \mapsto \int_{T_0}^t |\dot{a}(\tau)|\,d\tau$ whose derivative is non-negative.

Peralba's method associates with $(\mathcal{P})$ the regularized differential equation

$$(\mathcal{DE}^\lambda) \qquad \begin{cases} -\dot{x}_\lambda(t) = \nabla\varphi_\lambda(t,x_\lambda(t)) & \text{a.e. } t \in I,\\ x_\lambda(T_0) = x_0, \end{cases}$$

and shows that the net $(x_\lambda(\cdot))_{\lambda>0}$ of solutions of $(\mathcal{DE}^\lambda)$ converges uniformly when $\lambda \downarrow 0$ to a map $x(\cdot)$ which is an absolutely continuous solution of $(\mathcal{P})$, and $(\dot{x}_\lambda(\cdot))_\lambda$ converges in norm in $L^2_H(I)$ to $\dot{x}(\cdot)$. Moreover,

$$\begin{aligned} \|\dot{x}_\lambda\|^2_{L^2_H} \le{}& \sqrt{T-T_0}\,k(0)\|\dot{a}\|_{L^2_{\mathbb{R}}} \\ &+ \rho\|\dot{x}_\lambda\|_{L^2_H}\|\dot{a}\|_{L^2_{\mathbb{R}}} + \varphi_\lambda(T_0,x_0) - \varphi_\lambda(T,x_\lambda(T)). \end{aligned} \tag{3.3}$$

Thanks to this inequality, a similar estimate for the derivative of the solution to the problem $(\mathcal{P})$ can be obtained (see [13, Proposition 3.3]).

Proposition 3.3. *The unique absolutely continuous solution $x(\cdot)$ of $(\mathcal{P})$ satisfies*

$$\begin{aligned} \|\dot{x}\|_{L^2_H} \le{}& \frac{\rho}{2}\|\dot{a}\|_{L^2_{\mathbb{R}}} \\ &+ \bigg[\sqrt{T-T_0}k(0)\|\dot{a}\|_{L^2_{\mathbb{R}}} + \frac{\rho^2}{4}\|\dot{a}\|^2_{L^2_{\mathbb{R}}} + \varphi(T_0,x_0) - \varphi(T,x(T))\bigg]^{1/2}. \end{aligned} \tag{3.4}$$

Proof. Since $(x_\lambda(\cdot))_\lambda$ converges uniformly to $x(\cdot)$, by (2.1) we have

$$\varphi(T,x(T)) \le \liminf_{\lambda\downarrow 0} \varphi_\lambda(T,x_\lambda(T)).$$

Further, the pointwise convergence of $(\varphi_\lambda(t,\cdot))_\lambda$ to $\varphi(t,\cdot)$ leads to

$$\lim_{\lambda\downarrow 0} \varphi_\lambda(T_0,x_0) = \varphi(T_0,x_0).$$

Using these two facts along with the convergence in norm in $L^2_H(I)$ of $(\dot{x}_\lambda(\cdot))_\lambda$ to $\dot{x}(\cdot)$ recalled above, and taking the upper limit in (3.3), we obtain

$$\|\dot{x}\|^2_{L^2_H} \le \sqrt{T-T_0}\, k(0)\|\dot{a}\|_{L^2_{\mathbb{R}+}} + \rho\|\dot{x}\|_{L^2_H}\|\dot{a}\|_{L^2_{\mathbb{R}+}} + \varphi(T_0,x_0) - \varphi(T,x(T)),$$

and this is easily seen to yield the desired estimate. ∎

We recall the extension of Theorem 3.1 to the problem with a time-dependent perturbation, which is obtained in [13, Proposition 3.4] (see also [3, Remark 3.7]). Define, for any $h : I \to H$, the function $|h|$ of I into $\mathbb{R}$ by $|h|(t) := \|h(t)\|$ for all $t \in I$.

PROPOSITION 3.4. *Under the assumptions of Theorem* 3.1, *if* $h \in L^2_H(I)$ *and* $x_0 \in \operatorname{dom}\varphi(T_0,\cdot)$, *then the problem* $(\mathcal{P}_{h(\cdot)})$ *admits a unique absolutely continuous solution* $x(\cdot)$ *that satisfies*

$$(3.5)\qquad \|\dot{x}\|_{L^2_H} \le \frac{\rho+1}{2}\|\dot{a}+|h|\,\|_{L^2_{\mathbb{R}}} + \|h\|_{L^2_H}$$
$$+\bigg[\sqrt{T-T_0}\, k(0)\|\dot{a}+|h|\,\|_{L^2_{\mathbb{R}}} + \frac{(\rho+1)^2}{4}\|\dot{a}+|h|\,\|^2_{L^2_{\mathbb{R}}} + \varphi(T_0,x_0) - \varphi(T,x(T))\bigg]^{1/2}$$

and

$$(3.6)\qquad |\varphi(t_2,x(t_2)) - \varphi(t_1,x(t_1))|$$
$$\le \int_{t_1}^{t_2} \big[k(0) + (\rho+1)\|\dot{x}(t)+h(t)\|\big]\,[\dot{a}(t)+|h|(t)]\,dt + \int_{t_1}^{t_2} \|\dot{x}(t)+h(t)\|^2\,dt$$

for $T_0 \le t_1 \le t_2 \le T$.

4. Single-valued "Carathéodory" perturbation. In this section, we are interested in finding solutions for the problem $(\mathcal{P}_{f(\cdot,\cdot)})$, where $f(\cdot,\cdot)$ is a single-valued Carathéodory map, under a compactness assumption on φ.

THEOREM 4.1. *Assume the assumptions of Theorem* 3.1 *are satisfied, and* φ_t *is inf-ball-compact for a.e.* $t \in I$. *Let* $f : I\times H \to H$ *be a Carathéodory map such that there exists a non-negative function* $\beta(\cdot) \in L^2_{\mathbb{R}}(I)$ *which satisfies*

$$(4.1)\qquad \|f(t,x)\| \le \beta(t)(1+\|x\|) \quad \forall (t,x) \in I\times H.$$

Then, for each $x_0 \in \operatorname{dom}\varphi(T_0,\cdot)$, *the problem*

$$(4.2)\qquad \begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + f(t,x(t)) & a.e.\ t\in I,\\ x(T_0) = x_0, \end{cases}$$

has at least one absolutely continuous solution $x(\cdot)$.

Proof. We adopt the same steps and techniques as in [13, proof of Theorem 4.1]. Suppose first

$$\int_{T_0}^{T} \beta^2(s)\,ds < m, \qquad m = \frac{1}{4(T-T_0)(k^2(0)+3(\rho+1)^2+4)} > 0. \tag{4.3}$$

(A) *Construction of the sequence* $(x_n(\cdot))$. For every $n \in \mathbb{N}$, define a partition of $I = [T_0, T]$ by

$$t_i^n = T_0 + i\frac{T-T_0}{n} \qquad (0 \le i \le n).$$

Consider first the following differential inclusion on the interval $[t_0^n, t_1^n]$:

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + f(t, x_0) & \text{a.e. } t \in [t_0^n, t_1^n], \\ x(t_0^n) = x_0 \in \operatorname{dom}\varphi(T_0, \cdot), \end{cases}$$

and observe that the map $f(\cdot, x_0)$ depends only on t and is in $L^2_H([t_0^n, t_1^n])$ (by the assumption (4.1)). By Proposition 3.4, the last problem has a unique absolutely continuous solution that we denote by $x_0^n(\cdot) : [t_0^n, t_1^n] \to H$. According to (3.5) this solution satisfies

$$\begin{aligned} \|\dot{x}_0^n\|_{L^2_H} \le \frac{\rho+1}{2}\|\dot{a} + |h_0^n|\,\|_{L^2_{\mathbb{R}}} + \|f(\cdot, x_0)\|_{L^2_H} + \Big[\sqrt{t_1^n - T_0}\,k(0)\|\dot{a} + |h_0^n|\,\|_{L^2_{\mathbb{R}}} \\ + \frac{(\rho+1)^2}{4}\|\dot{a} + |h_0^n|\,\|^2_{L^2_{\mathbb{R}}} + \varphi(T_0, x_0) - \varphi(t_1^n, x_0^n(t_1^n))\Big]^{1/2}, \end{aligned}$$

where $|h_0^n| : t \mapsto \|f(t, x_0)\|$ for all $t \in [t_0^n, t_1^n]$.

Likewise, the differential inclusion

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + f(t, x_0^n(t_1^n)) & \text{a.e. } t \in [t_1^n, t_2^n], \\ x(t_1^n) = x_0^n(t_1^n) \in \operatorname{dom}\varphi(t_1^n, \cdot) \end{cases}$$

has a unique absolutely continuous solution that we denote by $x_1^n(\cdot) : [t_1^n, t_2^n] \to H$ with $x_1^n(t_1^n) = x_0^n(t_1^n)$, and it satisfies (3.5).

Similarly, for each n, there exists a finite sequence of absolutely continuous maps $x_i^n(\cdot) : [t_i^n, t_{i+1}^n] \to H$ $(0 \le i \le n-1)$ such that, for each $i \in \{0, \dots, n-1\}$,

$$\begin{cases} -\dot{x}_i^n(t) \in \partial\varphi(t, x_i^n(t)) + f(t, x_{i-1}^n(t_i^n)) & \text{a.e. } t \in [t_i^n, t_{i+1}^n], \\ x_i^n(t_i^n) = x_{i-1}^n(t_i^n) \in \operatorname{dom}\varphi(t_i^n, \cdot), \end{cases}$$

with

$$
(4.4)\qquad \|\dot{x}_i^n\|_{L^2_H} \le \frac{\rho+1}{2}\|\dot{a}+|h_i^n|\,\|_{L^2_{\mathbb{R}}} + \|f(\cdot,x_i^n(t_i^n))\|_{L^2_H} + \Big[\sqrt{t_{i+1}^n - t_i^n}\,k(0)\|\dot{a}+|h_i^n|\|_{L^2_{\mathbb{R}}} + \frac{(\rho+1)^2}{4}\|\dot{a}+|h_i^n|\,\|^2_{L^2_{\mathbb{R}}} + \varphi(t_i^n,x_i^n(t_i^n)) - \varphi(t_{i+1}^n,x_i^n(t_{i+1}^n))\Big]^{1/2},
$$

where $x_0^n(T_0)=x_0$, and $|h_i^n| : t\mapsto \|f(t,x_i^n(t_i^n))\|$ for all $t\in[t_i^n,t_{i+1}^n]$.

Define $x_n : [T_0,T]\to H$ by

$$
x_n(t) = x_i^n(t) \qquad \forall t\in[t_i^n,t_{i+1}^n],\ i\in\{0,\dots,n-1\}.
$$

Obviously $x_n(\cdot)$ is absolutely continuous on $[T_0,T]$, and setting

$$
\begin{cases} \theta_n(T_0) = T_0, \\ \theta_n(t) = t_i^n \quad \text{if } t\in\,]t_i^n,t_{i+1}^n],\ i\in\{0,\dots,n-1\}, \end{cases}
$$

one has

$$
\begin{cases} -\dot{x}_n(t)\in\partial\varphi(t,x_n(t)) + f(t,x_n(\theta_n(t))) \quad \text{a.e. } t\in[T_0,T], \\ x_n(T_0) = x_0. \end{cases}
$$

Note that the map $f(\cdot,x_n(\theta_n(\cdot)))$ defined for $t\in[T_0,T]$ belongs to $L^2_H([T_0,T])$, because by the assumption (4.1) for each $i\in\{0,\dots,n-1\}$ the map $f(\cdot,x_i^n(t_i^n))$ belongs to $L^2_H([t_i^n,t_{i+1}^n])$. Set

$$
\begin{aligned} h_n(t) &= f(t,x_n(\theta_n(t))), && \forall t\in[T_0,T], \\ |h_n| &: t\mapsto\|h_n(t)\|, && \forall t\in[T_0,T]. \end{aligned}
$$

Then, for $I_i := [t_i^n,t_{i+1}^n]$,

$$
\|\dot{x}_n\|_{L^2_H(I_i)} \le \frac{\rho+1}{2}\|\dot{a}+|h_n|\,\|_{L^2_{\mathbb{R}}(I_i)} + \|h_n\|_{L^2_H(I_i)} + \Big[\sqrt{t_{i+1}^n-t_i^n}\,k(0)\|\dot{a}+|h_n|\,\|_{L^2_{\mathbb{R}}(I_i)} + \frac{(\rho+1)^2}{4}\|\dot{a}+|h_n|\,\|^2_{L^2_{\mathbb{R}}(I_i)} + \varphi(t_i^n,x_n(t_i^n)) - \varphi(t_{i+1}^n,x_n(t_{i+1}^n))\Big]^{1/2}.
$$

Observing that

$$
\begin{aligned} \sqrt{t_{i+1}^n - t_i^n}\,k(0)\|\dot{a}+|h_n|\,\|_{L^2_{\mathbb{R}}(I_i)} &= 2\sqrt{t_{i+1}^n - t_i^n}\Big(\frac{k(0)}{2}\|\dot{a}+|h_n|\,\|_{L^2_{\mathbb{R}}(I_i)}\Big) \\ &\le (t_{i+1}^n - t_i^n) + \frac{k^2(0)}{4}\|\dot{a}+|h_n|\,\|^2_{L^2_{\mathbb{R}}(I_i)}, \end{aligned}
$$

we then obtain

$$\|\dot{x}_n\|_{L^2_H(I_i)} \le \frac{\rho+1}{2}\|\dot{a}+|h_n|\,\|_{L^2_{\mathbb{R}}(I_i)} + \|h_n\|_{L^2_H(I_i)} + \bigg[(t^n_{i+1}-t^n_i) + \frac{k^2(0)+(\rho+1)^2}{4}\|\dot{a}+|h_n|\,\|^2_{L^2_{\mathbb{R}}(I_i)} + \varphi(t^n_i,x_n(t^n_i)) - \varphi(t^n_{i+1},x_n(t^n_{i+1}))\bigg]^{1/2},$$

and hence

$$\|\dot{x}_n\|^2_{L^2_H(I_i)} \le 2\bigg[\frac{\rho+1}{2}\|\dot{a}+|h_n|\,\|_{L^2_{\mathbb{R}}(I_i)} + \|h_n\|_{L^2_H(I_i)}\bigg]^2 + 2\bigg[(t^n_{i+1}-t^n_i) + \frac{k^2(0)+(\rho+1)^2}{4}\|\dot{a}+|h_n|\,\|^2_{L^2_{\mathbb{R}}(I_i)} + \varphi(t^n_i,x_n(t^n_i)) - \varphi(t^n_{i+1},x_n(t^n_{i+1}))\bigg].$$

We may also write

$$\|\dot{x}_n\|^2_{L^2_H(I_i)} \le (\rho+1)^2\|\dot{a}+|h_n|\,\|^2_{L^2_{\mathbb{R}}(I_i)} + 4\|h_n\|^2_{L^2_H(I_i)} + 2\big[(t^n_{i+1}-t^n_i) + \varphi(t^n_i,x_n(t^n_i)) - \varphi(t^n_{i+1},x_n(t^n_{i+1}))\big] + \frac{k^2(0)+(\rho+1)^2}{2}\|\dot{a}+|h_n|\,\|^2_{L^2_{\mathbb{R}}(I_i)}.$$

Setting

$$b_0 = \tfrac{1}{2}(k^2(0)+3(\rho+1)^2),$$
$$c_i = 2\big[(t^n_{i+1}-t^n_i) + \varphi(t^n_i,x_n(t^n_i)) - \varphi(t^n_{i+1},x_n(t^n_{i+1}))\big],$$

one has

$$\|\dot{x}_n\|^2_{L^2_H(I_i)} \le b_0\|\dot{a}+|h_n|\,\|^2_{L^2_{\mathbb{R}}} + 4\|h_n\|^2_{L^2_H} + c_i. \tag{4.5}$$

As $\|\dot{a}+|h_n|\,\|^2_{L^2_{\mathbb{R}}} \le 2\|\dot{a}\|^2_{L^2_{\mathbb{R}}} + 2\|h_n\|^2_{L^2_H}$, defining $\sigma = 2(b_0+2)$ we get

$$\|\dot{x}_n\|^2_{L^2_H(I_i)} \le 2b_0\|\dot{a}\|^2_{L^2_{\mathbb{R}}(I_i)} + \sigma\|h_n\|^2_{L^2_H(I_i)} + c_i.$$

Equivalently,

$$\int_{t^n_i}^{t^n_{i+1}} \|\dot{x}_n(t)\|^2\,dt \le 2b_0\int_{t^n_i}^{t^n_{i+1}} \dot{a}^2(t)\,dt + \sigma\int_{t^n_i}^{t^n_{i+1}} \|h_n(t)\|^2\,dt + c_i. \tag{4.6}$$

Since, by assumption, $\|f(t,x)\| \le \beta(t)(1+\|x\|)$ for a.e. $t \in I$ and for all $x \in H$, it follows that, for any $i \in \{0,\dots,n-1\}$,

$$\begin{aligned}\int_{t^n_i}^{t^n_{i+1}} \|\dot{x}_n(t)\|^2\,dt &\le 2b_0\int_{t^n_i}^{t^n_{i+1}} \dot{a}^2(t)\,dt + \sigma(1+\|x_n(t^n_i)\|)^2\int_{t^n_i}^{t^n_{i+1}} \beta^2(t)\,dt + c_i \\ &\le 2b_0\int_{t^n_i}^{t^n_{i+1}} \dot{a}^2(t)\,dt + \sigma\Big(1+\max_{t\in[t^n_i,t^n_{i+1}]}\|x_n(t)\|\Big)^2\int_{t^n_i}^{t^n_{i+1}} \beta^2(t)\,dt + c_i.\end{aligned}$$

By summing these inequalities we get

$$\sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \|\dot{x}_n(t)\|^2\,dt \le 2b_0 \int_{T_0}^{T} \dot{a}^2(t)\,dt + \sigma\Big(1+\sup_{t\in I}\|x_n(t)\|\Big)^2 \int_{T_0}^{T} \beta^2(t)\,dt + d_n,$$

where

$$d_n := \sum_{i=0}^{n-1} c_i = 2\big[T - T_0 + \varphi(T_0, x_0) - \varphi(T, x_n(T))\big]. \tag{4.7}$$

As $-\varphi(T, x_n(T)) \le 0$ because φ is non-negative, setting $d := 2(T - T_0 + \varphi(T_0, x_0))$, we may write

$$\int_{T_0}^{T} \|\dot{x}_n(t)\|^2\,dt \le 2b_0 \int_{T_0}^{T} \dot{a}^2(t)\,dt + 2\sigma(1+\|x_n(\cdot)\|_\infty^2) \int_{T_0}^{T} \beta^2(t)\,dt + d,$$

and hence

$$\int_{T_0}^{T} \|\dot{x}_n(t)\|^2\,dt \le b + c\|x_n(\cdot)\|_\infty^2, \tag{4.8}$$

where

$$b = 2b_0 \int_{T_0}^{T} \dot{a}^2(t)\,dt + 2\sigma \int_{T_0}^{T} \beta^2(t)\,dt + d \quad \text{and} \quad c = 2\sigma \int_{T_0}^{T} \beta^2(t)\,dt.$$

Using the Cauchy–Schwarz inequality and (4.8), for all $s \in I$ we obtain

$$\|x_n(s) - x_0\|^2 \le (s - T_0) \int_{T_0}^{s} \|\dot{x}_n(t)\|^2\,dt \le (T - T_0)(b + c\|x_n(\cdot)\|_\infty^2),$$

and hence

$$\begin{aligned}\|x_n(s)\|^2 &\le 2\|x_0\|^2 + 2\|x_n(s) - x_0\|^2 \\ &\le 2\|x_0\|^2 + 2(T - T_0)(b + c\|x_n(\cdot)\|_\infty^2).\end{aligned}$$

Consequently, for each n, we get

$$(1 - 2(T - T_0)c)\|x_n(\cdot)\|_\infty^2 \le 2(\|x_0\|^2 + (T - T_0)b).$$

According to (4.3), that is, $2(T - T_0)c < 1$, one has, for any t and any integer n,

$$\|x_n(\cdot)\|_\infty \le M, \tag{4.9}$$

$$\|h_n(t)\| = \|f(t, x_n(\theta_n(t)))\| \le \beta(t)(1 + M), \tag{4.10}$$

where

$$M = \left[\frac{2(\|x_0\|^2 + (T - T_0)b)}{1 - 2(T - T_0)c}\right]^{1/2}.$$

From (4.8) and (4.9) one has

$$\sup_{n\in\mathbb{N}} \int_{T_0}^{T} \|\dot{x}_n(t)\|^2\,dt \le b + cM^2. \tag{4.11}$$

It follows from (4.6) that

$$\sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \|\dot{x}_n(t)\|^2\,dt \le 2b_0 \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \dot{a}^2(t)\,dt + \sigma \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \|h_n(t)\|^2\,dt + \sum_{i=0}^{n-1} c_i.$$

Thus, according to (4.7), for all n we have

$$\int_{T_0}^{T} \|\dot{x}_n(t)\|^2\,dt \le 2b_0 \int_{T_0}^{T} \dot{a}^2(t)\,dt + \sigma \int_{T_0}^{T} \|f(t, x_n(\theta_n(t)))\|^2\,dt + d_n. \tag{4.12}$$

(B) *Convergence of the sequence* $(x_n(\cdot))$. In view of (4.11),

$$\mathcal{S} = \sup_{n\in\mathbb{N}} \|\dot{x}_n\|_{L^2_H([T_0,T])} < \infty,$$

that is, $(\dot{x}_n)_n$ is bounded in $L^2_H([T_0,T])$. Therefore

$$\begin{aligned} \|x_n(t) - x_n(s)\| &= \Big\|\int_s^t \dot{x}_n(\tau)\,d\tau\Big\| \\ &\le (t-s)^{1/2}\Big(\int_{T_0}^{T} \|\dot{x}_n(\tau)\|^2\,d\tau\Big)^{1/2} \le (t-s)^{1/2}\mathcal{S}, \end{aligned} \tag{4.13}$$

so along with (4.9), the set $\{(x_n(\cdot))_n\}$ is bounded and equicontinuous in $\mathcal{C}_H(I)$. Thanks to the inequality (3.6), for any fixed $t \in [T_0,T]$ and any n, one has

$$\begin{aligned} |\varphi(t, x_n(t)) - \varphi(T_0, x_0)| \le \sup_{n\in\mathbb{N}} \int_{T_0}^{t} \big[k(0) + (\rho+1)\|\dot{x}_n + h_n\|\big]\,[\dot{a} + |h_n|] \\ + \sup_{n\in\mathbb{N}} \int_{T_0}^{t} \|\dot{x}_n + h_n\|^2 < \infty. \end{aligned}$$

Since φ_t is inf-ball-compact by assumption, the set $\{x_n(t) : n \in \mathbb{N}\}$ is relatively compact in H. By Ascoli's theorem, we can extract a subsequence of $(x_n(\cdot))$ that converges uniformly on I to some map $x(\cdot) \in \mathcal{C}_H(I)$.

Recall that (by construction) $0 \le t - \theta_n(t) \le (T - T_0)n^{-1}$ for any $t \in [T_0,T]$ and any $n \in \mathbb{N}$. Consequently, for any $T_0 \le t \le T$, one has $\theta_n(t) \to t$. Then for $T_0 \le s \le t \le T$, letting $n \to \infty$ in (4.13), one easily deduces that $x(\cdot)$ is actually absolutely continuous on $[T_0,T]$. Hence, $\dot{x}(\cdot)$ exists for a.e. $t \in [T_0,T]$, and $\dot{x}(\cdot) \in L^1_H(I)$ with $x(t) = x(s) + \int_s^t \dot{x}(\tau)\,d\tau$ for any

$T_0 \le s \le t \le T$. Further, observing that

$$\begin{aligned}\|x_n(\theta_n(t)) - x(t)\| &\le \|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\| \\ &\le (t-\theta_n(t))^{1/2}\mathcal{S} + \|x_n(t) - x(t)\|\end{aligned}$$

and so

$$\|x_n(\theta_n(t)) - x(t)\| \le ((T-T_0)/n)^{1/2}\mathcal{S} + \|x_n(t) - x(t)\|,$$

we conclude that $\|x_n(\theta_n(t)) - x(t)\| \to 0$ as $n \to \infty$ for any $t \in I$.

Thus, the continuity of $f(\cdot,\cdot)$ with respect to its second variable entails that, for a.e. $t \in [T_0, T]$,

$$\lim_{n\to\infty} \|f(t, x_n(\theta_n(t))) - f(t, x(t))\| = 0,$$

along with

$$\lim_{n\to\infty} \int_{T_0}^{T} \|f(t, x_n(\theta_n(t))) - f(t, x(t))\|^2\, dt = 0, \tag{4.14}$$

by Lebesgue's convergence theorem. Then, of course,

$$\lim_{n\to\infty} \int_{T_0}^{T} \|f(t, x_n(\theta_n(t)))\|^2\, dt = \int_{T_0}^{T} \|f(t, x(t))\|^2\, dt. \tag{4.15}$$

Furthermore, in view of (4.11), up to a subsequence that we do not relabel, $(\dot{x}_n)_n$ converges weakly in $L^2_H([T_0, T])$ to some element z. Since weak convergence in L^2 implies weak convergence in L^1, we conclude that $z(t) = x(t)$ for a.e. t, and

$$\dot{x}_n \to \dot{x} \quad \text{weakly in } L^2_H([T_0, T]). \tag{4.16}$$

Taking the upper limit in (4.12) as $n \to \infty$, and using (4.15) and (4.16) we obtain

$$\int_{T_0}^{T} \|\dot{x}(t)\|^2\, dt \le 2b_0 \int_{T_0}^{T} \dot{a}^2(t)\, dt + \sigma \int_{T_0}^{T} \|f(t, x(t))\|^2\, dt + \limsup_n d_n.$$

Since $x_n(t) \to x(t)$, by (4.7) and the lower semicontinuity of $\varphi(t,\cdot)$, we have

$$\begin{aligned}\limsup_n\, d_n &= 2\Big[T - T_0 + \varphi(T_0, x_0) - \liminf_n \varphi(T, x_n(T))\Big] \\ &\le 2[T - T_0 + \varphi(T_0, x_0) - \varphi(T, x(T))].\end{aligned}$$

Hence,

$$\int_{T_0}^{T} \|\dot{x}(t)\|^2\,dt \le \alpha + \sigma \int_{T_0}^{T} \|f(t,x(t))\|^2\,dt, \tag{4.17}$$

where

$$\alpha = (k^2(0) + 3(\rho+1)^2) \int_{T_0}^{T} \dot{a}^2(t)\,dt + 2\big[T - T_0 + \varphi(T_0,x_0) - \varphi(T,x(T))\big].$$

(C) *We prove that* $x(\cdot)$ *is a solution of* (4.2). Recall that, for each $n \in \mathbb{N}$,

$$\begin{cases} -\dot{x}_n(t) \in \partial\varphi(t,x_n(t)) + f(t,x_n(\theta_n(t))) & \text{a.e. } t \in [T_0,T],\\ x_n(T_0) = x_0. \end{cases}$$

To see that

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + f(t,x(t)) \quad \text{a.e. } t \in [T_0,T],$$

it is enough to follow the corresponding arguments in [9, proof of Theorem 1].

Now, we address the general case when $\int_{T_0}^{T} \beta^2(s)\,ds \ge m$. We fix some $\delta > 0$ such that for each subinterval J of I with length$(J) < \delta$ one has $\int_J \beta^2(s)\,ds < m$, and we also fix some integer N such that $(T-T_0)/N < \delta$. Set $T_i := T_0 + \frac{i}{N}(T-T_0)$ for $i = 0,\ldots,N$, and observe that for each $i = 1,\ldots,N$ we have

$$\int_{T_{i-1}}^{T_i} \beta^2(s)\,ds < m < \frac{1}{4(T_i - T_{i-1})(k^2(0) + 3(\rho+1)^2 + 4)},$$

and hence the condition (4.3) is fulfilled in each interval $[T_{i-1},T_i]$. Consequently, we may apply what precedes to the intervals $[T_0,T_1]$, $[T_1,T_2]$, $\ldots$, and $[T_{N-1},T]$, and we obtain absolutely continuous solutions $y_1(\cdot)$ on $[T_0,T_1]$ with $y_1(T_0) = x_0$, $y_2(\cdot)$ on $[T_1,T_2]$ with $y_2(T_1) = y_1(T_1), \ldots, y_N(\cdot)$ on $[T_{N-1},T]$ with $y_N(T_{N-1}) = y_{N-1}(T_{N-1})$. So the mapping $x(\cdot)$ from $I = [T_0,T]$ into H defined by $x(t) = y_i(t)$ for all $t \in [T_{i-1},T_i]$, $i = 1,\ldots,N$, is obviously an absolutely continuous solution on I of (4.2). ∎

As in [13], we have the following property:

PROPOSITION 4.2. *Any absolutely continuous solution* $x(\cdot)$ *of* (4.2) *satisfies*

$$\int_{T_0}^{T} \|\dot{x}(t)\|^2\,dt \le \alpha + \sigma \int_{T_0}^{T} \|f(t,x(t))\|^2\,dt, \tag{4.18}$$

where α *and* σ *are the same constants as defined before. Further,*

$$\|f(t,x(t))\| \le \beta(t)(1+K) \quad \textit{a.e. } t \in I,$$

and

$$\|x(\cdot)\|_\infty \le K \quad \textit{and} \quad \int_{T_0}^{T} \|\dot{x}(t)\|^2\,dt \le \alpha + \sigma(1+K)^2 \int_{T_0}^{T} \beta^2(t)\,dt$$

with $K = \|x_0\| + [\xi(T)]^{1/2}$, *where* $\xi(\cdot)$ *is the increasing, continuous, and non-negative function defined on* $[T_0, T]$ *by*

$$\xi(s) = (s - T_0)\Big[\alpha + 2\sigma(1+\|x_0\|)^2 \int_{T_0}^{s} \beta^2(\tau)\,d\tau\Big]$$
$$+ 2\sigma(s-T_0) \int_{T_0}^{s} b(\tau)\beta^2(\tau)\exp\Big(2\sigma \int_{\tau}^{s} \theta\beta^2(\theta)\,d\theta\Big)\,d\tau.$$

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