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The natural operators lifting projectable vector fields to some fiber product preserving bundles

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Abstract. Admissible fiber product preserving bundle functors F on $\mathcal{FM}_m$ are defined. For every admissible fiber product preserving bundle functor F on $\mathcal{FM}_m$ all natural operators $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \to TF$ lifting projectable vector fields to F are classified.

Introduction. In [4], the authors classified all fiber product preserving bundle functors $F : \mathcal{FM}_m \to \mathcal{FM}$ from the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fiber preserving maps with local diffeomorphisms as base maps into the category $\mathcal{FM}$ of fibered manifolds and fibered maps. All such functors of order r are in bijection with triples (A, H, t), where A is a Weil algebra of order r, H is a group homomorphism from the rth jet group G^r_m into the group $\mathrm{Aut}(A)$ of all automorphisms of A, and t is a G^r_m-invariant algebra homomorphism from the algebra $\mathcal{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ of all r-jets of $\mathbb{R}^m$ into $\mathbb{R}$ with source $0 \in \mathbb{R}^m$ into A. The natural transformations $F_1 \to F_2$ of two fiber product preserving bundle functors F_1 and F_2 on $\mathcal{FM}_m$ are in bijection with the morphisms between corresponding triples.

The most importrant example of such a functor F is the r-jet prolongation functor $J^r : \mathcal{FM}_m \to \mathcal{FM}$. The corresponding triple (A, H, t) is $(\mathcal{D}^r_m, \mathrm{id}_{G^r_m}, \mathrm{id}_{\mathcal{D}^r_m})$, where $H : G^r_m \to G^r_m \cong \mathrm{Aut}(\mathcal{D}^r_m)$ is the identity group homomorphism. Another example is the vertical Weil functor $V^A : \mathcal{FM}_m \to \mathcal{FM}$ corresponding to a Weil algebra A. The corresponding triple (A, H, t) is $(A, \mathrm{id}_A, \varepsilon)$, where $\varepsilon : \mathcal{D}^r_m \to A$ is the trivial algebra homomorphism and $\mathrm{id}_A : G^r_m \to \mathrm{Aut}(A)$ is the trivial group homomorphism. The functors J^r and V^A are admissible in the following sense: for every derivation $D \in \mathrm{Der}(A)$,

$$\text{if } H(j^r_0(\tau\,\mathrm{id}_{\mathbb{R}^m})) \circ D \circ H(j^r_0(\tau^{-1}\,\mathrm{id}_{\mathbb{R}^m})) \to 0 \text{ as } \tau \to 0 \text{ then } D = 0.$$

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Another example of an admissible fiber product preserving bundle functor is the non-holonomic r-jet prolongation bundle functor $\widetilde{J}^r : \mathcal{FM}_m \to \mathcal{FM}$ in the sense of C. Ehresmann [1]. All extensions of J^r and $\widetilde{J}^r$ in the sense of I. Kolář [2] are also admissible.

Let $\mathcal{FM}_{m,n} \subset \mathcal{FM}_m$ be the subcategory of all fibered manifolds with m-dimensional basis and n-dimensional fibers and local $\mathcal{FM}_m$-isomorphisms. In [5], Kolář and Slovák studied the problem of how a projectable vector field on an $\mathcal{FM}_{m,n}$-object Y induces a vector field $B(X)$ on J^rY. This problem is reflected in the notion of natural operators $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TJ^r$. They proved that every such B is a constant multiple of the flow operator $\mathcal{J}^r$. The similar problem with V^A playing the role of J^r has also been studied. Every natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TV^A$ is a constant multiple of the flow operator $\mathcal{V}^A$ plus an absolute operator $\mathrm{op}(D)$ for some $D \in \mathrm{Lie}(\mathrm{Aut}(A)) = \mathrm{Der}(A)$.

In the present paper we generalize the above results to a (large) class of admissible fiber product preserving bundle functors on $\mathcal{FM}_m$. The main result of this paper is that for an admissible fiber product preserving bundle functor $F : \mathcal{FM}_m \to \mathcal{FM}$ every natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TF$ is of the form

$$B = \lambda\mathcal{F} + \mathrm{op}(D)$$

for some $\lambda \in \mathbb{R}$ and $D \in \mathrm{Lie}(\mathrm{Aut}(A, H, t))$, where $\mathcal{F}$ is the flow operator of F. For $F = J^r$ and $F = V^A$ we recover the above-mentioned results of [5], [3].

We also present a conterexample showing that the assumption of admissibility of F is essential.

All manifolds are assumed to be without boundary, finite-dimensional and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

1. Fiber product preserving bundle functors. Suppose $F : \mathcal{FM}_m \to \mathcal{FM}$ is a bundle functor. We say that F is *fiber product preserving* if $F(Y_1 \times_M Y_2)_x \cong F(Y_1)_x \times F(Y_2)_x$ for any $\mathcal{FM}_m$-objects $Y_1 \to M$ and $Y_2 \to M$ and every $x \in M$.

The most important example of a fiber product preserving bundle functor F on $\mathcal{FM}_m$ is the r-jet prolongation functor $J^r : \mathcal{FM}_m \to \mathcal{FM}$. Another example is the vertical Weil functor $V^A : \mathcal{FM}_m \to \mathcal{FM}$ corresponding to a Weil algebra A. One more example is the non-holonomic r-jet prolongation bundle functor $\widetilde{J}^r : \mathcal{FM}_m \to \mathcal{FM}$ in the sense of C. Ehresmann [1]. The extensions of J^r and $\widetilde{J}^r$ in the sense of I. Kolář [2] are also fiber product preserving bundle functors on $\mathcal{FM}_m$.

A complete description of the fiber product preserving bundle functors on $\mathcal{FM}_m$ has been given in [4]. We will recall it in Sections 2, 3 and 4.

2. Fiber product preserving bundle functors and induced triples. Suppose F is a fiber product preserving bundle functor on $\mathcal{FM}_m$ of finite order r. The functor F induces both a product preserving bundle functor G^F on the category $\mathcal{M}f$ of manifolds by

$$G^F N = F_0(\mathbb{R}^m \times N), \quad G^F f = F_0(\mathrm{id}_{\mathbb{R}^m} \times f) : G^F N \to G^F P,$$

for every manifold N and every smooth map $f : N \to P$, and a group homomorphism $H^F : G^r_m \to \mathrm{Aut}(G^F)$ by

$$H^F(\xi)_N = F_0(\varphi \times \mathrm{id}_N) : G^F N \to G^F N$$

for every $\xi = j^r_0\varphi \in G^r_m$ and every manifold N, where $\mathrm{Aut}(G^F)$ is the group of natural automorphisms (equivalences) of G^F into itself and $G^r_m = \mathrm{inv}\, J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ is the r-jet group. By the general theory of product preserving bundle functors on $\mathcal{M}f$ (see [3]), we obtain a Weil algebra A^F by setting

$$A^F = (G^F\mathbb{R}, G^F(+), G^F(\cdot), G^F(0), G^F(1)),$$

and a group homomorphism $H^F : G^r_m \to \mathrm{Aut}(A^F)$ by defining

$$H^F(\xi) = H^F(\xi)_{\mathbb{R}} : A^F \to A^F$$

for every $\xi \in G^r_m$. Moreover the functor F defines an algebra homomorphism $t^F : \mathcal{D}^r_m \to A^F$ by

$$\{t^F(\xi)\} = \mathrm{im}(F_0(\mathrm{id}_{\mathbb{R}^m}, f))$$

for every $\xi = j^r_0 f \in \mathcal{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$.

Thus every fiber product preserving bundle functor F on $\mathcal{FM}_m$ of order r determines a triple (A^F, H^F, t^F), where A^F is a Weil algebra of order r, H^F is a group homomorphism from G^r_m into $\mathrm{Aut}(A^F)$, and t^F is a G^r_m-invariant algebra homomorphism from $\mathcal{D}^r_m$ into A^F.

If $\overline{F}$ is another fiber product preserving bundle functor on $\mathcal{FM}_m$ of order r and $\eta : F \to \overline{F}$ is a natural transformation, then we have a morphism $\sigma^\eta : (A^F, H^F, t^F) \to (A^{\overline{F}}, H^{\overline{F}}, t^{\overline{F}})$ of triples, where $\sigma^\eta : A^F \to A^{\overline{F}}$ is the restriction and corestriction of $\eta_{\mathbb{R}^m \times \mathbb{R}} : F(\mathbb{R}^m \times \mathbb{R}) \to \overline{F}(\mathbb{R}^m \times \mathbb{R})$.

3. Triples and induced fiber product preserving bundle functors. Conversely, suppose we have a triple (A, H, t), where A is a Weil algebra of order r, H is a group homomorphism from G^r_m into $\mathrm{Aut}(A)$, and t is a G^r_m-invariant algebra homomorphism from $\mathcal{D}^r_m$ into A. By the general theory of product preserving bundle functors on $\mathcal{M}f$, A determines the Weil functor $T^A : \mathcal{M}f \to \mathcal{FM}$ which is product preserving, H determines the group homomorphism $H : G^r_m \to \mathrm{Aut}(T^A)$ from G^r_m into the group $\mathrm{Aut}(T^A)$

of all natural automorphisms of T^A into itself, and t determines the natural transformation $t : T^r_m \to T^A$, where $T^r_m = T^{\mathcal{D}^r_m} = J^r_0(\mathbb{R}^m, \cdot) : \mathcal{M}f \to \mathcal{FM}$ is the Weil functor (corresponding to $\mathcal{D}^r_m$) of (m, r)-velocities. For every $\mathcal{FM}_m$-object $p : Y \to M$ we have the bundle

$$F^{(A,H,t)}Y = \{\{u, X\} \in P^r M[T^A Y, H] \mid t_M(u) = T^A p(X)\}$$

over Y, where $P^r M = \operatorname{inv} J^r_0(\mathbb{R}^m, M) \subset T^r_m M$ is the principal fiber bundle with structure group G^r_m acting on $P^r M$ by jet composition, $T^A Y$ is the left G^r_m-space by means of H, and $P^r M[T^A Y, H]$ is the associated fiber bundle. For every $\mathcal{FM}_m$-map $f : Y_1 \to Y_2$ covering $\varphi : M_1 \to M_2$ we have the induced map $P^r\varphi[T^A f, H] : P^r M_1[T^A Y_1, H] \to P^r M_2[T^A Y_2, H]$ sending $F^{(A,H,t)}Y_1$ into $F^{(A,H,t)}Y_2$, and (by restriction and corestriction) we have the fibered map $F^{(A,H,t)}f : F^{(A,H,t)}Y_1 \to F^{(A,H,t)}Y_2$ covering f.

Thus every triple (A, H, t), where A is a Weil algebra of order r, H is a group homomorphism from G^r_m into $\operatorname{Aut}(A)$, and t is a G^r_m-invariant algebra homomorphism from $\mathcal{D}^r_m$ into A, induces a fiber product preserving bundle functor $F^{(A,H,t)} : \mathcal{FM}_m \to \mathcal{FM}$ of order r.

If $(\overline{A}, \overline{H}, \overline{t})$ is another triple of order r and $\sigma : (A, H, t) \to (\overline{A}, \overline{H}, \overline{t})$ is a morphism of triples then we have a natural transformation $\eta^\sigma : F^{(A,H,t)} \to F^{(\overline{A},\overline{H},\overline{t})}$, where $\eta^\sigma_Y : F^{(A,H,t)}Y \to F^{(\overline{A},\overline{H},\overline{t})}Y$ is the restriction and corestriction of $\mathrm{id}_{P^r M}[\sigma_Y] : P^r M[T^A Y, H] \to P^r M[T^{\overline{A}} Y, \overline{H}]$ for any $\mathcal{FM}_m$-object $Y \to M$.

4. Classification of fiber product preserving bundle functors. The main result of [4] is the following classification theorem.

THEOREM 1 ([4]). (i) *Every fiber product preserving bundle functor F on $\mathcal{FM}_m$ is of some finite order r.*

(ii) *The correspondence $F \mapsto (A^F, H^F, t^F)$ induces a bijection between the equivalence classes of fiber product preserving bundle functors on $\mathcal{FM}_m$ of order r and the equivalence classes of triples of order r. The inverse bijection is determined by $(A, H, t) \mapsto F^{(A,H,t)}$.*

(iii) *The natural transformations $F_1 \to F_2$ of two fiber product preserving bundle functors F_1 and F_2 on $\mathcal{FM}_m$ of order r are in bijection with the morphisms between corresponding triples. An example of such a bijection is $\eta \mapsto \sigma^\eta$.*

5. The Lie algebra of $\operatorname{Aut}(A, H, t)$**.** Consider a triple (A, H, t), where A is a Weil algebra of order r, H is a group homomorphism from G^r_m into $\operatorname{Aut}(A)$, and t is a G^r_m-invariant algebra homomorphism from $\mathcal{D}^r_m$ into A. We note that $\operatorname{Aut}(A, H, t)$ is a closed (and hence Lie) subgroup in $GL(A)$.

PROPOSITION 1. $\operatorname{Lie}(\operatorname{Aut}(A, H, t)) = \{D \in \operatorname{Der}(A) \mid D \circ t = 0,\ H(\xi) \circ D = D \circ H(\xi)$ *for all* $\xi \in G^r_m\}$.

Proof. By [3], $\mathrm{Lie}(\mathrm{Aut}(A)) = \mathrm{Der}(A)$. Clearly, $\sigma \in \mathrm{Aut}(A, H, t)$ iff $\sigma \in \mathrm{Aut}(A)$ and $\sigma \circ t = t$ and $H(\xi) \circ \sigma = \sigma \circ H(\xi)$ for any $\xi \in G^r_m$. Analysing 1-parameter subgroups in $\mathrm{Aut}(A, H, t)$ we end the proof. ■

6. Natural transformations of $F_{|\mathcal{FM}_{m,n}}$ into itself. In this section we prove the following theorem.

THEOREM 2. *Let $F : \mathcal{FM}_m \to \mathcal{FM}$ be a fiber product preserving bundle functor and (A, H, t) be its triple. Every natural transformation η of $F_{|\mathcal{FM}_{m,n}}$ into itself can be extended to a unique natural transformation of F into itself. In particular*, $\mathrm{Aut}(F_{|\mathcal{FM}_{m,n}}) = \mathrm{Aut}(A, H, t)$.

Proof. Let $x^1, \dots, x^m, y^1, \dots, y^n$ be the coordinates on the $\mathcal{FM}_{m,n}$-object $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, the trivial bundle.

Consider a natural transformation η of $F_{|\mathcal{FM}_{m,n}}$ into itself. Since F is fiber product preserving, $F_0(\mathbb{R}^m \times \mathbb{R}^n) = A^n$. Thus η is uniquely determined by the restriction and corestriction $\eta : A^n \to A^n$. Write $\eta(a_1, \dots, a_n) = (\eta^1(a_1, \dots, a_n), \dots, \eta^n(a_1, \dots, a_n))$ for $a_1, \dots, a_n \in A$.

By the invariance of η with respect to the $\mathcal{FM}_{m,n}$-morphisms $(x^1, \dots, x^m, \tau_1 y^1, \dots, \tau_n y^n)$ for $\tau_1, \dots, \tau_n \in \mathbb{R}_+$ we get the homogeneity conditions $\tau_j \eta^j(a_1, \dots, a_n) = \eta^j(\tau_1 a_1, \dots, \tau_n a_n)$ for $j = 1, \dots, n$ and any $a_1, \dots, a_n \in A$ and any $\tau_1, \dots, \tau_n \in \mathbb{R}_+$. This type of homogeneity implies that η^j depends linearly on a_j by the homogeneous function theorem [3].

Using permutations of fibered coordinates we deduce that $\eta = \sigma \times \dots \times \sigma$ for $\sigma = \eta_1 : A \to A$.

We prove that $\sigma \in \mathrm{Morph}(A, H, t)$.

STEP 1. *σ is an algebra homomorphism.*

We know that σ is $\mathbb{R}$-linear. Using the invariance of η with respect to the local $\mathcal{FM}_{m,n}$-morphism $(x^1, \dots, x^m, y^1 + (y^1)^2, y^2, \dots, y^n)$ we derive that $\sigma(a + a^2) = \sigma(a) + (\sigma(a))^2$, i.e. $\sigma(a^2) = (\sigma(a))^2$ for any $a \in A$. Then $\sigma((a_1 + a_2)^2) = (\sigma(a_1 + a_2))^2$, i.e. $\sigma(a_1 a_2) = \sigma(a_1)\sigma(a_2)$ for any $a_1, a_2 \in A$. So, σ is multiplicative.

Using the invariance of η with respect to the $\mathcal{FM}_{m,n}$-map $(x^1, \dots, x^m, y^1 + 1, y^2, \dots, y^n)$ we derive that $\sigma(a + 1) = \sigma(a) + 1$, i.e. $\sigma(1) = 1$.

These facts show that σ is an algebra homomorphism.

STEP 2. *σ is G^r_m-equivariant.*

Using the invariance of η with respect to the $\mathcal{FM}_{m,n}$-maps $\varphi \times \mathrm{id}_{\mathbb{R}^n}$ for $\varphi \in \mathrm{Diff}(\mathbb{R}^m, \mathbb{R}^m)$ with $\varphi(0) = 0$ we obtain $H(\xi) \circ \sigma = \sigma \circ H(\xi)$ for any $\xi = j^r_0 \varphi \in G^r_m$. So, σ is G^r_m-equivariant.

STEP 3. *$\sigma \circ t = t$.*

By the invariance of η with respect to the $\mathcal{FM}_{m,n}$-morphisms $(x^1,\dots,$ x^m, $f(x^1,\dots,x^m)+\tau y^1,\dots,\tau y^n)$ for any $\tau\in\mathbb{R}_+$ and any $f:\mathbb{R}^n\to\mathbb{R}$ and next letting $\tau\to 0$ we get $\sigma\circ t(j^r_0 f)=t(j^r_0 f)$. Hence $\sigma\circ t=t$.

We have proved that $\sigma\in\mathrm{Morph}(A,H,t)$. By Theorem 1 we have the natural transformation $\eta^\sigma:F\to F$ corresponding to σ. Clearly, η is the restriction of η^σ.

If $\widetilde{\eta}:F\to F$ is another such transformation, then $\widetilde{\eta}=\eta^\sigma$ because $\widetilde{\eta}$ coincides with η^σ on $A=A\times\{0\}\subset A^n$. ∎

7. Absolute operators. Let $F:\mathcal{FM}_m\to\mathcal{FM}$ be a fiber product preserving bundle functor and let (A,H,t) be its triple. We have the following example of absolute natural operators $T_{\mathrm{proj}|\mathcal{FM}_{m,n}}\rightsquigarrow TF$.

Example 1 (The operators $\mathrm{op}(D)$). Let $D\in\mathrm{Lie}(\mathrm{Aut}(A,H,t))$. Let σ_τ be the 1-parameter subgroup in $\mathrm{Aut}(A,H,t)$ corresponding to D. By Theorem 1 we have the corresponding 1-parameter subgroup η^{σ_τ} of natural equivalences of F. So, for every $\mathcal{FM}_{m,n}$-object Y we have a flow η^{σ_τ} on FY. This flow defines a vector field $\mathrm{op}(D)$ on FY. The correspondence $\mathrm{op}(D):T_{\mathrm{proj}|\mathcal{FM}_{m,n}}\rightsquigarrow TF$ is an *absolute* (i.e. constant) natural operator.

We have the following classification of absolute operators.

Proposition 2. *Let $F:\mathcal{FM}_m\to\mathcal{FM}$ be a fiber product preserving bundle functor and (A,H,t) be its triple. Every absolute natural operator $B:T_{\mathrm{proj}|\mathcal{FM}_{m,n}}\rightsquigarrow TF$ is $\mathrm{op}(D)$ for some $D\in\mathrm{Lie}(\mathrm{Aut}(A,H,t))$.*

Proof. Consider an absolute natural operator $B:T_{\mathrm{proj}|\mathcal{FM}_{m,n}}\rightsquigarrow TF$. For every $\mathcal{FM}_{m,n}$-object Y we have a vector field B on FY invariant with respect to $\mathcal{FM}_{m,n}$-maps. The flow Fl^B_τ of B is $\mathcal{FM}_{m,n}$-invariant. Using Theorem 1 we can easily show that there exists $v\in FY$ such that FY is the orbit of U with respect to $\mathcal{FM}_{m,n}$-maps for any open neighbourhood $U\subset FY$ of v. This implies that B is complete, i.e. the flow Fl^B_τ is global. Hence the flow corresponds to some 1-parameter subgroup in $\mathrm{Aut}(F_{|\mathcal{FM}_{m,n}})$. By Theorem 2, we have the corresponding 1-parameter subgroup $\sigma^{\mathrm{Fl}^B_\tau}$ in $\mathrm{Aut}(A,H,t)$. This subgroup corresponds to some $D\in\mathrm{Lie}(\mathrm{Aut}(A,H,t))$. Thus $B=\mathrm{op}(D)$. ∎

The triple corresponding to J^r is $(\mathcal{D}^r_m,\mathrm{id}_{G^r_m},\mathrm{id}_{\mathcal{D}^r_m})$, where $\mathrm{id}_{G^r_m}:G^r_m\to G^r_m\cong\mathrm{Aut}(\mathcal{D}^r_m)$ is the identity. By Proposition 1, $\mathrm{Lie}(\mathrm{Aut}(\mathcal{D}^r_m,\mathrm{id}_{G^r_m},\mathrm{id}_{\mathcal{D}^r_m}))=\{0\}$. Therefore we have the following corollary.

Corollary 1. *Every absolute operator on $J^r_{|\mathcal{FM}_{m,n}}$ is* 0.

Let $J^r_v:\mathcal{FM}_m\to\mathcal{FM}$ be the vertical extension of J^r (see [2]). We recall that $J^r_v(Y)=\bigcup_{x\in M}J^r_x(M,Y_x)$ for every $\mathcal{FM}_m$-object $Y\to M$. The triple of J^r_v is $(\mathcal{D}^r_m,\mathrm{id}_{G^r_m},\varepsilon)$, where $\varepsilon:\mathcal{D}^r_m\to\mathcal{D}^r_m$ is the trivial algebra homomorphism (equal to 0 on the nilpotent ideal; see [4]). By Proposition 1 we have $\mathrm{Lie}(\mathrm{Aut}(\mathcal{D}^r_m, \mathrm{id}_{G^r_m}, \varepsilon)) = \{D \in \mathrm{Der}(\mathcal{D}^r_m) \mid \xi \circ D = D \circ \xi \text{ for all } \xi \in G^r_m\}$. As an easy exercise one can compute that $\{D \in \mathrm{Der}(\mathcal{D}^r_m) \mid \xi\circ D = D\circ\xi$ for all $\xi \in G^r_m\} = \{0\}$ if $r \geq 2$. For $r = 1$ we have $\{D \in \mathrm{Der}(\mathcal{D}^1_m) \mid \xi \circ D = D\circ\xi \text{ for all } \xi \in G^1_m\} = \mathbb{R}D^1_m$, where $D^1_m \in \mathrm{Der}(\mathcal{D}^1_m)$ is the unique derivation such that $D^1_m(j^1_0(x^i)) = j^1_0(x^i)$ for $i = 1, \ldots, m$. (Here $x^1, \ldots, x^m$ are the usual coordinates on $\mathbb{R}^m$.) Therefore we have the following corollary.

Corollary 2. (i) *Every absolute operator on* $J^1_{v\,|\mathcal{FM}_{m,n}}$ *is a constant multiple of* $\mathrm{op}(D^1_m)$.

(ii) *For* $r \geq 2$ *every absolute operator on* $J^r_{v\,|\mathcal{FM}_{m,n}}$ *is* 0.

Remark 1. We have the following geometrical interpretation of $\mathrm{op}(D^1_m)$. For every $\mathcal{FM}_{m,n}$-object $p : Y \to M$, $J^1_v Y = \bigcup_{w\in Y}(T^*_{p(w)}M \otimes T_w Y_{p(w)})$ is a vector bundle over Y. The Liouville vector field L on the vector bundle $J^1_v Y$ is $\mathrm{op}(D^1_m)$.

The triple corresponding to V^A is $(A, \mathrm{id}_A, \varepsilon)$, where $\mathrm{id}_A : G^r_m \to \mathrm{Aut}(A)$ is the trivial group homomorphism and $\varepsilon : \mathcal{D}^r_m \to A$ is the trivial algebra homomorphism. By Proposition 1 we obtain $\mathrm{Lie}(\mathrm{Aut}(A, \mathrm{id}_A, \varepsilon)) = \mathrm{Lie}(\mathrm{Aut}(A)) = \mathrm{Der}(A)$. So, we have the following corollary.

Corollary 3. *Every absolute operator on* $V^A_{|\mathcal{FM}_{m,n}}$ *is* $\mathrm{op}(D)$ *for some* $D \in \mathrm{Der}(D) = \mathrm{Lie}(\mathrm{Aut}(A)) = \mathrm{Lie}(\mathrm{Aut}(A, \mathrm{id}_A, \varepsilon))$.

Since every natural transformation of $\widetilde{J}^2 = J^1 \circ J^1$ into itself is the identity (see [3]), then $\mathrm{Lie}(\mathrm{Aut}(A^{\widetilde{J}^2}, H^{\widetilde{J}^2}, t^{\widetilde{J}^2})) = \{0\}$. So, we have the following corollary.

Corollary 4. *Every absolute operator on* $\widetilde{J}^2_{|\mathcal{FM}_{m,n}}$ *is* 0.

8. Admissible fiber product preserving bundle functors. Suppose that $F : \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor of order r and (A, H, t) is its corresponding triple. We say that F is *admissible* if the following condition is satisfied: for every derivation $D \in \mathrm{Der}(A)$,

$$\text{if } H(j^r_0(\tau\,\mathrm{id}_{\mathbb{R}^m})) \circ D \circ H(j^r_0(\tau^{-1}\,\mathrm{id}_{\mathbb{R}^m})) \to 0 \text{ as } \tau \to 0 \text{ then } D = 0.$$

Lemma 1. (i) *The functors* J^r *and* $\widetilde{J}^r$ *and their extensions in the sense of* [2] *are admissible.*

(ii) *All vertical Weil functors* V^A *are admissible.*

Proof. The triple corresponding to J^r is $(\mathcal{D}^r_m, \mathrm{id}_{G^r_m}, \mathrm{id}_{\mathcal{D}^r_m})$. Consider $D \in \mathrm{Der}(\mathcal{D}^r_m)$ such that $j^r_0(\tau\,\mathrm{id}_{\mathbb{R}^m}) \circ D \circ j^r_0(\tau^{-1}\,\mathrm{id}_{\mathbb{R}^m}) \to 0$ as $\tau \to 0$. Let $x^1, \ldots, x^m$ be the usual coordinates on $\mathbb{R}^m$. For $i = 1, \ldots, m$ we can write $D(j^r_0 x^i) = \sum a^i_\alpha j^r_0(x^\alpha)$ for some real numbers a^i_α, where the sum is over all

$\alpha \in (\mathbb{N} \cup \{0\})^m$ with $0 \le |\alpha| \le r$. We have

$$j^r_0(\tau \operatorname{id}_{\mathbb{R}^m}) \circ D \circ j^r_0(\tau^{-1} \operatorname{id}_{\mathbb{R}^m})(j^r_0(x^i)) = \sum a^i_\alpha \frac{1}{\tau^{|\alpha|-1}} j^r_0(x^\alpha).$$

Then from the assumption on D it follows that $a^i_\alpha = 0$ for all $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $1 \le |\alpha| \le r$, i.e. $D(j^r_0 x^i) = a^i_{(0)} j^r_0 1$ for $i = 1, \ldots, m$. Then (since $j^r_0((x^i)^{r+1}) = 0 \in \mathcal{D}^r_m$ and D is a differentiation) we have

$$\begin{aligned} 0 &= D(j^r_0((x^i)^{r+1})) = D((j^r_0 x^i)^{r+1}) = (r+1)(j^r_0 x^i)^r D(j^r_0 x^i) \\ &= (r+1) a^i_{(0)} j^r_0((x^i)^r). \end{aligned}$$

Then $a^i_{(0)} = 0$ as $j^r_0((x^i)^r) \neq 0 \in \mathcal{D}^r_m$. Then $D(j^r_0 x^i) = 0$ for $i = 1, \ldots, m$. Then $D = 0$ because the $j^r_0 x^i$ for $i = 1, \ldots, m$ generate the algebra $\mathcal{D}^r_m$. Hence J^r is admissible.

The proof of the admissibility of $\widetilde{J}^r$ is left to the reader. First observe that the triple (A, H, t) of $\widetilde{J}^r$ has the following properties: (1) $A = \bigotimes^r \mathcal{D}^1_m$ (see [2]); (2) $H(\xi) = \bigotimes^r \xi$ for $\xi = j^r_0(\tau \operatorname{id}_{\mathbb{R}^m}) \in G^1_m \subset G^r_m$. Then the proof is similar to that for J^r.

Every extension F of J^r is admissible because $(A^F, H^F) = (A^{J^r}, H^{J^r})$ and J^r is admissible. By the same argument every extension of $\widetilde{J}^r$ is admissible.

The admissibility of V^A is a consequence of the fact that the triple of V^A is $(A, \operatorname{id}_A, \varepsilon)$. ∎

In Section 11 we will exhibit a non-admissible fiber product preserving bundle functor.

9. The main result

EXAMPLE 2 (The flow operator). In general, if $E : \mathcal{FM}_{m,n} \to \mathcal{FM}$ is a bundle functor then we have the flow operator $\mathcal{E} : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TE$ lifting projectable vector fields to E. More precisely, if X is a projectable vector field on an $\mathcal{FM}_{m,n}$-object Y then its flow Fl^X_τ is formed by $\mathcal{FM}_{m,n}$-morphisms. The flow $E(\mathrm{Fl}^X_\tau)$ on EY generates $\mathcal{E}(X)$.

The main result of this paper is the following classification theorem.

THEOREM 3. *Let $F : \mathcal{FM}_m \to \mathcal{FM}$ be an admissible fiber product preserving bundle functor and let (A, H, t) be its triple. Every natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TF$ is of the form*

$$B = \lambda \mathcal{F} + \mathrm{op}(D)$$

for some $\lambda \in \mathbb{R}$ and $D \in \mathrm{Lie}(\mathrm{Aut}(A, H, t))$, where $\mathcal{F}$ is the flow operator.

Proof. We assume $F = F^{(A,H,t)}$. We have $\mathbb{R}^m \times T^A \mathbb{R}^n \cong F(\mathbb{R}^m \times \mathbb{R}^n)$, where $\mathbb{R}^m \times \mathbb{R}^n$ is the standard $\mathcal{FM}_{m,n}$-object. The identification is given by

$(x,X)\cong\{j^r_0\tau_x,(t_{\mathbb{R}^m}(j^r_0(\tau_x)),X)\}\in F^{(A,H,t)}(\mathbb{R}^m\times\mathbb{R}^n)$, $x\in\mathbb{R}^m$, $X\in T^A\mathbb{R}^n$, where $\tau_x:\mathbb{R}^m\to\mathbb{R}^m$ is the translation by x. The homothety $\tau\,\mathrm{id}_{\mathbb{R}^m}\times\mathrm{id}_{\mathbb{R}^n}$ for $\tau\neq 0$ sends (x,X) into $(\tau x,H(j^r_0(\tau\,\mathrm{id}_{\mathbb{R}^m}))(X))$. An $\mathcal{FM}_{m,n}$-morphism of the form $\mathrm{id}_{\mathbb{R}^m}\times\psi$ with $\psi\in\mathrm{Diff}(\mathbb{R}^n,\mathbb{R}^n)$ sends (x,X) into $(x,T^A\psi(X))$.

Let $x^1,\dots,x^n,y^1,\dots,y^n$ be the usual coordinates on $\mathbb{R}^m\times\mathbb{R}^n$. The operator B is determined by $B(\partial/\partial x^1)$. We can write

$$B\Big(\mu\,\frac{\partial}{\partial x^1}\Big)_{(x,X)}=\sum_{i=1}^m a^i(\mu,x,X)\,\frac{\partial}{\partial x^i}_{|x}+E(\mu,x,X),$$

where $E_{\mu,x}=E(\mu,x,\cdot)$ is a vector field on $T^A\mathbb{R}^n$ for any $\mu\in\mathbb{R}$ and $x\in\mathbb{R}^m$. The functions a^i and E are smooth.

Using the invariance of $B(\mu\partial/\partial x^1)$ with respect to the $\mathcal{FM}_{m,n}$-morphisms of the form $\mathrm{id}_{\mathbb{R}^m}\times\psi$ with $\psi\in\mathrm{Diff}(\mathbb{R}^n,\mathbb{R}^n)$ we see that $E_{\mu,x}$ is $T^A\psi$-invariant. So, $E_{\mu,x}$ corresponds to some absolute operator on $T^A_{|\mathcal{M}f_n}$. By the result of [3], $E_{\mu,x}=\mathrm{op}(D_{\mu,x})$ for some $D_{\mu,x}\in\mathrm{Der}(A)$. The family $D_{\mu,x}$ depends smoothly on μ and x.

Using the invariance of $B(\mu\partial/\partial x^1)$ with respect to the fiber homotheties $\mathrm{id}_{\mathbb{R}^m}\times\tau\,\mathrm{id}_{\mathbb{R}^n}$ for $\tau\neq 0$ we deduce that $a^i(\mu,x,X)=a^i(\mu,x,\tau X)$, i.e. $a^i(\mu,x,X)=a^i(\mu,x)$ for $i=1,\dots,m$.

Using the invariance of $B(\mu\partial/\partial x^1)$ with respect to the base homotheties $\tau\,\mathrm{id}_{\mathbb{R}^m}\times\mathrm{id}_{\mathbb{R}^n}$ for $\tau\neq 0$ we deduce that $a^i(\tau\mu,\tau x)=\tau a^i(\mu,x)$, i.e. $a^i(\mu,x)$ depends linearly on μ and x by the homogeneous function theorem.

Clearly, $B(0)$ is an absolute operator. So, $B(0)=\mathrm{op}(D)$ for $D\in\mathrm{Lie}(\mathrm{Aut}(A,H,t))$ by Proposition 2.

Now, replacing B by $B-B(0)$ we can write

$$B\Big(\mu\,\frac{\partial}{\partial x^1}\Big)_{(x,X)}=\sum_{i=1}^m a^i\mu\,\frac{\partial}{\partial x^i}_{|x}+\mathrm{op}(D_{\mu,x})_X,$$

where $a^i\in\mathbb{R}$ and $D_{\mu,x}\in\mathrm{Der}(A)$ is a smoothly parametrized family of derivations with $D_{0,0}=0$.

Using the invariance of $B(\mu\partial/\partial x^1)$ with respect to the $\mathcal{FM}_{m,n}$-morphisms $(x^1,\tau x^2,\dots,\tau x^m,y^1,\dots,y^n)$ for $\tau\neq 0$ we get $a^2=\dots=a^m=0$. So replacing B by $B-a^1\mathcal{F}$ we can write

$$B\Big(\mu\,\frac{\partial}{\partial x^1}\Big)_{(x,X)}=\mathrm{op}(D_{\mu,x})_X,$$

where $D_{\mu,x}\in\mathrm{Der}(A)$ is a smoothly parametrized family of derivations with $D_{0,0}=0$.

Using the invariance of $B(\mu\partial/\partial x^1)$ with respect to the base homotheties $\tau\,\mathrm{id}_{\mathbb{R}^m}\times\mathrm{id}_{\mathbb{R}^n}$ we get $\mathrm{op}(D_{\tau\mu,\tau x})_X=(H(j^r_0(\tau\,\mathrm{id}_{\mathbb{R}^m}))_*\,\mathrm{op}(D_{\mu,x}))_X$, i.e.

$$D_{\tau\mu,\tau x}=H(j^r_0(\tau\,\mathrm{id}_{\mathbb{R}^m}))\circ D_{\mu,x}\circ H(j^r_0(\tau^{-1}\,\mathrm{id}_{\mathbb{R}^m}))$$

for any $\tau\neq 0$.

If $\tau \to 0$, then $D_{\tau\mu,\tau x} \to 0$ because $D_{0,0} = 0$. Thus $D_{\mu,x} = 0$ for all $x \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$ because F is admissible. ■

REMARK 2. Observe that from Theorem 3 it follows that under the assumption of the theorem any natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TF$ has order less than or equal to the order of F. This order estimation also follows by the general method from [6].

10. Corollaries. We have the following immediate corollaries of Theorem 3, Corollaries 1–4 and Lemma 1.

COROLLARY 5 ([5]). *Every natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TJ^r$ is a constant multiple of the flow operator $\mathcal{J}^r$.*

COROLLARY 6. (i) *Every natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TJ^1_v$ is a linear combination of the flow operator $\mathcal{J}^1_v$ and $\mathrm{op}(D^1_m)$ with real coefficients.*

(ii) *For $r \geq 2$ every natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TJ^r_v$ is a constant multiple of the flow operator $\mathcal{J}^r_v$.*

COROLLARY 7. *Every natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow TV^A$, where A is a Weil algebra, is of the form $B = \lambda\mathcal{V}^A + \mathrm{op}(D)$ for some $\lambda \in \mathbb{R}$ and $D \in \mathrm{Der}(D) = \mathrm{Lie}(\mathrm{Aut}(A)) = \mathrm{Lie}(\mathrm{Aut}(A, \mathrm{id}_A, 0))$.*

COROLLARY 8. *Every natural operator $B : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T\widetilde{J}^2$ is a constant multiple of the flow operator $\widetilde{\mathcal{J}}^2$.*

11. A counterexample. We show that the assumption of admissibility of F in Theorem 3 is essential.

EXAMPLE 3. Given a fibered manifold $p : Y \to M$ from $\mathcal{FM}_m$ we define a vector bundle

$$F^{\langle r\rangle}Y = \bigcup_{y\in Y} \{j^r_{p(y)}\sigma \mid \sigma : M \to T_yY_{p(y)}\}$$

over Y. For every $\mathcal{FM}_m$-map $f : Y_1 \to Y_2$ covering $\underline{f} : M_1 \to M_2$ we define the induced vector bundle map $F^{\langle r\rangle}f : F^{\langle r\rangle}Y_1 \to F^{\langle r\rangle}Y_2$ covering f by

$$F^{\langle r\rangle}f(j^r_{p(y)}\sigma) = j^r_{\underline{f}(p(y))}(Tf \circ \sigma \circ \underline{f}^{-1})$$

for any $\sigma : M \to T_yY_{1p(y)}$, $y \in Y_1$. Then $F^{\langle r\rangle} : \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor with values in the category $\mathcal{VB}$ of vector bundles.

Let $s = 0, \ldots, r$. Given a projectable vector field X on an $\mathcal{FM}_{m,n}$-object Y covering a vector field $\underline{X}$ on M we define a vertical vector field $V^{\langle s\rangle}(X)$

on $F^{\langle r\rangle}Y$ as follows. Let $j^r_{p(y)}\sigma \in F^{\langle r\rangle}_y Y$, $\sigma : M \to T_y Y_{p(y)}$, $y \in Y$. We put

$$V^{\langle s\rangle}(X)(y) = (y, j^r_{p(y)}(\underline{X}^{(s)}\sigma(p(y)))) \in \{y\} \times F^{\langle r\rangle}_y Y = V_y F^{\langle r\rangle} Y,$$

where $\underline{X}^{(s)} = X \circ \ldots \circ X$ (s times) and $\underline{X}^{(s)}\sigma(p(y)) : M \to T_y Y_{p(y)}$ is the constant map.

The correspondence $V^{\langle s\rangle} : T_{\mathrm{lin}|\mathcal{FM}_{m,n}} \rightsquigarrow TF^{\langle r\rangle}$ is a natural operator.

Of course if $s = 1, \ldots, r$ then $V^{\langle s\rangle}$ is not of the form as in Theorem 3, for $V^{\langle s\rangle}(0) = 0$ and $V^{\langle s\rangle}$ is not absolute.

References

[1] C. Ehresmann, *Extension du calcul des jets aux jets non holonomes*, C. R. Acad. Sci. Paris 239 (1954), 1762–1764.

[2] I. Kolář, *Bundle functors of the jet type*, in: Differential Geometry and Its Applications (Brno, 1988), Masaryk Univ., Brno, 1999, 231–237.

[3] I. Kolář, P. W. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer, 1993.

[4] I. Kolář and W. M. Mikulski, *On the fiber product preserving bundle functors*, Differential Geom. Appl. 11 (1999), 105–115.

[5] I. Kolář and J. Slovák, *Prolongation of vector field to jet bundles*, Suppl. Rend. Circ. Mat. Palermo 22 (1990), 103–111.

[6] A. Zajtz, *On the order of natural differential operators and liftings*, Ann. Polon. Math. 49 (1988), 169–175.

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