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# On strong chain recurrence for maps

by Katsuya Yokoi (Tokyo)

*Dedicated to Professor Yasunao Hattori on his 60th birthday*

**Abstract.** This paper is concerned with strong chain recurrence introduced by Easton. We investigate the depth of the transfinite sequence of nested, closed invariant sets obtained by iterating the process of taking strong chain recurrent points, which is a related form of the central sequence due to Birkhoff. We also note the existence of a Lyapunov function which is decreasing off the strong chain recurrent set. As an application, we give a necessary and sufficient condition for the coincidence of the strong chain recurrence set and the chain recurrence set. Several examples are given to illustrate the difference between the concepts of strong chain recurrence and chain recurrence.

**1. Introduction.** We study the concept of strong chain recurrence introduced by R. Easton.

Chains and chain recurrent points have been introduced and studied by C. Conley [4] (see §2 for definition). They play an important role in the theory of attractors and in several other aspects of the topological dynamics of a map $f$ defined on a space $X$. The key theorem is Conley's Decomposition Theorem which says that $X$ decomposes into the chain recurrent set and the rest, where the action is *gradient-like*. Moreover, the chain recurrent set is the intersection of $A \cup A^*$ over all attracting-repelling pairs $(A, A^*)$ (see [4]). Note that the chain recurrent set contains all nonwandering points including the recurrent points, minimal subsets and periodic orbits.

Easton [5] strengthened the notion of an $\varepsilon$-chain to that of a strong $\varepsilon$-chain by replacing the error estimate with $\sum_{i=1}^{n} d(f(x_{i-1}), x_i) < \varepsilon$. He obtained a relation between strong chain transitivity and Lipschitz ergodicity (namely, any Lipschitz function which is constant along orbits is globally constant). He also gave an example of an Anosov homeomorphism of the torus which was strong chain transitive on all of the space. Ghane and Fakhari [8] showed that an isolated strong chain class $S$ of a generic homeomorphism has a generic

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continuation in $C^0$-topology; and they deduced the persistency of Lipschitz ergodic behavior at $S$. They (with Sarizadeh) [6] exhibited general properties of the strong chain recurrent set, and studied strong chain transitivity for a map having a shadowing property. Some properties of strong chain transitive components are discussed by Hu [9]. We also note the paper of Zheng [14] in which the relationships between several variations of chain transitivity and ergodicity are considered.

Motivated by the results above, we continue the study of strong chain recurrence. In Sections 3 and 4 the differences between the concepts of strong chain recurrence and chain recurrence are emphasized. We give some standard facts and examples concerning strong chain recurrence. One example shows that strong chain recurrence is a metric property. We next turn our attention to the depth of the transfinite sequence of nested, closed invariant sets obtained by iterating the process of taking strong chain recurrent points, which is a related form of the central sequence due to Birkhoff [1]. On the other hand, in the last section it is shown that strong chain recurrence behaves very much like chain recurrence. The concept of Lyapunov functions with respect to strong chain recurrence is studied.

**2. Preliminaries and definitions.** We now give the terminology and notation needed in what follows. A *map on $X$* is a continuous function $f : X \to X$ from a space $X$ to itself; $f^0$ is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$.

We let $f : X \to X$ be a map from a compact metric space $(X, d)$ to itself. Let $x, y \in X$. A (strong) *$\varepsilon$-chain* from $x$ to $y$ is a finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ of $X$ such that $x_0 = x$, $x_n = y$ and $d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, \ldots, n$ ($\sum_{i=1}^{n} d(f(x_{i-1}), x_i) < \varepsilon$, respectively). We say *$x$ can be* (*strongly*) *chained to $y$* if for every $\varepsilon > 0$ there exists a (strong) $\varepsilon$-chain from $x$ to $y$, and we say $x$ is (*strong*) *chain recurrent* if it can be (strongly, respectively) chained to itself. The set of all (strong) chain recurrent points is called the (*strong*) *chain recurrent set* of $f$ and denoted by $\mathrm{CR}(f)$ ($\mathrm{SCR}(f)$, respectively). The (strong) chain recurrent set is closed in $X$ and $f$-invariant, and the set $\mathrm{CR}(f)$ depends only on the topology (this statement is not true of $\mathrm{SCR}(f)$, see Example 3.1).

An invariant set $I$ is said to be (*strong*) *chain transitive* if for any $x, y \in I$, $x$ can be (strongly, respectively) chained to $y$ in $I$. We define a relation $\sim$ on $\mathrm{SCR}(f)$ by $x \sim y$ if for every $\varepsilon > 0$ there exists a strong $\varepsilon$-chain from $x$ to $y$ in $\mathrm{SCR}(f)$ and another from $y$ to $x$. Then $\sim$ is an equivalence relation. The equivalence classes of $\sim$ in $\mathrm{SCR}(f)$ are called the *strong chain transitive components* of $\mathrm{SCR}(f)$. A map $f : (X, d_X) \to (Y, d_Y)$ is called *Lipschitz* if there exists a real constant $k \geq 0$ such that $d_Y(f(x_1), f(x_2)) \leq k d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. Such a $k$ is referred to as a *Lipschitz constant* for $f$.

We need the following lemma due to Block and Franke which gives a characteristic property of chain recurrent points.

LEMMA 2.1 ([3, Theorem A], [2]). *Let $f$ be a map on a metrizable compact space $X$ and $x \in X$. Then $x \notin \mathrm{CR}(f)$ if and only if there exists an open subset $U$ of $X$ such that $x \notin \mathrm{Cl}\, U$, $f(x) \in U$ and $f(\mathrm{Cl}\, U) \subseteq U$.*

Throughout this paper, $\mathbb{R}^n$ is *$n$-dimensional Euclidean space* with the standard metric $d$.

**3. Elementary properties of the strong chain recurrent set and examples.** It is worth pointing out that strong chain recurrence is a metric property.

EXAMPLE 3.1. The strong chain recurrent set may depend on the metric even if two metrics induce the same topology.

*Construction.* The compact metric space $X$ is defined by

$$X = \{(x,y) \mid (x-1/2)^2 + y^2 = (1/2)^2,\ y < 0\} \cup \bigcup_{p=0}^{\infty} I_p \subseteq \mathbb{R}^2,$$

where $I_0 = \{(x,0) \mid 0 \le x \le 1\}$ and $I_p = \{(q/2^p, 1/2^{p-1}) \mid q = 0,1,\ldots,2^p\}$ for $p \in \mathbb{N}$ (see Figure 1). Define a map $f$ on $X$ with the fixed point set $F(f) = I_0$ by $f(q/2^p, 1/2^{p-1}) = ((q-1)/2^p, 0)$ for $q = 1,2,\ldots,2^p$; $f(0, 1/2^{p-1}) = (0,0)$; and by a homeomorphism on the semicircle such that the first coordinate of $f(x,y)$ is greater than $x$ ($\neq 0,1$).

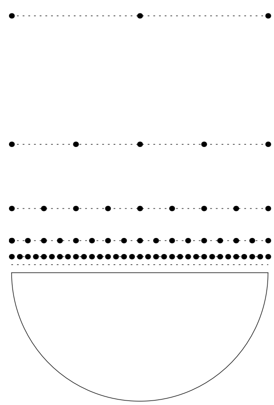

Fig. 1

We show that for small $\varepsilon > 0$ there exists no strong $\varepsilon$-chain from $(1,0)$ to $(0,0)$. Let $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be any chain from $\boldsymbol{x}_0 = (1,0)$ to $\boldsymbol{x}_n = (0,0)$; we may assume that the second coordinate of each $\boldsymbol{x}_i$ is nonnegative. Then

$$\begin{aligned}
&\{d(f(\boldsymbol{x}_0),\boldsymbol{x}_1)+d(f(\boldsymbol{x}_1),\boldsymbol{x}_2)+\cdots+d(f(\boldsymbol{x}_{n-1}),\boldsymbol{x}_n)\}\\
&\qquad\qquad +\{d_1((\boldsymbol{x}_1)_1,f(\boldsymbol{x}_1)_1)+\cdots+d_1((\boldsymbol{x}_{n-1})_1,f(\boldsymbol{x}_{n-1})_1)\}\\
&\ge \{d_1(f(\boldsymbol{x}_0)_1,(\boldsymbol{x}_1)_1)+d_1(f(\boldsymbol{x}_1)_1,(\boldsymbol{x}_2)_1)+\cdots+d_1(f(\boldsymbol{x}_{n-1})_1,(\boldsymbol{x}_n)_1)\}\\
&\quad +\{d_1((\boldsymbol{x}_1)_1,f(\boldsymbol{x}_1)_1)+\cdots+d_1((\boldsymbol{x}_{n-1})_1,f(\boldsymbol{x}_{n-1})_1)\}\\
&\ge d_1(f(\boldsymbol{x}_0)_1,(\boldsymbol{x}_n)_1)=1,
\end{aligned}$$

where $d_1$ is the 1-dimensional Euclidean metric, and $(\boldsymbol{x}_i)_1$, $f(\boldsymbol{x}_i)_1$ mean the first coordinate of $\boldsymbol{x}_i$, $f(\boldsymbol{x}_i)$, respectively. Using the inequalities

$$d_1((\boldsymbol{x}_i)_1,f(\boldsymbol{x}_i)_1)\le \tfrac{1}{2}(\text{the second coordinate of } \boldsymbol{x}_i)\le \tfrac{1}{2}d(f(\boldsymbol{x}_{i-1}),\boldsymbol{x}_i)$$

for $1\le i\le n-1$, it follows that

$$\sum_{i=1}^{n} d(f(\boldsymbol{x}_{i-1}),\boldsymbol{x}_i)\ge 2/3.$$

Hence $\mathrm{SCR}_d(f)=I_0$, and note that $\mathrm{CR}(f)$ is the circle part $S=I_0\cup\{(x,y)\mid (x-1/2)^2+y^2=(1/2)^2,\ y<0\}$ of $X$, because $(1,0)$ can be chained to $(0,0)$.

On the other hand, consider the new metric $d^*$ on $X$ defined by

$$d^*((z_1,z_2),(z_1',z_2'))=d((z_1,\xi(z_2)),(z_1',\xi(z_2'))),$$

where $\xi(t)=t^2$ for $0\le t\le 1$ and $\xi(t)=t$ for $-1/2\le t\le 0$. This metric induces the same topology as the original one. The sum

$$\sum_{q=1}^{2^p} d^*((q/2^p,0),(q/2^p,1/2^{p-1}))=1/2^{p-2}$$

shows that $(1,0)$ can be strongly chained to $(0,0)$ in $(X,d^*)$; hence, $\mathrm{SCR}_{d^*}(f)$ $=\mathrm{CR}(f)$ is the circle part $S$ of $X$. ∎

Remark. Using the circle part $S$ of $(X,d)$ and $f$ of Example 3.1, we also have a simple map whose strong chain recurrent set does not coincide with the chain recurrent set.

The following proposition will imply that the strong chain recurrent set of a map is unique if two metrics on a set are Lipschitz equivalent.

Proposition 3.2. *Let $f:(X,d_X)\to(X,d_X)$ and $g:(Y,d_Y)\to(Y,d_Y)$ be maps which are semi-conjugate by a Lipschitz map $\xi:(X,d_X)\to(Y,d_Y)$. Then $\xi(\mathrm{SCR}(f))\subseteq\mathrm{SCR}(g)$.*

*Proof.* Let $k$ be a Lipschitz constant of $\xi$, and let $x\in X$ be a strong chain recurrent point for $f$. For every $\varepsilon>0$, we take a strong chain $x_0=$

$x, x_1, \ldots, x_n = x$ with $\sum_{i=1}^n d_X(f(x_{i-1}), x_i) < \varepsilon/k$. Then

$$\begin{aligned}\sum_{i=1}^n d_Y(g(\xi(x_{i-1})), \xi(x_i)) &= \sum_{i=1}^n d_Y(\xi(f(x_{i-1})), \xi(x_i)) \\ &\le k \sum_{i=1}^n d_X(f(x_{i-1}), x_i) < \varepsilon;\end{aligned}$$

that is, $\xi(x_0), \xi(x_1), \ldots, \xi(x_n)$ is a strong $\varepsilon$-chain for $g$ from $\xi(x)$ to itself. This implies $\xi(x) \in \mathrm{SCR}(g)$. ■

Fakhari et al. [6] showed that if a homeomorphism $f : X \to X$ is *recurrent* (i.e., each point $x$ of $X$ is an accumulation point of the (positive) orbit of $x$), then $\mathrm{SCR}(f) = \mathrm{SCR}(f^n)$ for $n \ge 2$. The Lipschitz property is also a sufficient condition for the coincidence.

PROPOSITION 3.3. *Let $f : (X, d) \to (X, d)$ be a Lipschitz map. Then* $\mathrm{SCR}(f) = \mathrm{SCR}(f^n)$ *for $n \ge 2$.*

*Proof.* Let $k$ be a Lipschitz constant of $f$ with $d(f^i(x), f^i(y)) \le kd(x, y)$ for $i = 1, \ldots, n$.

Let $x \in \mathrm{SCR}(f)$. Given $\varepsilon > 0$, take, by concatenating a strong $\varepsilon/(kn)$-chain with itself $n$ times if necessary, a strong chain $x_0 = x, x_1, \ldots, x_{n\ell} = x$ with $\sum_{i=1}^{n\ell} d(f(x_{i-1}), x_i) < \varepsilon/k$. Then it is easily seen that

$$\sum_{j=1}^{\ell} d(f^n(x_{n(j-1)}), x_{nj}) \le \begin{cases} k \sum_{i=1}^{n\ell} d(f(x_{i-1}), x_i) < \varepsilon & \text{if } 1 \le k, \\ \sum_{i=1}^{n\ell} d(f(x_{i-1}), x_i) < \varepsilon/k & \text{if } 0 < k < 1. \end{cases}$$

It follows that $x \in \mathrm{SCR}(f^n)$. The reverse inclusion is clear. ■

EXAMPLE 3.4. For $n \ge 2$, there exists a map $f : X \to X$ such that $\mathrm{SCR}(f^n) \subsetneq \mathrm{SCR}(f)$.

*Construction.* By modifying the construction of Example 3.1 slightly, we consider it only for the case $n = 2$; the other cases are left to the reader. The compact metric space $X$ is drawn in Figure 2, and is defined analytically as follows.

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$; for $p = 0, 1, 2, \ldots$ and $q = 0, 1, \ldots, 2^p$, and let

$$a_{p,q} = \left(1 + \frac{1}{2^{2p}}\right) \exp\left[i\pi \frac{q}{2^{p+1}}\right], \qquad X = S^1 \cup \bigcup_{p=0}^{\infty} \{a_{p,q} \mid q = 0, 1, \ldots, 2^p\}.$$

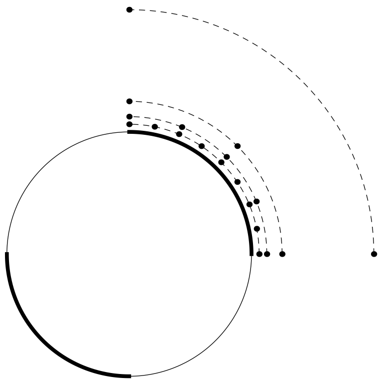

Fig. 2

Define the map $g$ on $X$ by

$$g(z) = \begin{cases} z & \text{if } z = \exp[i\theta],\ m\pi \le \theta \le (2m+1)\pi/2,\ m = 0, 1, \\ \exp[i\xi(\theta)] & \text{if } z = \exp[i\theta],\ (2m+1)\pi/2 \le \theta \le (m+1)\pi,\ m = 0, 1, \\ \exp[i\pi(q+1)/2^{p+1}] & \text{if } z = a_{p,q},\ q = 0, 1, \dots, 2^p - 1, \\ \exp[i\pi/2] & \text{if } z = a_{p,2^p}, \end{cases}$$

where $\xi(\theta) = \theta + \frac{\pi}{8}\sin(2\theta - \pi)$. Therefore, for $q = 0, 1, \dots, 2^p - 1$, $g(a_{p,q})$ is obtained from $a_{p,q}$ by rotating it about the origin through angle $\pi/2^{p+1}$, and then changing its length to the radius 1 of $S^1$; for $q = 2^p$, we simply change its length to 1. Then $g|_{S^1}$ is a homeomorphism which fixes $z$ or satisfies $\arg g(z) > \arg z$ for $z \in S^1$. The map $h$ (on $S^1$) is defined by rotation about the origin through angle $\pi$, that is,

$$h(\exp[i\theta]) = \exp[i(\theta + \pi)] \quad \text{for } 0 \le \theta \le 2\pi.$$

Consider the composite map $f = h \circ g$. Using

$$d(f^2(e^{i0}), a_{p,0}) + d(f^2(a_{p,0}), a_{p,1}) + \cdots + d(f^2(a_{p,2^p-2}), a_{p,2^p-1}) \\ + d(f^2(a_{p,2^p-1}), e^{i\pi/2}) = 1/2^{2p} \times 2^p = 1/2^p,$$

we find that $e^{i0}$ can be strongly chained to $e^{i\pi/2}$ by $f^2$ (hence by $f$); thus $e^{i\pi}$ can be strongly chained to $e^{i3\pi/2}$ by $f$. On the other hand, $e^{i\pi}$ cannot be strongly chained to $e^{i3\pi/2}$ by $f^2$. Hence $\mathrm{SCR}(f) = S^1$ and $\mathrm{SCR}(f^2) = \{\exp[i\theta] \mid 0 \le \theta \le \pi/2 \text{ or } \pi \le \theta \le 3\pi/2\}$. ∎

The following fact is needed later.

Proposition 3.5. *Let $f : X \to X$ be a map on a compact metric space. If $x \in \mathrm{SCR}(f)$, then $f^k(x)$ can be strongly chained to $x$ for every $k \in \mathbb{N}$.*

*Proof.* Uniform continuity yields this statement. The details are left to the reader. ∎

**4. The restriction property and the depth of the transfinite sequence.** The depth of the centre of a map has been widely studied. Here we extend this concept to the strong chain recurrent case. Before stating the definition and results, we note, by using the map $f$ on the compact metric space $(X, d^*)$ of Example 3.1, that:

EXAMPLE 4.1. The strong chain recurrent set does not generally satisfy the restriction property $\mathrm{SCR}(f|_{\mathrm{SCR}(f)}) = \mathrm{SCR}(f)$.

We introduce the notion of the $*$-depth of the transfinite sequence of nested, closed $f$-invariant sets obtained by iterating the process of taking strong chain recurrent points, which is a related form of the central sequence due to Birkhoff [1]. We consider a map $f$ on a compact metric space $X$. Let $\mathrm{SCR}_0(f) = X$ and $\mathrm{SCR}_1(f) = \mathrm{SCR}(f)$. For any ordinal $\lambda \geq 1$, we define $\mathrm{SCR}_\lambda(f)$ as follows: If $\lambda = \alpha + 1$, then we set $\mathrm{SCR}_\lambda(f) = \mathrm{SCR}(f|_{\mathrm{SCR}_\alpha(f)})$. If $\lambda$ is a limit ordinal, we set $\mathrm{SCR}_\lambda(f) = \bigcap_{\alpha<\lambda} \mathrm{SCR}_\alpha(f)$. We note that there exists a countable ordinal $\beta$ such that $\mathrm{SCR}_\beta(f) = \mathrm{SCR}_{\beta+1}(f)$, since $X$ has a countable open base. The minimal such $\beta$ is called the *$*$-depth* of the transfinite sequence generated by $f$, and is denoted by $D(f)$.

THEOREM 4.2. *For any countable ordinal $\alpha$, there exists a map $f : X \to X$ with $D(f) = \alpha$.*

REMARK. Kato [11] showed the existence of a homeomorphism $f_\alpha$ on a compactum $Z_\alpha$ for which the depth of the centre is $\alpha$ for any $\alpha < \omega_1$. Since the compactum is countable, it follows that the $*$-depth of $f_\alpha$ is at most 1 (see Corollary 5.6 below).

REMARK. Xiong [13] proved that if $f$ is a self-map of the closed interval, then the depth of the centre of $f$ is at most 2. Mai and Sun [12] showed that this is also true for maps on graphs. We do not know whether or not it is true for the $*$-depth.

*Proof of Theorem 4.2.* We construct $f$ by a process similar to that used in [11, Proposition 2.1]. In this proof, $L = \mathbb{R}$ means the Euclidean line, and $\xi$ a homeomorphism on $\mathbb{R}$ with $\xi(n) = n + 1$ for $n \in \mathbb{Z}$.

The construction is by transfinite induction on $\alpha$. For $\alpha = 0$, we let

$$K_0 = \{(x, 0) \mid 0 \leq x \leq 4\} \subseteq \mathbb{R}^2$$

and $f_0$ be the identity map on $K_0$; then $D(f_0) = 0$. For $\alpha = 1$, we let

$$K_1 = K_0 \cup L_1 \subseteq \mathbb{R}^2$$

and $f_1$ be a homeomorphism on $K_1$ satisfying

($1_1$) $L_1$ is a topological copy of $L$ with $\lim_{n\to\infty} z_1(n) = \{(0,0)\}$ and $\lim_{n\to-\infty} z_1(n) = \{(2,0)\}$, where $\{z_1(n)\}_{n\in\mathbb{Z}}$ is the bi-sequence in $L_1$ corresponding to $\{n\}_{n\in\mathbb{Z}}$ in $L$;

($2_1$) $f_1|_{K_0} = f_0$, and $f_1|_{L_1}$ is topologically conjugate to $\xi$.

We find that $\mathrm{SCR}_0(f_1) = K_1$ and $\mathrm{SCR}_1(f_1) = \mathrm{SCR}_2(f_1) = K_0$; that is, $D(f_1) = 1$.

Let $\alpha = 2$. We let

$$K_2 = K_1 \cup L_2 \subseteq \mathbb{R}^2$$

and $f_2$ be a homeomorphism on $K_2$ such that

($1_2$) $L_2$ is a topological copy of $L$ with $\lim_{n\to-\infty} z_2(n) = \{(3,0)\}$ and the Hausdorff distance $d_{\mathrm{H}}(\mathrm{Cl}\{z_2(n) \mid n \geq i\}, [0,2]\times\{0\})$ converges to zero as $i \to \infty$, where $\{z_2(n)\}_{n\in\mathbb{Z}}$ is the bi-sequence in $L_2$ corresponding to $\{n\}_{n\in\mathbb{Z}}$ in $L$;

($2_2$) $f_2|_{K_1} = f_1$, and $f_2|_{L_2}$ is topologically conjugate to $\xi$.

We find that $\mathrm{SCR}_0(f_2) = K_2$, $\mathrm{SCR}_1(f_2) = K_1$ and $\mathrm{SCR}_2(f_2) = \mathrm{SCR}_3(f_2) = K_0$; that is, $D(f_2) = 2$.

We continue in this fashion obtaining $K_\alpha$ and $f_\alpha$ for $\alpha < \omega_0$. Suppose that we have constructed $K_\alpha$ and $f_\alpha$ for $\alpha \leq m < \omega_0$.

We take

$$K_{m+1} = K_m \cup L_{m+1} \subseteq \mathbb{R}^2$$

and a homeomorphism $f_{m+1}$ on $K_{m+1}$ (see Figure 3) such that

($1_{m+1}$) $L_{m+1}$ is a topological copy of $L$ with

$$\lim_{n\to-\infty} z_{m+1}(n) = \{(4 - 1/2^{m-1}, 0)\}$$

and

$$d_{\mathrm{H}}\big(\mathrm{Cl}\{z_{m+1}(n) \mid n \geq i\}, [4 - 1/2^{m-3}, 4 - 1/2^{m-2}] \times \{0\}\big) \to 0$$

as $i \to \infty$, where $\{z_{m+1}(n)\}_{n\in\mathbb{Z}}$ is the bi-sequence in $L_{m+1}$ corresponding to $\{n\}_{n\in\mathbb{Z}}$ in $L$;

($2_{m+1}$) $f_{m+1}|_{K_m} = f_m$, and $f_{m+1}|_{L_{m+1}}$ is topologically conjugate to $\xi$.

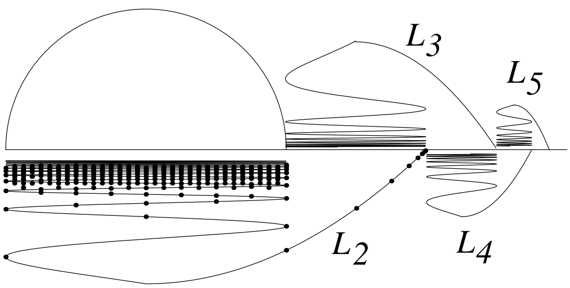


Fig. 3. $K_5$

We find that

$$\mathrm{SCR}_0(f_{m+1}) = K_{m+1}, \dots, \mathrm{SCR}_m(f_{m+1}) = K_1,$$
$$\mathrm{SCR}_{m+1}(f_{m+1}) = \mathrm{SCR}_{m+2}(f_{m+1}) = K_0;$$

that is, $D(f_{m+1}) = m+1$.

Through the construction above, we may represent $K_{m+1}$ as $K_{m+1} = K_1 \cup L_2 \cup \cdots \cup L_{m+1}$; and furthermore, one can take $L_2 \cup \cdots \cup L_{m+1}$ close to $[0, 4 - 1/2^{m-1}] \times \{0\}$ as needed.

Let $\alpha$ be a countable limit ordinal. Take a sequence of ordinals $\alpha_1 < \alpha_2 < \cdots$ converging to $\alpha$. We set $K_\alpha = \bigcup_{i=1}^\infty K_{\alpha_i} \cup \{*\}$ in $\mathbb{R}^3$, which is the one-point compactification of $\bigcup_{i=1}^\infty K_{\alpha_i}$, and define a homeomorphism $f_\alpha$ on $K_\alpha$ by $f_\alpha|_{K_{\alpha_i}} = f_{\alpha_i}$ and $f_\alpha(*) = *$. Then $D(f_\alpha) = \alpha$.

Let $\alpha = \omega_0 + s$ ($s \in \mathbb{N}$). We construct naturally

$$K_{\omega_0+s} = \Big(\bigcup_{i=s+1}^{\infty} K_i\Big) \cup K_s$$

in $\mathbb{R}^3$ and a homeomorphism $f_{\omega_0+s}$ on $K_{\omega_0+s}$ (see Figure 4) such that

(1) $K_s \subseteq \mathbb{R}^2 \times \{0\}$ and $K_i \subseteq \mathbb{R}^2 \times \{1/2^i\}$ for $i \geq s+1$;
(2) $d_{\mathrm{H}}(K_i, K_s) \to 0$ as $i \to \infty$ (note the remark following the construction for the case $m+1$);
(3) $f_{\omega_0+s}|_{K_j} = f_j$ for $j \geq s$.

(We call $K_s$ the *base space* of $K_{\omega_0+s}$.)

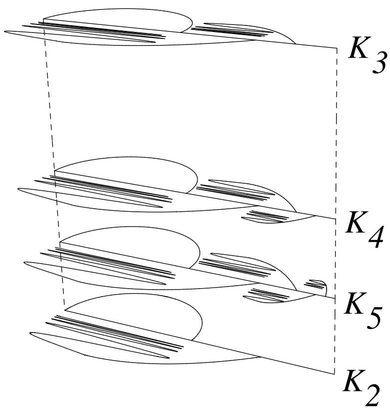


Fig. 4. $K_{\omega_0+2}$

Then we see that $\mathrm{SCR}_{\omega_0}(f_{\omega_0+s}) = (\bigcup_{i=s+1}^\infty K_0^i) \cup K_s$ and $\mathrm{SCR}_{\omega_0+s}(f_{\omega_0+s}) = \mathrm{SCR}_{\omega_0+s+1}(f_{\omega_0+s}) = (\bigcup_{i=s+1}^\infty K_0^i) \cup K_0^s$, where $K_0^i$ is the subset of $K_i$ corresponding to $K_0$ for $i = s, s+1, \dots$; hence, $D(f_{\omega_0+s}) = \omega_0 + s$.

Let $\alpha = \beta + s$ ($\beta$ is a countable limit ordinal with $\omega_0 < \beta$, $s \in \mathbb{N}$). Take a sequence of nonlimit ordinals $\omega_0 < \alpha_1 < \alpha_2 < \cdots$ which converges to $\beta$ satisfying $\alpha_i = \beta_i + m_i$, where $\beta_i$ is a limit ordinal and $\omega_0 > m_i \geq s+1$ for $i = 1, 2, \ldots$. We construct naturally

$$K_\alpha = \Big(\bigcup_{i=1}^{\infty} K_{\alpha_i}\Big) \cup K_s$$

in $\mathbb{R}^3$ and a homeomorphism $f_\alpha$ on $K_\alpha$ (see Figure 5) such that

(1) $K_s \subseteq \mathbb{R}^2 \times \{0\}$ and $(K_{\alpha_i}, K_{m_i}) \subseteq (\mathbb{R}^2 \times [1/2^i, 1/2^{i-1}), \mathbb{R}^2 \times \{1/2^i\})$ for $i = 1, 2, \ldots$, where $K_{m_i}$ is the base space of $K_{\alpha_i}$;
(2) $d_{\mathrm{H}}(K_{\alpha_i}, K_s) \to 0$ as $i \to \infty$;
(3) $f_\alpha|_{K_\gamma} = f_\gamma$ for $\gamma = \alpha_i$ $(i = 1, 2, \ldots)$ or $\gamma = s$.

(We also call $K_s$ the base space of $K_\alpha = K_{\beta+s}$.)

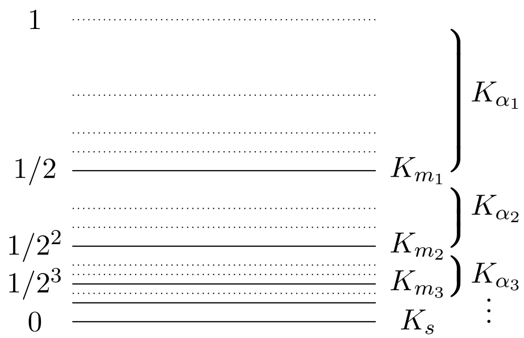


Fig. 5. The height map of $K_{\beta+s}$

The construction shows that $D(f_{\beta+s}) = \beta + s = \alpha$. ∎

**5. Lyapunov functions.** This section examines real-valued functions defined on the phase space of a dynamical system which reflect the dynamical behavior.

Definition 5.1. A *pseudo-complete Lyapunov function* for the dynamical system $f : X \to X$ is a continuous map $\varphi : X \to [0, \infty)$ satisfying the following conditions:

$(L_1)$ $\varphi(f(x)) \leq \varphi(x)$ for all $x \in X$;
$(L_2)$ $\varphi(f(x)) = \varphi(x)$ if and only if $x$ is strong chain recurrent for $f$;
$(L_3)$ $\varphi$ is constant on each strong chain transitive component.

The following theorem is essentially a result of C. Conley. We have changed the setting from "chain recurrence" to "strong chain recurrence".

Franks [7] had a proof of Conley's results for discrete dynamical systems. Hurley [10] also extended those results to noncompact spaces. Since the statement is effective and the key to the proof of Theorem 5.3, we include a proof for completeness.

THEOREM 5.2 (Conley). *For a map $f : X \to X$, there exists a pseudo-complete Lyapunov function for $f$.*

*Proof.* We define a real-valued function $L$ on $X \times X$ by setting

$$L(x, y) = \inf\Big\{\sum_{i=1}^{n} d(f(x_{i-1}), x_i) \;\Big|\; x_0 = x, x_1, \ldots, x_n = y,\ x_i \in X,\ n \in \mathbb{N}\Big\}.$$

This function has been used previously by Zheng [14]. Following an argument in [14, proof of Theorem 1], we find that

(1) $L(x, y) = 0$ if and only if for each $\varepsilon > 0$, there exists a strong $\varepsilon$-chain from $x$ to $y$;
(2) $L(x, y) \le L(x, z) + L(z, y)$;
(3) $L$ is continuous.

(1) and (2) are obvious. It follows from $|L(x, y)-L(x', y)| \le d(f(x), f(x'))$ and $|L(x', y) - L(x', y')| \le d(y, y')$ for $(x, y), (x', y') \in X \times X$ that $|L(x, y) - L(x', y')| \le d(f(x), f(x')) + d(y, y')$. This shows (3).

Let $\{z_i \mid i \in \mathbb{N}\}$ be a countable dense set in $X$. We define a real-valued function $\varphi$ on $X$ by setting

$$\varphi(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} L(z_i, x).$$

We show that $\varphi$ is a pseudo-complete Lyapunov function for $f$.

$(L_1)$: Since $L(z, f(x)) \le L(z, x) + L(x, f(x))$ and $L(x, f(x)) = 0$, it follows that $\varphi(f(x)) \le \varphi(x)$ for all $x \in X$.

$(L_2)$: Let $x \notin \mathrm{SCR}(f)$. Since $L(x, f(x)) = 0 < L(x, x)$, there is $i_0$ with $L(z_{i_0}, f(x)) < L(z_{i_0}, x)$ by (3). It follows that $\varphi(f(x)) = \sum_{i=1}^{\infty} 2^{-i} L(z_i, f(x)) < \sum_{i=1}^{\infty} 2^{-i} L(z_i, x) = \varphi(x)$.

Let $x \in \mathrm{SCR}(f)$. Then using $L(z, x) \le L(z, f(x))$ (note that $L(f(x), x) = 0$ by Proposition 3.5) together with $(L_1)$, we find that $\varphi(f(x)) = \varphi(x)$.

$(L_3)$: If $x$ and $x'$ are strong chain transitive, then $L(z, x) = L(z, x')$ for each $z \in X$, because $L(x, x') = L(x', x) = 0$. Hence $\varphi$ must be constant on each strong chain transitive component. ∎

REMARK ([1]). One might hope that a pseudo-complete Lyapunov function $\varphi$ takes different values on different strong chain transitive components [7, 10]. However, for example, let $f$ be the identity map on $S^1$. Then every

([1]) This remark was suggested by the referee.

point is strong chain recurrent, and each strong chain transitive component is a single point. Since there exist no continuous one-to-one maps from $S^1$ to $\mathbb{R}$, the condition above cannot be satisfied.

As an application of the theorem above, we obtain a necessary and sufficient condition for the coincidence of $\mathrm{SCR}(f)$ and $\mathrm{CR}(f)$.

THEOREM 5.3. *For a map $f : X \to X$, there exists a pseudo-complete Lyapunov function $\varphi$ with $\varphi(\mathrm{SCR}(f))$ totally disconnected if and only if*

$$\mathrm{SCR}(f) = \mathrm{CR}(f).$$

*Proof.* The "if" part is a direct consequence of the existence of a Lyapunov function $\varphi$ with $\varphi(\mathrm{CR}(f))$ totally disconnected (see [7, 10]).

Now, we show the "only if" part. The inclusion $\mathrm{SCR}(f) \subseteq \mathrm{CR}(f)$ is obvious. Conversely, let $x \notin \mathrm{SCR}(f)$. Since $\varphi(x) > \varphi(f(x))$ and $\varphi(\mathrm{SCR}(f))$ is totally disconnected, we can take a real number $t_0$ satisfying $\varphi(f(x)) < t_0 < \varphi(x)$ and $t_0 \notin \varphi(\mathrm{SCR}(f))$. Set $U = \varphi^{-1}([0, t_0))$. Then we can easily see that $U$ is open in $X$, $x \notin \mathrm{Cl}\, U$, $f(x) \in U$ and $f(\mathrm{Cl}\, U) \subseteq U$. Thus $x \notin \mathrm{CR}(f)$ by Lemma 2.1. ∎

EXAMPLE 5.4. There may exist a pseudo-complete Lyapunov function $\varphi$ such that $\varphi(\mathrm{SCR}(f))$ is an interval.

Let $f$ be the identity map on $I = [0, 1]$. Take a dense set $\{a_n \mid n \in \mathbb{N}\}$ in $[0, 1/2]$, and set $z_i = a_n$ for $i = 2n - 1$ and $z_i = 1 - a_n$ for $i = 2n$; then $\{z_i \mid i \in \mathbb{N}\}$ is dense in $[0, 1]$. As in the proof of Theorem 5.2, construct a pseudo-complete Lyapunov function $\varphi$ for $f$. Then it is easily seen that $\varphi(1) - \varphi(0) > 0$ (note that $L(z, x) = |z - x|$ for each $z, x \in I$). Thus, $\varphi(\mathrm{SCR}(f)) = \varphi(I)$ is an interval.

An immediate consequence of Theorem 5.3 is the following.

COROLLARY 5.5. *If $f$ is a map on a compact metric space $X$ and* $\mathrm{SCR}(f)$ *is countable, then* $\mathrm{SCR}(f) = \mathrm{CR}(f)$. *In particular, if $X$ is countable, then* $\mathrm{SCR}(f) = \mathrm{CR}(f)$.

COROLLARY 5.6. *If $f$ is a map on a countable compact metric space $X$, then $D(f) \leq 1$.*

*Proof.* Corollary 5.5 guarantees $\mathrm{SCR}(f) = \mathrm{CR}(f) = \mathrm{CR}(f|_{\mathrm{CR}(f)}) = \mathrm{SCR}(f|_{\mathrm{SCR}(f)})$. ∎

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## References


[1] G. D. Birkhoff, *Über gewisse Zentralbewegungen dynamischer Systeme*, Nachrichten Göttingen 1926, 81–92.

[2] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math. 1513, Springer, Berlin, 1992.

[3] L. Block and J. E. Franke, *The chain recurrent set, attractors, and explosions*, Ergodic Theory Dynam. Systems 5 (1985), 321–327.

[4] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Reg. Conf. Ser. Math. 38, Amer. Math. Soc., Providence, RI, 1978.

[5] R. Easton, *Chain transitivity and the domain of influence of an invariant set*, in: Lecture Notes in Math. 668, Springer, Berlin, 1978, 95–102.

[6] A. Fakhari, F. H. Ghane, and A. Sarizadeh, *Some properties of the strong chain recurrent set*, Comm. Korean Math. Soc. 25 (2010), 97–104.

[7] J. Franks, *A variation on the Poincaré–Birkhoff theorem*, in: Contemp. Math. 81, Amer. Math. Soc., Providence, RI, 1988, 111–117.

[8] F. H. Ghane and A. Fakhari, *On strong chain classes of homeomorphisms of compact metric spaces*, J. Dynam. Systems Geom. Theories 5 (2007), 33–39.

[9] F.-N. Hu, *Some properties of the chain recurrent set of dynamical systems*, Chaos Solitons Fractals 14 (2002), 1309–1314.

[10] M. Hurley, *Chain recurrence and attraction in noncompact spaces*, Ergodic Theory Dynam. Systems 11 (1991), 709–729.

[11] H. Kato, *The depth of centres of maps on dendrites*, J. Austral. Math. Soc. Ser. A 64 (1998), 44–53.

[12] J.-H. Mai and T.-X. Sun, *Non-wandering points and the depth for graph maps*, Sci. China Ser. A 50 (2007), 1818–1824.

[13] J.-C. Xiong, $\Omega(f\,|\,\Omega(f)) = \overline{P(f)}$ *for every continuous self-map* $f$ *of the interval*, Kexue Tongbao (English Ed.) 28 (1983), 21–23.

[14] Z. Zheng, *Chain transitivity and Lipschitz ergodicity*, Nonlinear Anal. 34 (1998), 733–744.


Katsuya Yokoi
Department of Mathematics
Jikei University School of Medicine
Chofu, Tokyo 182-8570, Japan
E-mail: yokoi@jikei.ac.jp

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