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A regularity criterion for the 3D density-dependent incompressible flow of liquid crystals with vacuum

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Abstract. We consider the Cauchy problem for the 3D density-dependent incompressible flow of liquid crystals with vacuum, and provide a regularity criterion in terms of $\boldsymbol{u}$ and $\nabla \boldsymbol{d}$ in the Besov spaces of negative order. This improves recent result of Fan–Li [Comm. Math. Sci. 12 (2014), 1185–1197].

1. Introduction. The present paper concerns the following Cauchy problem for the density-dependent liquid crystal flows [2, 8, 17, 22]:

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0,\\
&\partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \Delta \boldsymbol{u} + \nabla \pi = -\nabla \cdot (\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}),\\
&\partial_t \boldsymbol{d} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{d} - \Delta \boldsymbol{d} = \boldsymbol{d} |\nabla \boldsymbol{d}|^2,\\
&\nabla \cdot \boldsymbol{u} = 0, \quad |\boldsymbol{d}| = 1,\\
&(\rho, \boldsymbol{u}, \boldsymbol{d})|_{t=0} = (\rho_0, \boldsymbol{u}_0, \boldsymbol{d}_0).
\end{aligned} \tag{1.1}$$

Here, ρ is the fluid density; $\boldsymbol{u}$ is the fluid velocity field; $\boldsymbol{d}$ satisfying $|\boldsymbol{d}| = 1$ is the macroscopic molecular orientation; π is the scalar pressure; $\mu > 0$ is the kinematic viscosity; and $\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}$ is a 3×3 matrix with (i, j) entry

$$\partial_i \boldsymbol{d} \cdot \partial_j \boldsymbol{d} = \sum_{k=1}^{3} \partial_i d_k \partial_j d_k.$$

When $\boldsymbol{d}$ is a constant vector with $|\boldsymbol{d}| = 1$, the system $(1.1)_{1,2,4,5}$ reduces to the density-dependent Navier–Stokes equations, and has attracted many authors' attention (see [4, 5, 6, 7, 12] for example). On the other hand, if $\rho \equiv 1$, the system $(1.1)_{2,3,4,5}$ reduces to the incompressible liquid crystals system (see [18, 20, 21, 22, 23, 24] etc.).

Recently, Li–Wang [18, 19] considered the system (1.1) in a bounded domain, and showed the local (global) existence and uniqueness of strong

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solutions with large (small) initial data when the initial density has a positive lower bound. Fan–Gao–Guo [9] considered the system with an additional term $\boldsymbol{d}\times\Delta\boldsymbol{d}$ in $(1.1)_3$, and provided some regularity criteria when the initial density is away from vacuum. Fan–Li [10] then established some uniform a priori estimates that do not depend on $\mu>0$ when $\inf_{\mathbb{R}^3}\rho_0>0$.

Now, we consider the system (1.1) when the initial density may vanish on an open set of $\mathbb{R}^3$. Assume the initial data satisfy

$$\begin{aligned} &0\le\rho_0\le M<\infty, && \nabla\rho_0\in L^2\cap L^q(\mathbb{R}^3), \quad 3<q\le 6,\\ &\boldsymbol{u}_0\in H^2(\mathbb{R}^3), && \nabla\cdot\boldsymbol{u}_0=0,\\ &\nabla\boldsymbol{d}_0\in H^2(\mathbb{R}^3), && |\boldsymbol{d}_0|=1,\end{aligned}\tag{1.2}$$

and the compatibility condition

$$\mu\Delta\boldsymbol{u}_0-\nabla\cdot(\nabla\boldsymbol{d}_0\otimes\nabla\boldsymbol{d}_0)-\nabla\pi_0=\rho_0^{1/2}\boldsymbol{g}\quad\text{for some } \boldsymbol{g}\in L^2(\mathbb{R}^3),\tag{1.3}$$

Then it is standard (see [3, 15] for the non-homogeneous incompressible Navier–Stokes equations) that there exists a $T^*>0$ such that the Cauchy problem (1.1) has a unique strong solution $(\rho,\boldsymbol{u},\pi,\boldsymbol{d})$ on $(0,T^*)$ (^{1}). However, whether or not this local strong solution can be a global one remains open. Fan–Li first established the following fundamental optimal (see [10, Remark 1.4]) regularity criterion:

$$\begin{aligned} &\boldsymbol{u}\in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q\le\infty,\\ &\nabla\boldsymbol{d}\in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)),\end{aligned}\tag{1.4}$$

that is, if (1.4) holds, then the solution $(\rho,\boldsymbol{u},\pi,\boldsymbol{d})$ can be extended smoothly beyond T. Here, $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$ denotes the homogeneous Besov space, and will be introduced in Section 2. Notice that (see Section 2 again)

$$L^{3/r}(\mathbb{R}^3)\subsetneq\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3),\quad 0\le r\le 1.$$

We would like to refine (1.4) to involve Besov spaces of negative order. Precisely, our result reads

THEOREM 1.1. *Assume the initial data $(\rho_0,\boldsymbol{u}_0,\boldsymbol{d}_0)$ satisfy* (1.2) *and the compatibility condition* (1.3). *Let $(\rho,\boldsymbol{u},\pi,\boldsymbol{d})$ be a strong solution to the system* (1.1) *on* $(0,T)$. *If*

$$\boldsymbol{u},\nabla\boldsymbol{d}\in L^{2/(1-r)}(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)),\quad 0<r<1,\tag{1.5}$$

then the solution $(\rho,\boldsymbol{u},\pi,\boldsymbol{d})$ can be extended smoothly beyond T.

(^{1}) On the evolution of $\boldsymbol{d}$, we deduce from $(1.1)_3$ that

$$\partial_t f-\Delta f+(\boldsymbol{u}\cdot\nabla)f-2|\nabla\boldsymbol{d}|^2f=0,$$

where $f=|\boldsymbol{d}|^2-1$. Thus applying the maximum principle for this parabolic system with $|\nabla\boldsymbol{d}|^2\in L^\infty(0,T;L^1))$, we see $|\boldsymbol{d}|=1$ for all $t>0$.

Readers interested in regularity criteria for other density-dependent incompressible models may refer to [11, 13, 14, 27].

The rest of the paper is organized as follows. In Section 2, we set up some notation, recall the definition of Besov spaces and some often used inequalities. Section 3 is devoted to proving Theorem 1.1.

2. Preliminaries. First, we set up the notation used throughout the paper. The standard Lebesgue spaces and Sobolev spaces

$$L^p(\mathbb{R}^3), \quad W^{k,p}(\mathbb{R}^3), \quad H^k(\mathbb{R}^3) = W^{k,2}(\mathbb{R}^3)$$

are endowed with the norms

$$\|\cdot\|_{L^p}, \quad \|\cdot\|_{W^{k,p}}, \quad \|\cdot\|_{H^k}$$

respectively. A generic constant C may change from line to line. For simplicity, we shall omit the spatial domain $\mathbb{R}^3$ in all integrals, and we do not distinguish between the spaces X and their N-dimensional vector analogues X^N; however, all vector- and tensor-valued functions are printed in boldface.

Next, we introduce the Littlewood–Paley decomposition. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. For $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}\,dx.$$

Let us choose a non-negative radial function $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \le \hat{\varphi}(\xi) \le 1, \quad \hat{\varphi}(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1, \\ 0 & \text{if } |\xi| \ge 2, \end{cases}$$

and let

$$\psi(x) = \varphi(x) - 2^{-3}\varphi(x/2),\ \varphi_j(x) = 2^{3j}\varphi(2^j x),\ \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, the Littlewood–Paley projection operators S_j and Δ_j are, respectively, defined by

$$S_j f = \varphi_j * f, \quad \Delta_j f = \psi_j * f.$$

Observe that $\Delta_j = S_j - S_{j-1}$. Also, it is easy to check that if $f \in L^2(\mathbb{R}^3)$, then

$$S_j f \to 0 \quad \text{as } j \to -\infty, \quad S_j f \to f \quad \text{as } j \to \infty,$$

in the L^2 sense. By telescoping the series, we have the Littlewood–Paley decomposition

$$f = \sum_{j=-\infty}^{\infty} \Delta_j f$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is in the L^2 sense.

Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by means of the full dyadic decomposition as

$$\dot{B}^s_{p,q} = \big\{ f \in \mathcal{Z}'(\mathbb{R}^3);\ \|f\|_{\dot{B}^s_{p,q}} = \big\| \{2^{js}\|\Delta_j f\|_{L^p}\}_{j=-\infty}^{\infty} \big\|_{\ell^q} < \infty \big\},$$

where $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \big\{ f \in \mathcal{S}(\mathbb{R}^3);\ D^\alpha \hat{f}(0) = 0,\ \forall \alpha \in \mathbb{N}^3 \big\}.$$

Also, the homogeneous Triebel–Lizorkin space $\dot{F}^s_{p,q}$ is defined as

$$\dot{F}^s_{p,q} = \big\{ f \in \mathcal{Z}'(\mathbb{R}^3);\ \|f\|_{\dot{F}^s_{p,q}} = \big\| \|\{2^{js}|\Delta_j f\}_{j=-\infty}^{\infty}\|_{\ell^q} \big\|_{L^p} < \infty \big\}.$$

It is well-known that (see [26] for example) for all $s \in \mathbb{R}$,

$$\dot{H}^s(\mathbb{R}^3) = \dot{B}^s_{2,2}(\mathbb{R}^3) = \dot{F}^s_{2,2}(\mathbb{R}^3), \quad \dot{B}^s_{\infty,\infty}(\mathbb{R}^3) = \dot{F}^s_{\infty,\infty}(\mathbb{R}^3). \tag{2.1}$$

Also, Kozono–Shimada [16] proved the bilinear estimates

$$\|f \cdot g\|_{\dot{F}^s_{p,q}} \le C\big(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}} \|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}} \|g\|_{\dot{F}^{s+\beta}_{r_2,q}} \big), \tag{2.2}$$

where

$$s > 0, \quad \alpha > 0, \quad \beta > 0, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}.$$

With (2.2), the following regularity criterion for the 3D incompressible Navier–Stokes equations was proved in [16]:

$$\boldsymbol{u} \in L^{2/(1-r)}(0, T; \dot{B}^{-r}_{\infty,\infty}) \quad (0 < r < 1).$$

The main obstacle in utilizing (2.2) for system (1.1) is that we do not have any incompressibility condition on $\nabla \boldsymbol{d}$, and this leads us to derive a new bilinear estimate, which corresponds to (2.2) with $s = 0$. Before stating its precise form, let us recall a refined Sobolev embedding theorem (see [1, Theorem 2.42]):

$$\|f\|_{L^p} \le C\|f\|^{1-2/p}_{\dot{B}^{-r}_{\infty,\infty}} \|f\|^{2/p}_{\dot{H}^\beta} \tag{2.3}$$

with

$$r > 0, \quad \beta = r\bigg(\frac{p}{2} - 1\bigg), \quad 2 < p < \infty.$$

Now, our new bilinear estimate is as follows.

Lemma 2.1. *Let $0 < r \le 1$, $f \in \dot{B}^{-r}_{\infty,\infty} \cap \dot{H}^2$, $g \in \dot{B}^{-r}_{\infty,\infty} \cap \dot{H}^1 \cap \dot{H}^2$. Then there exists a constant $C = C(r)$ such that*

$$\|f \nabla g\|_{L^2} \le C\|(f, g)\|_{\dot{B}^{-r}_{\infty,\infty}} \|\nabla g\|^{1-r}_{L^2} \|(\nabla^2 f, \nabla^2 g)\|^r_{L^2}. \tag{2.4}$$

Proof. By the Hölder inequality, (2.3), and the interpolation inequality, we deduce

$$\begin{aligned}\|f\nabla g\|_{L^2} &\le \|f\|_{L^{2+4/r}}\|\nabla g\|_{L^{r+2}}\\ &\le C\|f\|_{\dot B^{-r}_{\infty,\infty}}^{\frac{2}{r+2}}\|f\|_{\dot H^2}^{\frac{r}{r+2}}\cdot\|\nabla g\|_{\dot B^{-1-r}_{\infty,\infty}}^{\frac{r}{r+2}}\|\nabla g\|_{\dot H^{\frac{r(r+1)}{2}}}^{\frac{2}{r+2}}\\ &\le C\|f\|_{\dot B^{-r}_{\infty,\infty}}^{\frac{2}{r+2}}\|\nabla^2 f\|_{L^2}^{\frac{r}{r+2}}\|g\|_{\dot B^{-r}_{\infty,\infty}}^{\frac{r}{r+2}}\|\nabla g\|_{L^2}^{1-r}\|\nabla^2 g\|_{L^2}^{\frac{r(r+1)}{r+2}}. \blacksquare\end{aligned}$$

3. Proof of Theorem 1.1. As remarked in [10], to prove Theorem 1.1, we only need to establish sufficient regularity estimates of the solutions under the condition (1.5).

First, it follows from $(1.1)_{1,4}$ that

$$\partial_t\rho + (\boldsymbol{u}\cdot\nabla)\rho = 0.$$

The maximum principle then yields

$$0 \le \rho \le M < \infty. \tag{3.1}$$

Taking the inner product of $(1.1)_2$ with $\boldsymbol{u}$ and of $(1.1)_3$ with $-\Delta\boldsymbol{d}$ in $L^2(\mathbb{R}^3)$, we find

$$\frac{1}{2}\frac{d}{dt}\int\rho|\boldsymbol{u}|^2\,dx + \int\mu|\nabla\boldsymbol{u}|^2\,dx + \int[(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}]\cdot\Delta\boldsymbol{d}\,dx = 0, \tag{3.2}$$

as well as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla\boldsymbol{d}|^2\,dx &+ \int|\Delta\boldsymbol{d}|^2\,dx - \int[(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}]\cdot\Delta\boldsymbol{d}\,dx\\ &= -\int|\nabla\boldsymbol{d}|^2\boldsymbol{d}\cdot\Delta\boldsymbol{d}\,dx\\ &= \int|\boldsymbol{d}\cdot\Delta\boldsymbol{d}|^2\,dx \qquad \left(|\boldsymbol{d}| = 1 \Rightarrow 0 = \frac{1}{2}\Delta|\boldsymbol{d}|^2 = \boldsymbol{d}\cdot\Delta\boldsymbol{d} + |\nabla\boldsymbol{d}|^2\right)\\ &\le \int|\Delta\boldsymbol{d}|^2\,dx.\end{aligned} \tag{3.3}$$

Adding (3.2) and (3.3) then yields

$$\frac{1}{2}\frac{d}{dt}\int[\rho|\boldsymbol{u}|^2 + |\nabla\boldsymbol{d}|^2]\,dx + \int\mu|\nabla\boldsymbol{u}|^2\,dx \le 0. \tag{3.4}$$

Integrating in time gives

$$\sup_{0\le t\le T}\int[\rho|\boldsymbol{u}|^2 + |\nabla\boldsymbol{d}|^2]\,dx + 2\int\mu|\nabla\boldsymbol{u}|^2\,dx \le C. \tag{3.5}$$

Taking the inner product of $(1.1)_2$ with $\partial_t\boldsymbol{u}$ in $L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\int|\nabla\boldsymbol{u}|^2\,dx &+ \int\rho|\partial_t\boldsymbol{u}|^2\,dx\\ &= \int[(\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{u}\,dx + \int\nabla\partial_t\boldsymbol{u} : (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx =: I_1 + I_2.\end{aligned} \tag{3.6}$$

For I_1,

(3.7)

$$\begin{aligned} I_1 &\le C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}\big\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\big\|_{L^2} && \text{(Hölder inequality)}\\ &\le C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{\dot{B}^{-r}_{\infty,\infty}}\|\nabla\boldsymbol{u}\|_{L^2}^{1-r}\|\nabla^2\boldsymbol{u}\|_{L^2}^{r} && \text{(by (2.4))}\\ &\le \varepsilon\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}^2+\frac{\varepsilon}{4}\|\Delta\boldsymbol{u}\|_{L^2}^2+C\|\boldsymbol{u}\|_{\dot{B}^{-r}_{\infty,\infty}}^{2/(1-r)}\|\nabla\boldsymbol{u}\|_{L^2}^2 && \text{(Young inequality)}, \end{aligned}$$

where $0<\varepsilon\ll 1$ is to be determined.

For I_2, we integrate by parts:

$$\begin{aligned} I_2 &= \sum_{i,j=1}^{3}\int \partial_t\partial_j u_i\partial_i d_k\partial_j d_k\,dx\\ &= \sum_{i,j=1}^{3}\bigg[\frac{d}{dt}\int\partial_j u_i\partial_i d_k\partial_j d_k\,dx-\int\partial_j u_i\partial_i\partial_t d_k\partial_j d_k\,dx\\ &\qquad\qquad -\int\partial_j u_i\partial_i d_k\partial_j\partial_t d_k\,dx\bigg]\\ &= \frac{d}{dt}\int\nabla\boldsymbol{u}:(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx+\sum_{i,j=1}^{3}\int\partial_j u_i\partial_t d_k\partial_i\partial_j d_k\,dx\\ &\quad+\sum_{i,j=1}^{3}\int\Delta u_i\partial_i d_k\partial_t d_k\,dx+\sum_{i,j=1}^{3}\int\partial_j u_i\partial_i\partial_j d_k\partial_t d_k\,dx\\ &= \frac{d}{dt}\int\nabla\boldsymbol{u}:(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx\\ &\quad+2\sum_{i,j,k=1}^{3}\int\partial_j u_i\partial_i\partial_j d_k[d_k|\nabla\boldsymbol{d}|^2-(\boldsymbol{u}\cdot\nabla)d_k+\Delta d_k]\,dx\\ &\quad+\sum_{i,k=1}^{3}\int\Delta u_i\partial_i d_k[d_k|\nabla\boldsymbol{d}|^2-(\boldsymbol{u}\cdot\nabla)d_k+\Delta d_k]\,dx, \end{aligned}$$

and then control it as follows:

(3.8)
$$\begin{aligned} I_2 &\le \frac{d}{dt}\int\nabla\boldsymbol{u}:(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx+2\int(|\nabla\boldsymbol{u}|\cdot|\nabla\boldsymbol{d}|)\cdot(|\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|)\,dx\\ &\quad+2\int(|\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|)\cdot(|\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|)\,dx\\ &\quad-2\sum_{i,k=1}^{3}\int u_i(\partial_i\Delta d_k\Delta d_k+\partial_i\partial_j d_k\partial_j\Delta d_k)\,dx\\ &\quad+\sum_{i=1}^{3}\int\Delta u_i\partial_i\frac{|\boldsymbol{d}|^2}{2}|\nabla\boldsymbol{d}|^2\,dx \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,k=1}^{3}\int \Delta u_i d_k[(\partial_i \boldsymbol{u}\cdot\nabla)d_k+(\boldsymbol{u}\cdot\nabla)\partial_i d_k]\,dx+\int|\Delta\boldsymbol{u}|\cdot|\nabla\boldsymbol{d}|\cdot|\Delta\boldsymbol{d}|\,dx\\
\le{}&\frac{d}{dt}\int\nabla\boldsymbol{u}:(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx+\big\||\nabla\boldsymbol{u}|\cdot|\nabla\boldsymbol{d}|\big\|_{L^2}\cdot\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}\\
&+\big\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\big\|_{L^2}\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}+4\|\nabla\Delta\boldsymbol{d}\|_{L^2}\big\||\boldsymbol{u}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}\\
&+\|\Delta\boldsymbol{u}\|_{L^2}\big(\big\||\nabla\boldsymbol{u}|\cdot|\nabla\boldsymbol{d}|\big\|_{L^2}+\big\||\boldsymbol{u}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}\big)+\|\Delta\boldsymbol{u}\|_{L^2}\big\||\nabla\boldsymbol{d}|\cdot|\Delta\boldsymbol{d}|\big\|_{L^2}\\
\le{}&\frac{d}{dt}\int\nabla\boldsymbol{u}:(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx+\frac{\varepsilon}{4}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2+\frac{\varepsilon}{8}\|\Delta\boldsymbol{u}\|_{L^2}^2\\
&+C\big\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\big\|_{L^2}^2+C\big\||\boldsymbol{u}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}^2\\
&+C\big\||\nabla\boldsymbol{u}|\cdot|\nabla\boldsymbol{d}|\big\|_{L^2}^2+C\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}^2.
\end{aligned}
$$

By (2.4), we may dominate the last four terms as follows:

$$
\begin{aligned}
C\big\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\big\|_{L^2}^2&\le C\|\boldsymbol{u}\|_{\dot B^{-r}_{\infty,\infty}}^2\|\nabla\boldsymbol{u}\|_{L^2}^{2(1-r)}\|\nabla^2\boldsymbol{u}\|_{L^2}^{2r}\\
&\le\frac{\varepsilon}{16}\|\Delta\boldsymbol{u}\|_{L^2}^2+C\|\boldsymbol{u}\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|\nabla\boldsymbol{u}\|_{L^2}^2,
\end{aligned}\tag{3.9}
$$

$$
\begin{aligned}
C\big\||\boldsymbol{u}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}^2&\le C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}\|\nabla^2\boldsymbol{d}\|_{L^2}^{1-r}\|(\nabla^2\boldsymbol{u},\nabla^3\boldsymbol{d})\|_{L^2}^{r}\\
&\le\frac{\varepsilon}{16}\|(\Delta\boldsymbol{u},\nabla\Delta\boldsymbol{d})\|_{L^2}^2+C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|\Delta\boldsymbol{d}\|_{L^2}^2,
\end{aligned}\tag{3.10}
$$

$$
\begin{aligned}
C\big\||\nabla\boldsymbol{u}|\cdot|\nabla\boldsymbol{d}|\big\|_{L^2}^2&=C\big\||\nabla\boldsymbol{d}|\cdot|\nabla\boldsymbol{u}|\big\|_{L^2}^2\\
&\le C\|(\nabla\boldsymbol{d},\boldsymbol{u})\|_{\dot B^{-r}_{\infty,\infty}}\|\nabla\boldsymbol{u}\|_{L^2}^{1-r}\|(\nabla^3\boldsymbol{d},\nabla^2\boldsymbol{u})\|_{L^2}^{r}\\
&\le\frac{\varepsilon}{16}\|(\Delta\boldsymbol{u},\nabla\Delta\boldsymbol{d})\|_{L^2}^2+C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|\nabla\boldsymbol{u}\|_{L^2}^2,
\end{aligned}\tag{3.11}
$$

$$
\begin{aligned}
C\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}^2&\le C\|\nabla\boldsymbol{d}\|_{\dot B^{-r}_{\infty,\infty}}\|\nabla^2\boldsymbol{d}\|_{L^2}^{1-r}\|\nabla^3\boldsymbol{d}\|_{L^2}^{r}\\
&\le\frac{\varepsilon}{16}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2+C\|\nabla\boldsymbol{d}\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|\Delta\boldsymbol{d}\|_{L^2}^2.
\end{aligned}\tag{3.12}
$$

Plugging these four estimates into (3.8), we find

$$
\begin{aligned}
I_2\le{}&\frac{d}{dt}\int\nabla\boldsymbol{u}:(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx+\frac{\varepsilon}{4}\|(\Delta\boldsymbol{u},\nabla\Delta\boldsymbol{d})\|_{L^2}^2\\
&+C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|(\nabla\boldsymbol{u},\Delta\boldsymbol{d})\|_{L^2}^2.
\end{aligned}\tag{3.13}
$$

Inserting (3.7) and (3.13) into (3.6), we obtain

$$
\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int|\nabla\boldsymbol{u}|^2\,dx+\int\rho|\partial_t\boldsymbol{u}|^2\,dx\\
&\quad\le\frac{d}{dt}\int\nabla\boldsymbol{u}:(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx+\varepsilon\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}^2+\frac{\varepsilon}{2}\|(\Delta\boldsymbol{u},\nabla\Delta\boldsymbol{d})\|_{L^2}^2\\
&\qquad+C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|(\nabla\boldsymbol{u},\Delta\boldsymbol{d})\|_{L^2}^2.
\end{aligned}\tag{3.14}
$$

Taking the inner product of $(1.1)_3$ with $\Delta^2\boldsymbol{d}$ in $L^2(\mathbb{R}^3)$, we get

$$\begin{aligned}(3.15)\qquad \frac{1}{2}\frac{d}{dt}\int|\Delta\boldsymbol{d}|^2\,dx+\int|\nabla\Delta\boldsymbol{d}|^2\,dx&=\int[\boldsymbol{d}|\nabla\boldsymbol{d}|^2-(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}]\cdot\Delta^2\boldsymbol{d}\,dx\\&=-\sum_{k=1}^{3}\int\partial_k(\boldsymbol{d}|\nabla\boldsymbol{d}|^2)\cdot\partial_k\Delta\boldsymbol{d}\,dx-\int\Delta[(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}]\cdot\Delta\boldsymbol{d}\,dx=:J_1+J_2.\end{aligned}$$

We successively dominate J_1 and J_2 as follows:

(3.16)

$$\begin{aligned}J_1&=-\sum_{k=1}^{3}\int\partial_k\boldsymbol{d}|\nabla\boldsymbol{d}|^2\cdot\partial_k\Delta\boldsymbol{d}\,dx-\sum_{i,j,k=1}^{3}\int(\boldsymbol{d}\cdot\partial_k\Delta\boldsymbol{d})\cdot(2\partial_j d_i\partial_k\partial_j d_i)\,dx\\&=\sum_{k=1}^{3}\int\partial_k(\partial_k\boldsymbol{d}|\nabla\boldsymbol{d}|^2)\cdot\Delta\boldsymbol{d}\,dx-\sum_{i,j,k=1}^{3}\int(\boldsymbol{d}\cdot\partial_k\Delta\boldsymbol{d})\cdot(2\partial_j d_i\partial_k\partial_j d_i)\,dx\\&\le 3\int|\nabla\boldsymbol{d}|^2\cdot|\nabla^2\boldsymbol{d}|^2\,dx+2\int|\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\cdot|\nabla\Delta\boldsymbol{d}|\,dx\\&\le C\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}^2+\frac{\varepsilon}{8}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2\\&\le C\|\nabla\boldsymbol{d}\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|\Delta\boldsymbol{d}\|_{L^2}^2+\frac{\varepsilon}{4}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2\qquad(\text{by }(3.12))\end{aligned}$$

and

(3.17)

$$\begin{aligned}J_2&=\int[(\Delta\boldsymbol{u}\cdot\nabla)\boldsymbol{d}]\cdot\Delta\boldsymbol{d}\,dx+2\sum_{k=1}^{3}\int[(\partial_k\boldsymbol{u}\cdot\nabla)\partial_k\boldsymbol{d}]\cdot\Delta\boldsymbol{d}\,dx\\&\qquad\qquad(\text{by }\Delta(fg)=f\Delta g+g\Delta f+2\nabla f\cdot\nabla g)\\&\le\|\Delta\boldsymbol{u}\|_{L^2}\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}-2\sum_{i,j,k=1}^{3}\int u_j\partial_k(\partial_j\partial_k d_i\Delta d_i)\,dx\\&\le\|\Delta\boldsymbol{u}\|_{L^2}\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}+C\big\||\boldsymbol{u}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}\|\nabla\Delta\boldsymbol{d}\|_{L^2}\\&\le\frac{\varepsilon}{4}\|\Delta\boldsymbol{u}\|_{L^2}^2+\frac{\varepsilon}{8}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2+C\big\||\nabla\boldsymbol{d}|\cdot|\Delta\boldsymbol{d}|\big\|_{L^2}^2+C\big\||\boldsymbol{u}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}^2\\&\le\frac{\varepsilon}{2}\|\Delta\boldsymbol{u}\|_{L^2}^2+\frac{\varepsilon}{4}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2+C\|\nabla\boldsymbol{d}\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|\Delta\boldsymbol{d}\|_{L^2}^2\\&\quad+C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|(\nabla\boldsymbol{u},\Delta\boldsymbol{d})\|_{L^2}^2\qquad(\text{by }(3.12),\ (3.10)).\end{aligned}$$

Combining (3.16) and (3.17) with (3.15), we deduce

$$\begin{aligned}(3.18)\qquad &\frac{1}{2}\frac{d}{dt}\int|\Delta\boldsymbol{d}|^2\,dx+\int|\nabla\Delta\boldsymbol{d}|^2\,dx\\&\qquad\le\frac{\varepsilon}{2}\|(\Delta\boldsymbol{u},\nabla\Delta\boldsymbol{d})\|_{L^2}^2+C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|(\nabla\boldsymbol{u},\Delta\boldsymbol{d})\|_{L^2}^2.\end{aligned}$$

Putting (3.14) and (3.18) together, we have

$$\begin{aligned}(3.19)\qquad &\frac{1}{2}\frac{d}{dt}\int \mu|\nabla \boldsymbol{u}|^2 + |\Delta \boldsymbol{d}|^2\,dx + \int \rho|\partial_t \boldsymbol{u}|^2\,dx + \int |\nabla\Delta \boldsymbol{d}|^2\,dx\\ &\le \frac{d}{dt}\int \nabla \boldsymbol{u} : (\nabla \boldsymbol{d}\odot\nabla \boldsymbol{d})\,dx + \varepsilon\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}^2 + \varepsilon\|(\Delta\boldsymbol{u},\nabla\Delta\boldsymbol{d})\|_{L^2}^2\\ &\quad + C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|(\nabla\boldsymbol{u},\Delta\boldsymbol{d})\|_{L^2}^2.\end{aligned}$$

To close the estimate, we need to bound $\|\Delta\boldsymbol{u}\|_{L^2}$. Utilizing $(1.1)_1$, we may rewrite $(1.1)_2$ as

$$-\mu\Delta\boldsymbol{u} + \nabla\pi = -\nabla\cdot(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) - \rho\partial_t\boldsymbol{u} - (\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u},$$

and invoke the H^2-theory of the Stokes system (see [25] for example) to deduce

$$\begin{aligned}(3.20)\quad \|\Delta\boldsymbol{u}\|_{L^2} &\le C\|-\nabla\cdot(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) - \rho\partial_t\boldsymbol{u} - (\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{L^2}\\ &\le C\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2} + C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2} + C\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\|_{L^2}\\ &\le \frac{1}{2}\|\nabla\Delta\boldsymbol{d}\|_{L^2} + C\|\nabla\boldsymbol{d}\|_{\dot B^{-r}_{\infty,\infty}}^{1/(1-r)}\|\Delta\boldsymbol{d}\|_{L^2} + C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}\\ &\quad + \frac{1}{2}\|\Delta\boldsymbol{u}\|_{L^2} + C\|\boldsymbol{u}\|_{\dot B^{-r}_{\infty,\infty}}^{1/(1-r)}\|\nabla\boldsymbol{u}\|_{L^2}\qquad \text{(by (3.12), (3.9))}.\end{aligned}$$

Consequently,

$$\begin{aligned}(3.21)\qquad \|\Delta\boldsymbol{u}\|_{L^2} &\le \|\nabla\Delta\boldsymbol{d}\|_{L^2} + C\|\nabla\boldsymbol{d}\|_{\dot B^{-r}_{\infty,\infty}}^{1/(1-r)}\|\Delta\boldsymbol{d}\|_{L^2}\\ &\quad + C\|\sqrt{\rho}\partial_t\boldsymbol{u}\|_{L^2} + C\|\boldsymbol{u}\|_{\dot B^{-r}_{\infty,\infty}}^{1/(1-r)}\|\nabla\boldsymbol{u}\|_{L^2}.\end{aligned}$$

Substituting (3.21) into (3.19), we readily see that

$$\begin{aligned}(3.22)\qquad &\frac{1}{2}\frac{d}{dt}\int \mu|\nabla \boldsymbol{u}|^2 + |\Delta \boldsymbol{d}|^2\,dx + \int \rho|\partial_t \boldsymbol{u}|^2\,dx + \int |\nabla\Delta \boldsymbol{d}|^2\,dx\\ &\le \frac{d}{dt}\int \nabla \boldsymbol{u} : (\nabla \boldsymbol{d}\odot\nabla \boldsymbol{d})\,dx + C\varepsilon\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}^2 + C\varepsilon\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2\\ &\quad + C\|(\boldsymbol{u},\nabla\boldsymbol{d})\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|(\nabla\boldsymbol{u},\Delta\boldsymbol{d})\|_{L^2}^2.\end{aligned}$$

Taking $\varepsilon = (2(C+1))^{-1}$, then integrating in time, noticing

$$\begin{aligned}\int \nabla\boldsymbol{u} : (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\,dx &\le \|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{d}\|_{L^4}^2 \le C\|\nabla\boldsymbol{u}\|_{L^2}\|\boldsymbol{d}\|_{L^\infty}\|\Delta\boldsymbol{d}\|_{L^2}\\ &\le C\|\nabla\boldsymbol{u}\|_{L^2}\|\Delta\boldsymbol{d}\|_{L^2},\end{aligned}$$

we may apply the Gronwall inequality to (3.22) to get

$$(3.23)\qquad \begin{aligned}&\|\nabla\boldsymbol{u}\|_{L^\infty(0,T;L^2)} \le C, \qquad \|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2(0,T;L^2)} \le C,\\ &\|\nabla\boldsymbol{d}\|_{L^\infty(0,T;H^1)} \le C, \qquad \|\nabla\boldsymbol{d}\|_{L^2(0,T;H^2)} \le C.\end{aligned}$$

and infer from (3.21) and $(1.1)_3$ that

$$\|\nabla \boldsymbol{u}\|_{L^2(0,T;H^1)} \le C, \quad \|\partial_t \boldsymbol{d}\|_{L^\infty(0,T;L^2)} \le C, \quad \|\partial_t \boldsymbol{d}\|_{L^2(0,T;H^1)} \le C. \tag{3.24}$$

Applying the time derivative operator ∂_t to $(1.1)_2$, multiplying by $\partial_t \boldsymbol{u}$, and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\partial_t \boldsymbol{u}|^2\,dx + \mu\int|\nabla\partial_t\boldsymbol{u}|^2\,dx \\
&= \int \partial_t(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) : \nabla\partial_t \boldsymbol{u}\,dx - \int \partial_t\rho[\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{u}\,dx \\
&\quad - \int \rho[(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{u}\,dx \\
&\le \int |\nabla \boldsymbol{d}|\cdot|\nabla\partial_t\boldsymbol{d}|\cdot|\nabla\partial_t\boldsymbol{u}|\,dx + \int \rho\boldsymbol{u}\cdot\nabla\{[\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{u}\}\,dx \\
&\quad + \int |\sqrt{\rho}\,\partial_t\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\cdot|\partial_t\boldsymbol{u}|\,dx \\
&\le C\|\nabla\boldsymbol{d}\|_{L^\infty}\|\nabla\partial_t\boldsymbol{d}\|_{L^2}\|\nabla\partial_t\boldsymbol{u}\|_{L^2} + C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^\infty}\|\nabla\partial_t\boldsymbol{u}\|_{L^2} \\
&\quad + C\|\boldsymbol{u}\|_{L^6}^2\|\Delta\boldsymbol{u}\|_{L^2}\|\partial_t\boldsymbol{u}\|_{L^6} + C\|\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{u}\|_{L^3}^2\|\partial_t\boldsymbol{u}\|_{L^6} \\
&\quad + C\|\boldsymbol{u}\|_{L^6}^2\|\nabla\boldsymbol{u}\|_{L^6}\|\nabla\partial_t\boldsymbol{u}\|_{L^2} + \|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^3}\|\partial_t\boldsymbol{u}\|_{L^6} \\
&\le \frac{\mu}{2}\|\nabla\partial_t\boldsymbol{u}\|_{L^2}^2 + \|\nabla\boldsymbol{d}\|_{H^2}^2\|\nabla\partial_t\boldsymbol{d}\|_{L^2}^2 + \|\boldsymbol{u}\|_{L^6}\|\Delta\boldsymbol{u}\|_{L^2}\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}^2 \\
&\quad + \|\nabla\boldsymbol{u}\|_{L^2}^2\|\Delta\boldsymbol{u}\|_{L^2}^2 + C\|\nabla\boldsymbol{u}\|_{L^2}^2\|\nabla\boldsymbol{u}\|_{L^2}\|\Delta\boldsymbol{u}\|_{L^2} + \|\nabla\boldsymbol{u}\|_{L^2}^4\|\Delta\boldsymbol{u}\|_{L^2}^2 \\
&\quad + \|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}^2\|\nabla\boldsymbol{u}\|_{L^2}\|\Delta\boldsymbol{u}\|_{L^2} \\
&\qquad\qquad\qquad \text{(Gagliardo–Nirenberg, Sobolev and Young inequalities)} \\
&\le \frac{\mu}{2}\|\nabla\partial_t\boldsymbol{u}\|_{L^2}^2 + C(1+\|\Delta\boldsymbol{u}\|_{L^2}^2)\big(1+\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}^2 + \|\nabla\partial_t\boldsymbol{d}\|_{L^2}^2\big),
\end{aligned} \tag{3.25}$$

where (3.23) was invoked in the above last estimate.

Applying ∂_t to $(1.1)_3$, multiplying by $\partial_t\Delta\boldsymbol{d}$, and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla\partial_t\boldsymbol{d}|^2\,dx + \int|\Delta\partial_t\boldsymbol{d}|^2\,dx \\
&= \int[\partial_t\boldsymbol{d}|\nabla\boldsymbol{d}|^2 + \boldsymbol{d}\partial_t|\nabla\boldsymbol{d}|^2 - (\partial_t\boldsymbol{u}\cdot\nabla)\boldsymbol{d} - (\boldsymbol{u}\cdot\nabla)\partial_t\boldsymbol{d}]\cdot\Delta\partial_t\boldsymbol{d}\,dx \\
&\le \|\partial_t\boldsymbol{d}\|_{L^6}\|\nabla\boldsymbol{d}\|_{L^6}^2\|\Delta\partial_t\boldsymbol{d}\|_{L^2} + 2\|\boldsymbol{d}\|_{L^\infty}\|\nabla\boldsymbol{d}\|_{L^6}\|\nabla\partial_t\boldsymbol{d}\|_{L^3}\|\Delta\partial_t\boldsymbol{d}\|_{L^2} \\
&\quad + \|\partial_t\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{d}\|_{L^3}\|\Delta\partial_t\boldsymbol{d}\|_{L^2} + \|\boldsymbol{u}\|_{L^6}\|\nabla\partial_t\boldsymbol{d}\|_{L^3}\|\Delta\partial_t\boldsymbol{d}\|_{L^2} \\
&\le C\|\nabla\partial_t\boldsymbol{d}\|_{L^2}\|\Delta\boldsymbol{d}\|_{L^2}^2\|\Delta\partial_t\boldsymbol{d}\|_{L^2} + C\|\Delta\boldsymbol{d}\|_{L^2}\|\nabla\partial_t\boldsymbol{d}\|_{L^2}^{1/2}\|\Delta\partial_t\boldsymbol{d}\|_{L^2}^{3/2} \\
&\quad + C\|\nabla\partial_t\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{d}\|_{L^2}^{1/2}\|\Delta\boldsymbol{d}\|_{L^2}^{1/2}\|\Delta\partial_t\boldsymbol{d}\|_{L^2} + \|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\partial_t\boldsymbol{d}\|_{L^2}^{1/2}\|\Delta\partial_t\boldsymbol{d}\|_{L^2}^{3/2} \\
&\le \frac{1}{2}\|\Delta\partial_t\boldsymbol{d}\|_{L^2}^2 + C\|\nabla\partial_t\boldsymbol{u}\|_{L^2}^2 + C\|\nabla\partial_t\boldsymbol{d}\|_{L^2}^2 \qquad \text{(by (3.23))}.
\end{aligned} \tag{3.26}$$

Multiplying (3.25) by $4C/\mu$ and adding it to (3.26) yields

$$\frac{d}{dt}\int \frac{2C}{\mu}\rho|\partial_t \boldsymbol{u}|^2 + |\nabla\partial_t \boldsymbol{d}|^2\,dx + C\int|\nabla\partial_t\boldsymbol{u}|^2\,dx + \int|\Delta\partial_t\boldsymbol{d}|^2\,dx \\ \le C(1+\|\Delta\boldsymbol{u}\|_{L^2}^2)\big(1+\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}^2+\|\nabla\partial_t\boldsymbol{d}\|_{L^2}^2\big).$$

Invoking the Gronwall inequality, we deduce

$$\begin{aligned}&\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^\infty(0,T;L^2)}\le C, \quad \|\nabla\partial_t\boldsymbol{u}\|_{L^2(0,T;L^2)}\le C,\\ &\|\partial_t\boldsymbol{d}\|_{L^\infty(0,T;H^1)}\le C, \quad \|\partial_t\boldsymbol{d}\|_{L^2(0,T;H^2)}\le C.\end{aligned} \tag{3.27}$$

With estimates (3.23) and (3.27), we may use $(1.1)_3$ and the Gagliardo–Nirenberg inequality to bound

$$\begin{aligned}\|\nabla\Delta\boldsymbol{d}\|_{L^2} &= \|\nabla\partial_t\boldsymbol{d}+(\nabla\boldsymbol{u}\cdot\nabla)\boldsymbol{d}+(\boldsymbol{u}\cdot\nabla)(\nabla\boldsymbol{d})-\nabla(\boldsymbol{d}|\nabla\boldsymbol{d}|^2)\|_{L^2}\\ &\le C\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{d}\|_{L^\infty}+\|\boldsymbol{u}\|_{L^6}\|\nabla^2\boldsymbol{d}\|_{L^3}\\ &\quad+\|\nabla\boldsymbol{d}\|_{L^6}^3+2\|\boldsymbol{d}\|_{L^\infty}\|\nabla\boldsymbol{d}\|_{L^\infty}\|\nabla^2\boldsymbol{d}\|_{L^2}\\ &\le C+C\|\nabla\boldsymbol{d}\|_{L^\infty}+C\|\Delta\boldsymbol{d}\|_{L^3}\\ &\le C+C\|\nabla\boldsymbol{d}\|_{L^6}^{1/2}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^{1/2}+C\|\Delta\boldsymbol{d}\|_{L^2}^{1/2}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^{1/2}\\ &\le C+C\|\nabla\Delta\boldsymbol{d}\|_{L^2}^{1/2}\le C+\frac{1}{2}\|\nabla\Delta\boldsymbol{d}\|_{L^2}.\end{aligned}$$

Consequently,

$$\|\nabla\boldsymbol{d}\|_{L^\infty(0,T;H^2)}\le C. \tag{3.28}$$

To get the uniform bound of $\boldsymbol{u}$ in $L^\infty(0,T;H^2(\mathbb{R}^3))$, we use (3.20) once more:

$$\begin{aligned}\|\Delta\boldsymbol{u}\|_{L^2} &\le C\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^2}+C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}+C\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\|_{L^2}\\ &\le C\|\nabla\boldsymbol{d}\|_{L^\infty}\|\Delta\boldsymbol{d}\|_{L^2}+C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}+C\|\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^2}\\ &\le C\|\nabla\boldsymbol{d}\|_{L^6}^{1/2}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^{1/2}+C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}+C\|\boldsymbol{u}\|_{L^6}^{1/2}\|\Delta\boldsymbol{u}\|_{L^2}^{1/2}\\ &\qquad\qquad(\text{by } \|f\|_{L^\infty}\le C\|f\|_{L^6}^{1/2}\|\Delta f\|_{L^2}^{1/2},\text{ and (3.23)})\\ &\le C\|\nabla\Delta\boldsymbol{d}\|_{L^2}^{1/2}+C\|\sqrt{\rho}\,\partial_t\boldsymbol{u}\|_{L^2}+C+\frac{1}{2}\|\Delta\boldsymbol{u}\|_{L^2},\end{aligned}$$

and hence by (3.27) and (3.28),

$$\|\nabla\boldsymbol{u}\|_{L^\infty(0,T;H^1)}\le C. \tag{3.29}$$

By the H^2-theory of the Stokes system again, we get

$$\begin{aligned}\|\Delta\boldsymbol{u}\|_{L^6} &\le C\big\||\nabla\boldsymbol{d}|\cdot|\nabla^2\boldsymbol{d}|\big\|_{L^6}+\|\partial_t\boldsymbol{u}\|_{L^6}+\big\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\big\|_{L^6}\\ &\le C\|\nabla\boldsymbol{d}\|_{L^\infty}\|\Delta\boldsymbol{d}\|_{L^6}+C\|\nabla\partial_t\boldsymbol{u}\|_{L^2}+C\|\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^6}\\ &\le C\|\nabla\boldsymbol{d}\|_{L^6}^{1/2}\|\nabla\Delta\boldsymbol{d}\|_{L^2}^{3/2}+C\|\nabla\partial_t\boldsymbol{u}\|_{L^2}+C\|\boldsymbol{u}\|_{L^6}^{1/2}\|\Delta\boldsymbol{u}\|_{L^6}^{3/2},\end{aligned}$$

and thus

$$\|\nabla\boldsymbol{u}\|_{L^2(0,T;W^{1,6})}\le C. \tag{3.30}$$

Applying ∇ to $(1.1)_1$, multiplying by $|\nabla\rho|^{q-2}\nabla\rho$, and integrating over $\mathbb{R}^3$, we conclude

$$\frac{d}{dt}\int|\nabla\rho|^q\,dx \le C\|\nabla\boldsymbol{u}\|_{L^\infty}\int|\nabla\rho|^q\,dx \le C\|\nabla\boldsymbol{u}\|_{W^{1,6}}\int|\nabla\rho|^q\,dx.$$

The Gronwall inequality together with (3.30) then yields

$$\|\nabla\rho\|_{L^\infty(0,T;L^2\cap L^q)} \le C. \tag{3.31}$$

Hence,

$$\partial_t\rho = -(\boldsymbol{u}\cdot\nabla)\rho \;\Rightarrow\; \|\partial_t\rho\|_{L^\infty(0,T;L^2\cap L^q)} \le C. \tag{3.32}$$

Now that we have the uniform estimates (3.23), (3.24), (3.27)–(3.32), it is standard to extend the strong solution smoothly beyond T, and thus conclude the proof of Theorem 1.1.

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