

Monge–Ampère boundary measures

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Abstract. We study swept-out Monge–Ampère measures of plurisubharmonic functions and boundary values related to those measures.

1. Introduction. The purpose of this paper is to study certain boundary measures related to plurisubharmonic functions on hyperconvex domains. These measures are obtained as swept-out Monge–Ampère measures and generalize the boundary measures studied by Demailly [14] (see Section 3). A number of properties of the measures, such as density, support and convergence, are given in Section 4. The idea is then to use these measures to define and study boundary values of plurisubharmonic functions on the given domain. This is done in Section 5, where we also describe some situations where this coincides with other notions of boundary values. Finally, in Section 6 we study more general boundary measures on a more restricted class of hyperconvex domains. Here we start with a measure on the boundary and find a sequence of Monge–Ampère measures approximating it.

Results from this paper have been applied in [2].

2. Preliminaries. We first recall some definitions needed in this paper. Let Ω be a domain in \mathbb{C}^n , $n \geq 2$. Denote by $\text{PSH}(\Omega)$ the plurisubharmonic functions on Ω and by $\text{PSH}^-(\Omega)$ the subclass of nonpositive functions. A set $\Omega \subset \mathbb{C}^n$ is said to be a *hyperconvex domain* if it is open, connected and if there exists a function $\varphi \in \text{PSH}^-(\Omega)$ such that $\{z \in \Omega : \varphi(z) < -c\} \subset\subset \Omega$ for all $c > 0$. If Ω is a bounded hyperconvex domain, then it can be shown that the exhaustion function φ can be chosen in $C^\infty(\Omega) \cap C(\bar{\Omega})$ and such that $\int_\Omega (dd^c\varphi)^n < \infty$ (see [11]). This implies for example that the classes defined below are nontrivial. Unless otherwise stated, throughout this paper, Ω will denote a bounded hyperconvex do-

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main in \mathbb{C}^n . Also, by a *measure* we mean a positive regular Borel measure.

Let $\mathcal{E}_0(\Omega)$, $\mathcal{F}(\Omega)$, $\mathcal{E}(\Omega)$ and $\mathcal{F}^a(\Omega)$ be the subclasses of $\text{PSH}^-(\Omega)$ defined as in [6] and [8], namely:

- $\mathcal{E}_0(\Omega)$ is the set of functions $u \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $\int_\Omega (dd^c u)^n < \infty$ and $\lim_{z \rightarrow \xi} u(z) = 0$ for all $\xi \in \partial\Omega$.
- $\mathcal{F}(\Omega)$ is the set of functions $u \in \text{PSH}(\Omega)$ such that there is a sequence $\{u_j\}$ in $\mathcal{E}_0(\Omega)$ with $u_j \searrow u$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$.
- $\mathcal{E}(\Omega)$ is the set of functions $u \in \text{PSH}(\Omega)$ such that for each $\omega \subset\subset \Omega$ there is a function $u_\omega \in \mathcal{F}(\Omega)$ with $u_\omega \geq u$ on Ω and $u_\omega = u$ on ω .
- $\mathcal{F}^a(\Omega)$ is the set of functions $u \in \mathcal{F}(\Omega)$ such that $\int_E (dd^c u)^n = 0$ for each pluripolar set $E \subset \Omega$.

For the convenience of the reader, we state some results concerning these classes, which we use most frequently in this paper. If not indicated otherwise, proofs can be found in [8].

First, observe that $\text{PSH}^-(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ is contained in $\mathcal{E}(\Omega)$ and that $\mathcal{E}_0(\Omega) \subset \mathcal{F}^a(\Omega) \subset \mathcal{F}(\Omega) \subset \mathcal{E}(\Omega)$. The following lemma explains why the functions in $\mathcal{E}_0(\Omega)$ are sometimes called *test functions*.

LEMMA 2.1. *If $\varphi \in C^\infty_0(\Omega)$, then there are $\varphi_1, \varphi_2 \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ such that $\varphi = \varphi_1 - \varphi_2$.*

If $u_1, \dots, u_n \in \mathcal{E}(\Omega)$, then $dd^c u_1 \wedge \dots \wedge dd^c u_n$ is defined as the limit measure obtained by combining the following two theorems.

THEOREM 2.2. *Suppose that $u \in \text{PSH}^-(\Omega)$. Then there is a sequence $\{u_j\} \subset \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ such that $u_j \searrow u$ on Ω and $\text{supp}(dd^c u_j)^n \subset\subset \Omega$ for each j .*

THEOREM 2.3. *For $k = 1, \dots, n$, let $u_k \in \mathcal{E}(\Omega)$ and $\{g_{kj}\}_{j=1}^\infty \subset \mathcal{E}_0(\Omega)$ be such that $g_{kj} \searrow u_k$ as $j \rightarrow \infty$. Then $dd^c g_{1j} \wedge \dots \wedge dd^c g_{nj}$ is weak*-convergent and the limit measure is independent of the sequences $\{g_{kj}\}$.*

A function $u \in \mathcal{E}(\Omega)$ is a maximal plurisubharmonic function if and only if $(dd^c u)^n = 0$ (see [5] and [7]). If $u \in \mathcal{F}(\Omega)$ and $(dd^c u)^n = 0$, then $u = 0$ (see Theorem 5.15 in [8]). Theorem 2.3 can be generalized as follows (see e.g. Lemma 3.2 in [10]).

LEMMA 2.4. *For $k = 1, \dots, n$, let $u_k \in \mathcal{E}(\Omega)$ and $\{g_{kj}\}_{j=1}^\infty \subset \mathcal{E}(\Omega)$ be such that $g_{kj} \geq u_k$ and g_{kj} tends weakly to u_k as $j \rightarrow \infty$. If $h \in \text{PSH}^-(\Omega) \cap L^\infty(\Omega)$, then $h dd^c g_{1j} \wedge \dots \wedge dd^c g_{nj}$ tends weak* to $h dd^c u_1 \wedge \dots \wedge dd^c u_n$. Moreover, if $u_k \in \mathcal{F}(\Omega)$ then $\lim_{j \rightarrow \infty} \int_\Omega h dd^c g_{1j} \wedge \dots \wedge dd^c g_{nj} = \int_\Omega h dd^c u_1 \wedge \dots \wedge dd^c u_n$.*

The next lemma contains some useful basic properties of the classes we use.

LEMMA 2.5. *Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}^a, \mathcal{F}, \mathcal{E}\}$.*

- (i) *If $u, v \in \mathcal{K}(\Omega)$ and $\alpha, \beta \geq 0$, then $\alpha u + \beta v \in \mathcal{K}(\Omega)$.*
- (ii) *If $u \in \mathcal{K}(\Omega)$ and $v \in \text{PSH}^-(\Omega)$, then $\max\{u, v\} \in \mathcal{K}(\Omega)$. In particular, if $u \in \mathcal{K}(\Omega)$, $v \in \text{PSH}^-(\Omega)$ and $v \geq u$, then $v \in \mathcal{K}(\Omega)$.*

Note that functions in $\mathcal{F}(\Omega)$ have finite total Monge–Ampère mass. Also, they have in some sense boundary values zero, which can be seen e.g. in the following formula for partial integration.

THEOREM 2.6. *Let $v, u_1, \dots, u_n \in \mathcal{F}(\Omega)$. Then*

$$\int_{\Omega} v dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_n = \int_{\Omega} u_1 dd^c v \wedge dd^c u_2 \wedge \dots \wedge dd^c u_n.$$

Since a bounded function cannot put Monge–Ampère mass on pluripolar sets (see e.g. [3]), we have $\mathcal{F}(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}^a(\Omega)$. Moreover, Theorems 5.5 and 5.8 of [8] gives:

LEMMA 2.7. *If $u_1, \dots, u_{n-1} \in \mathcal{F}(\Omega)$ and $v \in \mathcal{F}^a(\Omega)$ or $v \in \text{PSH}^-(\Omega) \cap L^\infty(\Omega)$, then $dd^c u_1 \wedge \dots \wedge dd^c u_{n-1} \wedge dd^c v$ vanishes on pluripolar sets.*

We conclude this section with some notation needed in this paper. Let Ω and $u \in \mathcal{E}(\Omega)$ be given and choose a fundamental sequence $\{\Omega_j\}$ of strictly pseudoconvex domains, i.e. $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ and $\bigcup_{j=1}^\infty \Omega_j = \Omega$. For each j define

$$(2.1) \quad u^j = \sup \{ \varphi \in \text{PSH}(\Omega) : \varphi|_{\Omega \setminus \Omega_j} \leq u|_{\Omega \setminus \Omega_j} \}.$$

Note that since Ω_j has C^2 boundary, it follows that u^j equals $(u^j)^*$, the smallest upper semicontinuous majorant of u^j , so u^j is plurisubharmonic. Moreover, $u \leq u^j \leq u^{j+1} \leq 0$, so each $u^j \in \mathcal{E}(\Omega)$ and the same holds for $\tilde{u} = (\lim_{j \rightarrow \infty} u^j)^*$. It follows that \tilde{u} is the smallest maximal plurisubharmonic majorant of u and that \tilde{u} is independent of the sequence $\{\Omega_j\}$ chosen. In [10] the following classes were defined:

$$\mathcal{N}(\Omega) = \{ u \in \mathcal{E}(\Omega) : \tilde{u} = 0 \}, \quad \mathcal{M}(\Omega) = \{ u \in \mathcal{E}(\Omega) : (dd^c u)^n = 0 \}.$$

Thus $\mathcal{M}(\Omega)$ is the class of maximal plurisubharmonic functions in $\mathcal{E}(\Omega)$. Note that $\mathcal{N}(\Omega)$ contains $\mathcal{F}(\Omega)$, since if $u \in \mathcal{F}(\Omega)$, then \tilde{u} is a maximal function in $\mathcal{F}(\Omega)$ so $\tilde{u} = 0$. It also follows that if $u \in \mathcal{F}(\Omega)$, then $u^j \nearrow 0$ outside a pluripolar subset of Ω (see [16] or [3]).

Finally, we say that $u \in \mathcal{E}(\Omega)$ has *boundary values* \tilde{u} if there is a function $\psi \in \mathcal{N}(\Omega)$ such that $\tilde{u} \geq u \geq \tilde{u} + \psi$. Given $H \in \mathcal{M}(\Omega)$ we define

$$\mathcal{F}(\Omega, H) = \{ u \in \text{PSH}(\Omega) : H \geq u \geq H + \psi \text{ for some } \psi \in \mathcal{F}(\Omega) \},$$

which is a subclass of $\mathcal{E}(\Omega)$. It follows that if $u \in \mathcal{F}(\Omega, H)$ then $\tilde{u} = H$. Also, $\mathcal{F}(\Omega, 0) = \mathcal{F}(\Omega)$.

3. Construction of the boundary measures μ_u . In this section we show that every function in $\mathcal{F}(\Omega)$ gives rise to a measure on the boundary of Ω . Let $u \in \mathcal{F}(\Omega)$ be given, choose a fundamental sequence $\{\Omega_j\}$ of strictly pseudoconvex domains and let u^j be defined by (2.1). Then $u \leq u^j \leq u^{j+1} \leq 0$, so each u^j is in $\mathcal{F}(\Omega)$. Moreover, Stokes' theorem implies that $\int_{\Omega} (dd^c u^j)^n = \int_{\Omega} (dd^c u)^n < \infty$, and by maximality $(dd^c u^j)^n$ is concentrated on $\Omega \setminus \Omega_j$.

THEOREM 3.1. *Suppose that $u \in \mathcal{F}(\Omega)$. Then $\{(dd^c u^j)^n\}$ is a weak*-convergent sequence, which defines a positive measure μ_u on $\partial\Omega$. Also $\lim_{j \rightarrow \infty} \int_{\Omega} \varphi (dd^c u^j)^n$ exists for all $\varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Choose W to be a strictly pseudoconvex set containing the closure of Ω . First assume that $\varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$ and $\varphi \leq 0$. Then

$$(3.1) \quad \begin{aligned} -\infty < \int_{\Omega} \varphi (dd^c u)^n &\leq \int_{\Omega} \varphi (dd^c u^j)^n \\ &\leq \int_{\Omega} \varphi (dd^c u^{j+1})^n \leq \sup_{\Omega} \varphi \int_{\Omega} (dd^c u)^n. \end{aligned}$$

To see this, approximate φ with functions in $\mathcal{E}_0(\Omega)$ and use partial integration in $\mathcal{F}(\Omega)$ (see Section 2). Since all Monge–Ampère measures involved have the same total mass, it follows that (3.1) holds for all $\varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$. Thus $\{\int_{\Omega} \varphi (dd^c u^j)^n\}$ is a bounded monotone sequence, so $\lim_{j \rightarrow \infty} \int_{\Omega} \varphi (dd^c u^j)^n$ exists for all $\varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$. In particular, the limit exists for $\varphi \in C_0^\infty(W)$ (see Lemma 2.1). Since each $(dd^c u^j)^n$ is a positive distribution on $C_0^\infty(W)$, it follows from standard distribution theory that the convergence in fact holds for all $\varphi \in C_0(W)$. Also the limit distribution itself is positive and thus defines a positive regular Borel measure μ_u on W , which by the construction is concentrated on $\partial\Omega$. ■

In this manner we may, to each $u \in \mathcal{F}(\Omega)$, associate a positive measure μ_u , and it follows for example that

$$(3.2) \quad \int_{\partial\Omega} \varphi d\mu_u = \lim_{j \rightarrow \infty} \int_{\Omega} \varphi (dd^c u^j)^n$$

holds for all $\varphi \in C_0(W)$, in particular for $\varphi \in C(\bar{\Omega})$. We also see that $\int_{\partial\Omega} d\mu = \int_{\Omega} (dd^c u)^n$, which implies that $\mu_u = 0$ if and only if $u = 0$ (since $u \in \mathcal{F}(\Omega)$). Note that μ_u does not depend on the sequence $\{\Omega_j\}$ chosen. Note also that by applying (3.1) to φ and $-\varphi$ we get

$$(3.3) \quad \int_{\Omega} \varphi (dd^c u^j)^n = \int_{\Omega} \varphi (dd^c u)^n, \quad \forall \varphi \in \text{PH}(\Omega) \cap L^\infty(\Omega),$$

where $\text{PH}(\Omega)$ denotes the pluriharmonic functions on Ω .

In [14] Demailly defines a set of Monge–Ampère boundary measures in the following setting. Let X be a Stein manifold of dimension n and $\Omega \subset\subset X$ an open hyperconvex subset. Assume that $\phi : \Omega \rightarrow [-\infty, 0]$ is a continuous plurisubharmonic exhaustion function such that $\int_{\Omega} (dd^c \phi)^n < \infty$. For each $r < 0$ define

$$\begin{aligned} B(r) &= \{z \in \Omega : \phi(z) < r\}, \\ S(r) &= \{z \in \Omega : \phi(z) = r\}, \\ \phi_r(z) &= \max\{\phi(z), r\}. \end{aligned}$$

It is then shown that

$$(3.4) \quad (dd^c \phi_r)^n = \chi_{\Omega \setminus B(r)} \cdot (dd^c \phi)^n + \mu_{\phi,r}$$

where $\mu_{\phi,r}$ is a positive measure concentrated on $S(r)$. Furthermore, if $r \rightarrow 0$ then $\mu_{\phi,r}$ converges in a weak sense to a positive measure $\tilde{\mu}_{\phi}$ concentrated on $\partial\Omega$. (More explicitly, it is shown that $\lim_{r \rightarrow 0} \int h d\mu_{\phi,r}$ exists for all $h \in C^2(X, \mathbb{R})$.)

Now consider the case when $X = \mathbb{C}^n$. Then the function ϕ is in $\mathcal{F}(\Omega)$ so we can define μ_{ϕ} according to Theorem 3.1. Choose a sequence $\{r_j\}$ such that $r_j \nearrow 0$ and let $\Omega_j = B(r_j)$. Then $\phi_{r_j} = \max\{\phi, r_j\}$ is equal to the function ϕ^j defined as in (2.1). Note that Ω_j is not necessarily strictly pseudoconvex in this setting, only hyperconvex. However, this is enough in the proof of Theorem 3.1, since we only use the smoothness of $\partial\Omega_j$ to ensure that the function ϕ^j is plurisubharmonic. Hence

$$(3.5) \quad (dd^c \phi_{r_j})^n = \chi_{\Omega \setminus B(r_j)} \cdot (dd^c \phi)^n + \mu_{\phi,r_j},$$

where the left hand side converges to the boundary measure μ_{ϕ} and the right hand side to $0 + \tilde{\mu}_{\phi}$ (since $\int_{\Omega} (dd^c \phi)^n < \infty$). This shows that $\mu_{\phi} = \tilde{\mu}_{\phi}$, so in particular Demailly’s boundary measures form a subset of those defined in Theorem 3.1, when $X = \mathbb{C}^n$.

Also, note that if $u \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ then u satisfies the conditions in Demailly’s definition, so for boundary measures corresponding to such functions we may use Demailly’s results.

The following theorem, where u^j is defined by (2.1), generalizes a formula considered by Demailly in [14].

THEOREM 3.2. *Assume that $u \in \mathcal{F}(\Omega)$, $h \in \mathcal{E}(\Omega)$, $\int_{\Omega} h (dd^c u)^n > -\infty$ and $dd^c h \wedge (dd^c u)^{n-1}$ vanishes on pluripolar sets. Then*

$$\lim_{j \rightarrow \infty} \int_{\Omega} h (dd^c u^j)^n = \int_{\Omega} h (dd^c u)^n - \int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1}.$$

Note that the conditions in this theorem are satisfied if for example $u \in \mathcal{F}(\Omega)$ and $h \in \text{PSH}^-(\Omega) \cap L^{\infty}(\Omega)$ (see Lemma 2.7). Actually, it is enough that $h \in \text{PSH}(\Omega) \cap L^{\infty}(\Omega)$, since $\int_{\Omega} (dd^c u^j)^n = \int_{\Omega} (dd^c u)^n$.

Proof of Theorem 3.2. First we claim the following:

- (i)
$$\int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1} > -\infty,$$
- (ii)
$$\lim_{j \rightarrow \infty} \int_{\Omega} u^j dd^c h \wedge (dd^c u)^{n-1} = 0,$$
- (iii)
$$\begin{aligned} \int_{\Omega} h (dd^c u^j)^{n-p+1} \wedge (dd^c u)^{p-1} &\geq \int_{\Omega} h (dd^c u^j)^{n-p} \wedge (dd^c u)^p \\ &\geq \int_{\Omega} h (dd^c u)^n, \quad 1 \leq p \leq n-1, \end{aligned}$$
- (iv)
$$\begin{aligned} &\int_{\Omega} h dd^c(u^j - u) \wedge (dd^c u^j)^{n-p} \wedge (dd^c u)^{p-1} \\ &= \int_{\Omega} u^j dd^c h \wedge dd^c(u^j - u) \wedge (dd^c u^j)^{n-p-1} \wedge (dd^c u)^{p-1} \\ &= \int_{\Omega} (u^j - u) dd^c h \wedge (dd^c u^j)^{n-p} \wedge (dd^c u)^{p-1} \geq 0, \quad 1 \leq p \leq n. \end{aligned}$$

For the proof of (i), choose a sequence $\{h_k\}$ in $\mathcal{E}_0(\Omega)$ decreasing to h on Ω . Then $dd^c h_k \wedge (dd^c u)^{n-1}$ converges weak* to $dd^c h \wedge (dd^c u)^{n-1}$ (Lemma 2.4). Combining this with the fact that u is upper semicontinuous shows that

$$\begin{aligned} \int_{\Omega} (-u) dd^c h \wedge (dd^c u)^{n-1} &\leq \limsup_{k \rightarrow \infty} \int_{\Omega} (-u) dd^c h_k \wedge (dd^c u)^{n-1} \\ &= \limsup_{k \rightarrow \infty} \int_{\Omega} (-h_k) (dd^c u)^n = \int_{\Omega} (-h) (dd^c u)^n < \infty \end{aligned}$$

(where we have used partial integration in $\mathcal{F}(\Omega)$). Since $u^j \nearrow 0$ outside a pluripolar set (see Section 2) and since $dd^c h \wedge (dd^c u)^{n-1}$ puts no mass there, (i) implies (ii) by the dominated convergence theorem. To see (iii), use the same technique as in Theorem 3.1. Finally, (iv) follows from partial integration, using the fact that h is locally in $\mathcal{F}(\Omega)$ and that $u^j - u$ is compactly supported in Ω . This proves the claim.

Now using (iv) we have

$$\begin{aligned} &\int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1} \\ &= \int_{\Omega} (u - u^j) dd^c h \wedge (dd^c u)^{n-1} + \int_{\Omega} u^j dd^c h \wedge (dd^c u)^{n-1} \\ &= \int_{\Omega} h dd^c(u - u^j) \wedge (dd^c u)^{n-1} + \int_{\Omega} u^j dd^c h \wedge (dd^c u)^{n-1}, \end{aligned}$$

so we can write

$$\begin{aligned} & \int_{\Omega} h (dd^c u^j)^n - \int_{\Omega} h (dd^c u)^n + \int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1} \\ &= \int_{\Omega} h (dd^c u^j)^n - \int_{\Omega} h dd^c u^j \wedge (dd^c u)^{n-1} + \int_{\Omega} u^j dd^c h \wedge (dd^c u)^{n-1} \end{aligned}$$

where the last integral tends to 0 according to (ii). Moreover,

$$\begin{aligned} & \int_{\Omega} h (dd^c u^j)^n - \int_{\Omega} h dd^c u^j \wedge (dd^c u)^{n-1} \\ &= \sum_{p=1}^{n-1} \left(\int_{\Omega} h (dd^c u^j)^{n-p+1} \wedge (dd^c u)^{p-1} - \int_{\Omega} h (dd^c u^j)^{n-p} \wedge (dd^c u)^p \right) \\ &= \sum_{p=1}^{n-1} a_p \end{aligned}$$

where $a_p \geq 0$ for all p by (iii). Using (iv) we have

$$\begin{aligned} a_p &= \int_{\Omega} h dd^c (u^j - u) \wedge (dd^c u^j)^{n-p} \wedge (dd^c u)^{p-1} \\ &= \int_{\Omega} u^j dd^c h \wedge dd^c (u^j - u) \wedge (dd^c u^j)^{n-p-1} \wedge (dd^c u)^{p-1} \\ &\leq - \int_{\Omega} u^j dd^c h \wedge (dd^c u^j)^{n-p-1} \wedge (dd^c u)^p. \end{aligned}$$

Now, the second expression in (iv) implies that $\int_{\Omega} u^j dd^c h \wedge (dd^c u^j)^{n-k} \wedge (dd^c u)^{k-1}$ is decreasing in k , so it follows that $0 \leq a_p \leq - \int_{\Omega} u^j dd^c h \wedge (dd^c u)^{n-1}$. Hence (ii) implies that each term a_p tends to 0 as $j \rightarrow \infty$ and the theorem is proved. ■

REMARK 1. Combining the preceding theorem with (3.2), we have the following formula. Given $u \in \mathcal{F}(\Omega)$,

$$(3.6) \quad \int_{\Omega} h (dd^c u)^n = \int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1} + \int_{\partial\Omega} h d\mu_u, \quad \forall h \in \text{PSH}(\Omega) \cap C(\bar{\Omega}).$$

In Section 4 (Corollary 4.10) we will show that there is a set $S \subset \partial\Omega$ such that $\text{supp } \mu_u = S$ for each $u \in \mathcal{F}(\Omega)$, $u \neq 0$. Hence (3.6) gives a partial integration formula for $h \in \text{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $h|_S = 0$. From Theorem 5.3 in Section 5 it follows that if $u \in \mathcal{F}^a(\Omega)$, then (3.6) is valid for $h \in \text{PSH}(W) \cap L^\infty(W)$, where W is some neighbourhood of $\bar{\Omega}$.

We also get a Jensen-type inequality: given $u \in \mathcal{F}(\Omega)$,

$$(3.7) \quad \int_{\Omega} h (dd^c u)^n \leq \int_{\partial\Omega} h d\mu_u, \quad \forall h \in \text{PSH}(\Omega) \cap C(\bar{\Omega}).$$

If $h \in \text{PSH}(W)$ for some neighbourhood W of $\bar{\Omega}$, then using convolution we

may find functions $h_k \in \text{PSH}(W') \cap C(W')$, where $\bar{\Omega} \subset W' \subset\subset W$, such that $h_k \searrow h$ on W' . Therefore (3.7) holds true if $h \in \text{PSH}(W)$ and $u \in \mathcal{F}(\Omega)$.

4. Some properties of the boundary measures μ_u . In this section we investigate some properties of the boundary measures μ_u defined in Section 3. Recall that a hyperconvex domain Ω is called *B-regular* if each continuous function on $\partial\Omega$ can be extended continuously to a plurisubharmonic function on Ω (see [19]).

THEOREM 4.1. *Let μ be a finite positive measure on $\partial\Omega$, where Ω is a bounded B-regular domain. Then μ is in the weak* closure of $\{\mu_u : u \in \mathcal{F}(\Omega)\}$.*

Proof. For simplicity, assume that $\mu(\partial\Omega) = 1$. Choose a sequence of measures

$$\mu_k = \sum_{j=1}^{N_k} a_j^k \delta_{z_j^k}, \quad \text{where } \{z_j^k\}_{j=1}^{N_k} \subset \Omega \quad \text{and} \quad \sum_{j=1}^{N_k} a_j^k = 1$$

such that

$$(4.1) \quad \lim_{k \rightarrow \infty} \int_{\Omega} h d\mu_k = \int_{\partial\Omega} h d\mu, \quad \forall h \in C(\bar{\Omega}),$$

where $\delta_{z_j^k}$ denotes the Dirac measure at z_j^k . Let e.g. $a_j^k = \mu(A_j^k)$ and $z_j^k \in A_j^k \cap \Omega$, where $\{A_j^k\}_{j=1}^{N_k}$ is a partition of $\bar{\Omega}$ such that $\text{diam}(A_j^k) \leq 1/2^k$, and use the fact that h is uniformly continuous on $\bar{\Omega}$. For each k , consider $g_k(z)$, the multipole pluricomplex Green's function for Ω with poles at $\{z_j^k\}$ with weights $\{(a_j^k)^{1/n}\}$ (see [17] and [18]). Then $g_k \in \mathcal{F}(\Omega)$ and $(dd^c g_k)^n = \mu_k$. Form $\tilde{\mu}_k = \lim_{i \rightarrow \infty} (dd^c(g_k)^i)^n$ as in Section 3. Then for each k ,

$$(4.2) \quad \int_{\partial\Omega} d\tilde{\mu}_k = \int_{\Omega} (dd^c g_k)^n = \int_{\Omega} d\mu_k = 1 = \int_{\partial\Omega} d\mu,$$

and from (3.2) and (3.1) it follows that

$$(4.3) \quad \int_{\partial\Omega} \varphi d\tilde{\mu}_k = \lim_{i \rightarrow \infty} \int_{\Omega} \varphi (dd^c(g_k)^i)^n \geq \int_{\Omega} \varphi (dd^c g_k)^n = \int_{\Omega} \varphi d\mu_k$$

for $\varphi \in \text{PSH}(\Omega) \cap C(\bar{\Omega})$. Let $\{\tilde{\mu}_{k_m}\}$ be any weak*-convergent subsequence of $\{\tilde{\mu}_k\}$. (Such a subsequence exists since the measures $\{\tilde{\mu}_k\}$ have uniformly bounded total mass.) Now let $t \in C(\partial\Omega)$ with $t \leq 0$ be given. Since Ω is B-regular there is a $\varphi \in \text{PSH}(\Omega) \cap C(\bar{\Omega})$ with $\varphi = t$ on $\partial\Omega$. Hence, by (4.1) and (4.3),

$$\int_{\partial\Omega} t d\mu = \lim_{m \rightarrow \infty} \int_{\Omega} \varphi d\mu_{k_m} \leq \lim_{m \rightarrow \infty} \int_{\partial\Omega} \varphi d\tilde{\mu}_{k_m} = \lim_{m \rightarrow \infty} \int_{\partial\Omega} t d\tilde{\mu}_{k_m}.$$

This shows that $\mu \geq \lim_{m \rightarrow \infty} \tilde{\mu}_{k_m}$. It then follows from (4.2) that they have the same total mass, so $\mu = \lim_{m \rightarrow \infty} \tilde{\mu}_{k_m}$ and the theorem is proved. Note that since the argument is valid for any weak*-convergent subsequence, it follows that $\{\tilde{\mu}_k\}$ itself tends weak* to μ . ■

Later in this section, we will show that not every positive measure on $\partial\Omega$ is in $\{\mu_u : u \in \mathcal{F}(\Omega)\}$ (see for example Proposition 4.7). Moreover, the assumption of B-regularity cannot be removed in Theorem 4.1 (see for example Corollary 4.10 and Example 4.11). Before we can prove this, we need the following convergence property.

PROPOSITION 4.2. *Suppose that $u \in \mathcal{F}(\Omega)$ and that $\{u_k\}$ is a decreasing sequence in $\mathcal{F}(\Omega)$ such that $u_k \searrow u$ on Ω . Then μ_{u_k} converges weak* to μ_u .*

Proof. Let $h \in \mathcal{E}_0(\Omega') \cap C(\overline{\Omega'})$ where $\Omega' \supset \overline{\Omega}$. Then (3.6) gives

$$\int_{\partial\Omega} h d\mu_u = \int_{\Omega} h (dd^c u)^n - \int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1},$$

and for each k ,

$$\int_{\partial\Omega} h d\mu_{u_k} = \int_{\Omega} h (dd^c u_k)^n - \int_{\Omega} u_k dd^c h \wedge (dd^c u_k)^{n-1}.$$

From Lemma 2.4 it follows that $\lim_{k \rightarrow \infty} \int_{\Omega} h (dd^c u_k)^n = \int_{\Omega} h (dd^c u)^n$. Moreover, $\lim_{k \rightarrow \infty} \int_{\Omega} u_k dd^c h \wedge (dd^c u_k)^{n-1} = \int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1}$ by the following calculations. Since $u \leq u_k$ for each k , Lemma 3.3 in [1] implies that

$$\int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1} \leq \int_{\Omega} u dd^c h \wedge (dd^c u_k)^{n-1} \leq \int_{\Omega} u_k dd^c h \wedge (dd^c u_k)^{n-1}$$

for each k . Hence, for fixed k_0 ,

$$\begin{aligned} \int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1} &\leq \liminf_{k \rightarrow \infty} \int_{\Omega} u_k dd^c h \wedge (dd^c u_k)^{n-1} \\ &\leq \limsup_{k \rightarrow \infty} \int_{\Omega} u_k dd^c h \wedge (dd^c u_k)^{n-1} \\ &\leq \limsup_{k \rightarrow \infty} \int_{\Omega} u_{k_0} dd^c h \wedge (dd^c u_k)^{n-1} \\ &\leq \int_{\Omega} u_{k_0} dd^c h \wedge (dd^c u)^{n-1}, \end{aligned}$$

where the last inequality follows since $dd^c h \wedge (dd^c u_k)^{n-1}$ is weak*-convergent to $dd^c h \wedge (dd^c u)^{n-1}$ (Lemma 2.4) and u_{k_0} is upper semicontinuous. Now, the claim follows if we let $k_0 \rightarrow \infty$.

Thus

$$(4.4) \quad \lim_{k \rightarrow \infty} \int_{\partial\Omega} h d\mu_{u_k} = \int_{\partial\Omega} h d\mu_u$$

holds true for $h \in \mathcal{E}_0(\Omega') \cap C(\overline{\Omega'})$ and therefore for $h \in C_0^\infty(\Omega')$. By standard distribution theory it follows that (4.4) holds for $h \in C_0(\Omega')$ and hence for $h \in C(\partial\Omega)$. ■

Recall from Section 3 that for functions in $\mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ we can apply the results of Demailly in [14]. We make use of this fact in the proof of the following proposition.

PROPOSITION 4.3. *If $u, v \in \mathcal{F}(\Omega)$ are such that $u \leq v$, then $\mu_u \geq \mu_v$.*

Proof. Take $\{u_k\}, \{w_k\} \subset \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ such that $u_k \searrow u$ and $w_k \searrow v$. Let $v_k = \max\{u_k, w_k\}$. Then $v_k \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$, $v_k \searrow v$ and $u_k \leq v_k$. By Theorem 3.4 in [14], $\mu_{u_k} \geq \mu_{v_k}$ for each k . From Proposition 4.2 it follows that $\mu_u \geq \mu_v$. ■

REMARK 2. When Ω is B-regular there is a slightly more direct proof of Proposition 4.3, without using Demailly’s results. If in that case $f \in C(\partial\Omega)$ with $f \leq 0$ is given, it may be extended to a function in $\text{PSH}^-(\Omega) \cap C(\overline{\Omega})$. Since $u \leq v$ we have $u^j \leq v^j$ for each j , which (see the proof of Theorem 3.1) implies that $\int_\Omega f (dd^c u^j)^n \leq \int_\Omega f (dd^c v^j)^n$ for each j . From (3.2) it follows that $\int_{\partial\Omega} f d\mu_u \leq \int_{\partial\Omega} f d\mu_v$, so the regularity of μ_u and μ_v implies that $\mu_u \geq \mu_v$.

COROLLARY 4.4. *Suppose that $u \in \mathcal{F}(\Omega)$. Then $\mu_u = \mu_{\max\{u, -1\}}$.*

Proof. Let $v = \max\{u, -1\}$. Then $\mu_u \geq \mu_v$ by Proposition 4.3. Take $\{u_k\} \subset \mathcal{E}_0(\Omega)$ such that $u_k \searrow u$ and let $v_k = \max\{u_k, -1\}$. Then $v_k \in \mathcal{E}_0(\Omega)$, $v_k \searrow v$ and $v_k = u_k$ on $\Omega \setminus \{u_k < -1\}$ (note that $\{u_k < -1\} \subset\subset \Omega$). From Theorem 5.1 in [8] and Stokes’ theorem, it follows that

$$\begin{aligned} \int_{\partial\Omega} d\mu_u &= \int_{\Omega} (dd^c u)^n = \lim_{k \rightarrow \infty} \int_{\Omega} (dd^c u_k)^n \\ &= \lim_{k \rightarrow \infty} \int_{\Omega} (dd^c v_k)^n = \int_{\Omega} (dd^c v)^n = \int_{\partial\Omega} d\mu_v, \end{aligned}$$

so $\mu_u = \mu_v$. ■

We will now use this corollary to show that each μ_u vanishes on pluripolar sets. We start with two technical lemmas.

LEMMA 4.5. *Suppose that $u \in \mathcal{F}(\Omega)$ and that $\varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$ is upper semicontinuous on some neighbourhood Ω' of $\overline{\Omega}$. Then*

$$\lim_{j \rightarrow \infty} \int_{\Omega} \varphi (dd^c u^j)^n \leq \int_{\partial\Omega} \varphi d\mu_u.$$

Proof. Choose Ω'' such that $\Omega \subset\subset \Omega'' \subset\subset \Omega'$. Then there is a decreasing sequence $\{\varphi_k\}$ of continuous functions on Ω'' that are bounded above and

that converge to φ on Ω'' . Using equality (3.2) we see that

$$\lim_{j \rightarrow \infty} \int_{\Omega} \varphi (dd^c u^j)^n \leq \lim_{j \rightarrow \infty} \int_{\Omega} \varphi_k (dd^c u^j)^n = \int_{\partial\Omega} \varphi_k d\mu_u$$

for each k . Hence the lemma follows by letting $k \rightarrow \infty$. ■

LEMMA 4.6. *Let $E \subset \partial\Omega$ be a pluripolar set and $u \in \mathcal{F}(\Omega)$. Suppose that there is a function $g \in \text{PSH}(\Omega')$, where $\Omega' \supset \Omega$, such that $E \subset \mathcal{S}_g = \{z : g(z) = -\infty\}$ and $(dd^c u)^n$ is concentrated on $\Omega \setminus \mathcal{S}_g$. Then $\mu_u(E) = 0$.*

Proof. By subtracting a suitable constant we may assume that $g \leq 0$ on $\bar{\Omega}$. For each positive integer k , define $h_k = \max\{(1/k) \cdot g, -1\}$. Then from (3.1) and Lemma 4.5 it follows that

$$\begin{aligned} -\infty < \int_{\Omega} h_k (dd^c u)^n &\leq \lim_{j \rightarrow \infty} \int_{\Omega} h_k (dd^c u^j)^n \\ &\leq \int_{\partial\Omega} h_k d\mu_u \leq \int_E h_k d\mu_u = -\mu_u(E), \end{aligned}$$

since $h_k \leq 0$ on $\bar{\Omega}$ and $h_k = -1$ on E . Moreover, $h_k(z) \nearrow 0$ for all $z \in \Omega \setminus \mathcal{S}_g$ as $k \rightarrow \infty$, so $\lim_{k \rightarrow \infty} \int_{\Omega} h_k (dd^c u)^n = 0$. Hence $\mu_u(E) = 0$. ■

PROPOSITION 4.7. *If $u \in \mathcal{F}(\Omega)$, then μ_u vanishes on pluripolar subsets of $\partial\Omega$.*

Proof. If $u \in \mathcal{F}(\Omega)$ then $v = \max\{u, -1\} \in \mathcal{F}^a(\Omega)$ and from Corollary 4.4 we know that $\mu_u = \mu_v$. Now, for functions in $\mathcal{F}^a(\Omega)$ the conditions in Lemma 4.6 are satisfied for each pluripolar set $E \subset \partial\Omega$, so the proposition follows. ■

The next proposition enables us to say more about the support of the μ_u -measures.

PROPOSITION 4.8. *Assume that $u, v \in \mathcal{E}_0(\Omega)$, $u \neq 0, v \neq 0$, are such that $\text{supp}(dd^c u)^n \subset\subset \Omega$ and $\text{supp}(dd^c v)^n \subset\subset \Omega$. Then there are constants $a, b > 0$ such that*

$$a\mu_u \leq \mu_v \leq b\mu_u.$$

In particular, $\text{supp } \mu_u = \text{supp } \mu_v$.

LEMMA 4.9. *Assume that $u \in \mathcal{F}^a(\Omega)$, $u \neq 0, v \in \mathcal{E}(\Omega)$ and $u \geq v$ on $\text{supp}(dd^c u)^n$. Then $u \geq v$ on Ω .*

Proof. Assume that $u(z_0) < v(z_0)$ for some $z_0 \in \Omega$. Let $\psi \in \mathcal{E}_0(\Omega) \cap C^\infty(\Omega)$ be a strictly plurisubharmonic exhaustion function and let $s > 0$ be such that $u(z_0) < s\psi(z_0) + v(z_0)$. Corollary 3.6 in [10] gives, with $A = \{u(z) < s\psi(z) + v(z)\}$,

$$\int_A (dd^c(s\psi + v))^n \leq \int_A (dd^c u)^n = 0.$$

Hence $s^n \int_A (dd^c \psi)^n = 0$, which implies that A has Lebesgue measure 0. Since the functions involved are plurisubharmonic, this means that $A = \emptyset$. This is a contradiction and the lemma is proved. ■

Proof of Proposition 4.8. Let $K = \text{supp}(dd^c u)^n$. Since K is compact, and since u and v are bounded upper semicontinuous functions, $\alpha > 0$ may be chosen such that $\alpha v \leq u$ on K . It then follows from Lemma 4.9 that $\alpha v \leq u$ holds on all of Ω . Similarly, there is $\beta > 0$ such that $\beta u \leq v$ on Ω . Then Proposition 4.3 implies that $\mu_{\alpha^{-1}u} \leq \mu_v \leq \mu_{\beta u}$. Hence, if we let $a = \alpha^{-n}$ and $b = \beta^n$, the proposition follows. ■

COROLLARY 4.10. *There is a set $S \subset \partial\Omega$ such that $\text{supp } \mu_u = S$ for each $u \in \mathcal{F}(\Omega)$, $u \neq 0$.*

Proof. Choose a function $v_0 \in \mathcal{E}_0(\Omega)$ with $\text{supp}(dd^c v_0)^n \subset\subset \Omega$, and let $S = \text{supp } \mu_{v_0}$. Let u be an arbitrary function in $\mathcal{F}(\Omega)$. Choose a sequence $\{u_j\} \subset \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ and $\text{supp}(dd^c u_j)^n \subset\subset \Omega$. Then Proposition 4.8 implies that $\text{supp } \mu_{u_j} = S$ for each j . Moreover, $\mu_{u_1} \leq \dots \leq \mu_u$ and μ_{u_j} tends weak* to μ_u , by Propositions 4.3 and 4.2. Hence $\text{supp } \mu_u = S$. ■

Note that if μ is in the weak* closure of $\{\mu_u : u \in \mathcal{F}(\Omega)\}$, then $\text{supp } \mu \subset S$. Hence if Ω is B-regular, then the support set S has to be all of $\partial\Omega$, because of Theorem 4.1.

On the other hand, if $\Omega = \omega_1 \times \omega_2 \subset \mathbb{C}^n = \mathbb{C}^{n_1+n_2}$, where $\omega_1 \subset \mathbb{C}^{n_1}$ and $\omega_2 \subset \mathbb{C}^{n_2}$ are bounded hyperconvex domains, then $S \subset \partial\omega_1 \times \partial\omega_2$. To see this, consider the function $u(z, w) = \max\{g_1(z), g_2(w)\}$ where g_k is the pluricomplex Green’s function for ω_k with pole at some point in ω_k . Note that g_k is continuous outside the pole and tends to zero at the boundary of ω_k . Then $u \in \mathcal{F}(\Omega)$ and $\text{supp}(dd^c u)^n \subset \{(z, w) \in \Omega : g_1(z) = g_2(w)\}$. Choose a sequence $\{\varepsilon_j\}$ such that $\varepsilon_j \searrow 0$. Then $\Omega_j = \{(z, w) \in \Omega : u(z, w) < -\varepsilon_j\}$ defines a fundamental sequence of Ω and $u^j := \sup\{\varphi \in \text{PSH}(\Omega) : \varphi|_{\Omega \setminus \Omega_j} \leq u|_{\Omega \setminus \Omega_j}\} = \max\{u, -\varepsilon_j\}$. It follows that $\text{supp}(dd^c u^j)^n \subset \{(z, w) \in \Omega : g_1(z) = g_2(z) \geq -\varepsilon_j\}$, which implies that $\text{supp } \mu_u \subset \partial\omega_1 \times \partial\omega_2$. Hence the claim follows from Corollary 4.10.

Using a similar argument, the following example shows that when $\Omega = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$, then we have equality, $S = \partial\mathbb{D} \times \partial\mathbb{D}$.

EXAMPLE 4.11. Let Ω be the unit bidisc $\mathbb{D} \times \mathbb{D}$ in \mathbb{C}^2 . Then $\text{supp } \mu_u$ is equal to the distinguished boundary $\partial\mathbb{D} \times \partial\mathbb{D}$ for each $u \in \mathcal{F}(\Omega)$, $u \neq 0$. This follows from Corollary 4.10, if we for example consider the pluricomplex Green’s function g for Ω with pole at the origin. We then see that $g(z, w) = m \max\{\log |z|, \log |w|\}$, where the constant $m > 0$ is chosen such that $\int_{\Omega} (dd^c g)^2 = 1$. This is a function in $\mathcal{F}(\Omega)$, and we can compute μ_g explicitly. For $j = 1, 2, \dots$, let $\Omega_j = \{(z, w) : |z| < r_j, |w| < r_j\}$

where $r_j = 1 - 1/j$. Then $g^j := \sup\{\varphi \in \text{PSH}(\Omega) : \varphi|_{\Omega \setminus \Omega_j} \leq g|_{\Omega \setminus \Omega_j}\} = m \max\{\log |z|, \log |w|, \log(r_j)\}$, from which it follows that

$$(dd^c g^j)^2 = m^2 dd^c(\max\{\log |z|, \log(r_j)\}) \wedge dd^c(\max\{\log |w|, \log(r_j)\}).$$

Since $\int_{\Omega} (dd^c g^j)^2 = 1$ for each j (see Section 3), we conclude that $(dd^c g^j)^2 = \sigma_j \times \sigma_j$, where σ_j is the normalized Lebesgue measure on the the circle $\partial\mathbb{D}(0, r_j)$. This implies that $\mu_g = \sigma \times \sigma$, where σ is the normalized Lebesgue measure on the unit circle.

REMARK 3. Recall from Remark 1 at the end of Section 3 that Corollary 4.10 and (3.6) together give the partial integration formula

$$(4.5) \quad h|_S = 0 \Rightarrow \int_{\Omega} h (dd^c u)^n = \int_{\Omega} u dd^c h \wedge (dd^c u)^{n-1}.$$

The implication (4.5) holds true for $h \in \text{PSH}(\Omega) \cap C(\bar{\Omega})$ if $u \in \mathcal{F}(\Omega)$, and for $h \in \text{PSH}(W) \cap L^\infty(W)$, $W \supset \bar{\Omega}$, if $u \in \mathcal{F}^a(\Omega)$ (using Theorem 5.3 of Section 5). Here S is the support set defined in Corollary 4.10.

Furthermore, (3.7) implies that

$$(4.6) \quad \sup_{\Omega} h \leq \sup_S h, \quad \forall h \in \text{PSH}(\Omega) \cap C(\bar{\Omega}).$$

To see this, let $h \in \text{PSH}(\Omega) \cap C(\bar{\Omega})$ be given. For $z \in \Omega$ fixed, let g_z be the pluricomplex Green’s function for Ω with pole at z . Then $(dd^c g_z)^n = \delta_z$, the Dirac measure at z , and we have $h(z) = \int_{\Omega} h (dd^c g_z)^n \leq \int_{\partial\Omega} h d\mu_{g_z} \leq \sup_S h$. By the same argument, (4.6) holds true if h is an upper bounded function in $\text{PSH}(W)$, where $W \supset \bar{\Omega}$.

REMARK 4. Another property of the measures μ_u is that they are *Henkin measures* (a kind of measure introduced by Henkin in [15]). This means that

$$\lim_{k \rightarrow \infty} \int_{\partial\Omega} f_k d\mu_u = 0$$

for each uniformly bounded sequence $\{f_k\}$ in $A(\Omega)$ such that $\lim_{k \rightarrow \infty} f_k(z) = 0$ for all $z \in \Omega$. Here $A(\Omega)$ denotes the functions that are holomorphic on Ω and continuous on $\bar{\Omega}$. To see this, take such a sequence $\{f_k\}$ and let $\{\varphi_k\} = \{\text{Re } f_k\}$. From (3.2) and (3.3) it follows that

$$\lim_{k \rightarrow \infty} \int_{\partial\Omega} \varphi_k d\mu_u = \lim_{k \rightarrow \infty} \left(\lim_{j \rightarrow \infty} \int_{\Omega} \varphi_k (dd^c u^j)^n \right) = \lim_{k \rightarrow \infty} \int_{\Omega} \varphi_k (dd^c u)^n = 0$$

for each $u \in \mathcal{F}(\Omega)$, since φ_k is uniformly bounded and $\int_{\Omega} (dd^c u)^n < \infty$. Since the same holds for $\{\psi_k\} = \{\text{Im } f_k\}$, it follows that $\lim_{k \rightarrow \infty} \int_{\partial\Omega} f_k d\mu_u = 0$.

This property can be used to show the following fact about the support of the measures μ_u . Suppose that $u \in \mathcal{F}(\Omega)$ and that $K \subset \partial\Omega$ is a peak set for $A(\Omega)$. Let $f \in A(\Omega)$ be a peak function for K and define $f_k(z) = (f(z))^k$ for $z \in \bar{\Omega}$ and $k = 1, 2, \dots$. Then $\{f_k\}$ satisfies the assumptions above, so

$\lim_{k \rightarrow \infty} \int_{\partial\Omega} f_k d\mu_u = 0$. But we also know that $\lim_{k \rightarrow \infty} \int_{\partial\Omega} f_k d\mu_u = \mu_u(K)$. Hence $\mu_u(K) = 0$ for each peak set K and each $u \in \mathcal{F}(\Omega)$.

5. Boundary values. In this section we define and study boundary values of plurisubharmonic functions with respect to the measures μ_u .

LEMMA 5.1. *Assume that $u \in \mathcal{F}(\Omega)$ and $g \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$. Then $\{g(dd^c u^j)^n\}$ is weak*-convergent.*

Proof. By the same argument as in Theorem 3.1 it is enough to prove that the limit $\lim_{j \rightarrow \infty} \int_{\Omega} \varphi g(dd^c u^j)^n$ exists for all $\varphi \in \text{PSH}^-(\Omega) \cap L^\infty(\Omega)$. Given such a function φ , take $M, N \geq 0$ such that $\varphi + M \geq 0$ and $g + N \geq 0$. Then $(\varphi + M)^2, (g + N)^2, (\varphi + M + g + N)^2 \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$, so if ψ is any of these then $\lim_{j \rightarrow \infty} \int_{\Omega} \psi(dd^c u^j)^n$ exists by Theorem 3.1. Expanding $((\varphi + M) + (g + N))^2$, it follows that the limit exists for $\psi = (\varphi + M)(g + N)$ and then finally for $\psi = \varphi g$ (using Theorem 3.1 again). ■

Using this lemma, together with standard measure theory, we can make the following definition.

DEFINITION 5.2. For $u \in \mathcal{F}(\Omega)$ and $g \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$, let g^u be the function in $L^\infty(\partial\Omega, \mu_u)$ such that $\lim_{j \rightarrow \infty} \int_{\Omega} g(dd^c u^j)^n = \int_{\partial\Omega} g^u d\mu_u$.

We may consider g^u as the boundary values of g with respect to μ_u . Note that, at least formally, g^u depends on both g and u . However, the following theorems describe some situations when this definition agrees with other notions of boundary values.

THEOREM 5.3. *Assume that $u \in \mathcal{F}^a(\Omega)$ and $g \in \text{PSH}(W) \cap L^\infty(W)$ where W is a bounded domain containing $\bar{\Omega}$. Then $g^u = g|_{\partial\Omega}$ a.e. (μ_u) .*

Proof. Note that if M is a constant then $(g - M)^u = g^u - M$, so we may assume that $g \leq 0$. Let $t \in C(\bar{\Omega})$ with $t \geq 0$ be given. Then it follows, in the same way as in the proof of Lemma 4.5, that

$$\int_{\partial\Omega} t g^u d\mu_u = \lim_{j \rightarrow \infty} \int_{\Omega} t g(dd^c u^j)^n \leq \int_{\partial\Omega} t g d\mu_u.$$

Thus $g^u \leq g$ a.e. (μ_u) , so it remains to prove that $\int_{\partial\Omega} g^u d\mu_u = \int_{\partial\Omega} g d\mu_u$. Choose K such that $\Omega \subset\subset K \subset\subset W$. Given $\varepsilon > 0$ there is an open set $U_\varepsilon \subset W$ and a function $g_\varepsilon \in C_0(W)$ such that $\inf_W g \leq g_\varepsilon \leq 0$, the relative capacity $\text{cap}(U_\varepsilon, W) < \varepsilon$ and $K \setminus U_\varepsilon \subset \{z \in W : g(z) = g_\varepsilon(z)\}$ (for definition and properties of relative capacity, see [3]). It follows that

$$\begin{aligned} \int_{\partial\Omega} g^u d\mu_u &= \lim_{j \rightarrow \infty} \int_{\Omega} g(dd^c u^j)^n \\ &= \lim_{j \rightarrow \infty} \int_{\Omega \cap U_\varepsilon} g(dd^c u^j)^n + \lim_{j \rightarrow \infty} \int_{\Omega \setminus U_\varepsilon} g_\varepsilon(dd^c u^j)^n \end{aligned}$$

$$\begin{aligned}
 &\geq \lim_{j \rightarrow \infty} \int_{\Omega \cap U_\varepsilon} g (dd^c u^j)^n + \int_{\partial \Omega} g_\varepsilon d\mu_u \\
 &= \lim_{j \rightarrow \infty} \int_{\Omega \cap U_\varepsilon} g (dd^c u^j)^n + \int_{\partial \Omega \cap U_\varepsilon} g_\varepsilon d\mu_u + \int_{\partial \Omega \setminus U_\varepsilon} g d\mu_u \\
 &\geq \lim_{j \rightarrow \infty} \int_{\Omega \cap U_\varepsilon} g (dd^c u^j)^n + \int_{\partial \Omega \cap U_\varepsilon} g_\varepsilon d\mu_u + \int_{\partial \Omega} g d\mu_u.
 \end{aligned}$$

Let $h_\varepsilon = \sup \{ \psi \in \text{PSH}^-(W) : \psi|_{U_\varepsilon} \leq -1 \}$. Then

$$\begin{aligned}
 0 &\geq \int_{\partial \Omega} g^u d\mu_u - \int_{\partial \Omega} g d\mu_u \\
 &\geq \lim_{j \rightarrow \infty} \int_{\Omega \cap U_\varepsilon} g (dd^c u^j)^n + \int_{\partial \Omega \cap U_\varepsilon} g_\varepsilon d\mu_u \\
 &\geq (\inf_W g) \left(\lim_{j \rightarrow \infty} \int_{\Omega \cap U_\varepsilon} (dd^c u^j)^n + \int_{\partial \Omega \cap U_\varepsilon} d\mu_u \right) \\
 &= (-\inf_W g) \left(\lim_{j \rightarrow \infty} \int_{\Omega \cap U_\varepsilon} h_\varepsilon (dd^c u^j)^n + \int_{\partial \Omega \cap U_\varepsilon} h_\varepsilon d\mu_u \right) \\
 &\geq (-\inf_W g) \left(\lim_{j \rightarrow \infty} \int_{\Omega} h_\varepsilon (dd^c u^j)^n + \int_{\partial \Omega} h_\varepsilon d\mu_u \right) \\
 &\geq 2(-\inf_W g) \int_{\Omega} h_\varepsilon (dd^c u)^n,
 \end{aligned}$$

where we have used (3.1) and Lemma 4.5 in the last inequality. From Lemma 1.9 in [13], using the fact that $u \in \mathcal{F}^a(\Omega)$ and $\text{cap}(U_\varepsilon, W) < \varepsilon$, it follows that this last integral tends to zero as $\varepsilon \searrow 0$, which completes the proof. ■

The following theorem may be compared with the definitions in Section 2.

THEOREM 5.4. *Suppose that $H \in \mathcal{M}(\Omega) \cap L^\infty(\Omega)$. Then, for every $u \in \mathcal{F}^a(\Omega)$ and every $g \in \mathcal{F}(\Omega, H)$ such that $\int_{\Omega} g (dd^c u)^n > -\infty$, $g (dd^c u^j)^n$ is weak*-convergent to $H^u d\mu_u$.*

Proof. By the same argument as in Theorem 3.1, it is enough to prove that

$$\lim_{j \rightarrow \infty} \int_{\Omega} t g (dd^c u^j)^n = \lim_{j \rightarrow \infty} \int_{\Omega} t H (dd^c u^j)^n, \quad \forall t \in \text{PSH}^-(\Omega) \cap L^\infty(\Omega).$$

Since $g \in \mathcal{F}(\Omega, H)$ there is a $\psi \in \mathcal{F}(\Omega)$ such that $\psi + H \leq g \leq H$. We may assume that $\psi \geq g$ (otherwise, take $\psi_0 = \max \{ \psi, g \}$). We may also (after dividing by suitable constants) assume that $-1 \leq t \leq 0$ and $-1 \leq H \leq 0$. Now,

$$\int_{\Omega} t g (dd^c u^j)^n = \int_{\Omega} t (g - H) (dd^c u^j)^n + \int_{\Omega} t H (dd^c u^j)^n$$

where

$$\begin{aligned} 0 &\leq \int_{\Omega} t(g - H) (dd^c u^j)^n = \int_{\Omega} (-t)(H - g) (dd^c u^j)^n \\ &\leq \int_{\Omega} (-t)(-\psi) (dd^c u^j)^n \leq \int_{\Omega} (-\psi) (dd^c u^j)^n. \end{aligned}$$

Using partial integration in $\mathcal{F}(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} (-\psi) (dd^c u^j)^n &= \int_{\Omega} (-u^j) dd^c \psi \wedge (dd^c u^j)^{n-1} \\ &\leq \int_{\Omega} (-u) dd^c \psi \wedge (dd^c u^j)^{n-1} = \int_{\Omega} (-u^j) dd^c \psi \wedge dd^c u \wedge (dd^c u^j)^{n-2} \\ &\leq \dots \leq \int_{\Omega} (-u^j) dd^c \psi \wedge (dd^c u)^{n-1} = I_j \leq \int_{\Omega} (-u) dd^c \psi \wedge (dd^c u)^{n-1} \\ &= \int_{\Omega} (-\psi) (dd^c u)^n \leq \int_{\Omega} (-g) (dd^c u)^n < \infty. \end{aligned}$$

Since u^j increases to zero outside a pluripolar set and $dd^c \psi \wedge (dd^c u)^{n-1}$ vanishes on pluripolar sets (see Lemma 2.7), it follows that $I_j \searrow 0$ as $j \rightarrow \infty$. This proves the theorem. ■

REMARK 5. If $g \in L^\infty(\Omega)$ then $\int_{\Omega} g (dd^c u)^n > -\infty$ for every $u \in \mathcal{F}(\Omega)$. Furthermore, $\psi \geq g$ implies that ψ is bounded as well, so $dd^c \psi \wedge (dd^c u)^{n-1}$ vanishes on pluripolar sets for every $u \in \mathcal{F}(\Omega)$ (Lemma 2.7). Thus for bounded functions g in $\mathcal{F}(\Omega, H)$, the conclusion $g^u d\mu_u = H^u d\mu_u$ holds for every $u \in \mathcal{F}(\Omega)$.

Suppose that we have a bounded plurisubharmonic function on Ω and want to approximate it with plurisubharmonic functions that are continuous on $\bar{\Omega}$. The following theorem gives a condition for this to imply weak* convergence on the boundary.

THEOREM 5.5. *Assume that $u \in \mathcal{F}(\Omega)$ and $\mu_u = \lim_{j \rightarrow \infty} (dd^c u^j)^n$. Let $\{\varphi_j\}$ be a sequence in $\text{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $0 \leq \varphi_j \leq 1$. If φ_j tends to $\varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$ in the sense of distributions, then $\varphi_j d\mu_u$ tends weak* to $\varphi^u d\mu_u$ if and only if $\lim_{j \rightarrow \infty} \int \varphi_j d\mu_u = \int \varphi^u d\mu_u$.*

Proof. By Corollary 4.4 we may assume that $u \in \mathcal{F}^a(\Omega)$. The condition in the theorem is obviously necessary; we prove it is also sufficient. First, note that for $\{\psi_k\} \subset \text{PSH}(\Omega) \cap C(\bar{\Omega})$ with $\psi_k \geq 0$, the following holds. For k fixed, $(\sup_{l \geq k} \psi_l)^* \in \text{PSH}(\Omega) \cap L^\infty(\Omega)$, therefore $(\sup_{l \geq k} \psi_l)^* (dd^c u^j)^n$ is weak*-convergent (as $j \rightarrow \infty$) by Lemma 5.1. Furthermore, since $\sup_{l \geq k} \psi_l = (\sup_{l \geq k} \psi_l)^*$ outside a pluripolar set and $u^j \in \mathcal{F}^a(\Omega)$ (since $u \in \mathcal{F}^a(\Omega)$), the

star may be removed. We claim that

$$(5.1) \quad \lim_{j \rightarrow \infty} (\sup_{l \geq k} \psi_l) (dd^c u^j)^n = (\sup_{l \geq k} \psi_l) d\mu_u.$$

Given $f \in C(\bar{\Omega})$, $f \geq 0$, it follows from (3.2) that for each m ,

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\Omega} f(\sup_{l \geq k} \psi_l) (dd^c u^j)^n &\geq \lim_{j \rightarrow \infty} \int_{\Omega} f(\sup_{m \geq l \geq k} \psi_l) (dd^c u^j)^n \\ &= \int_{\partial \Omega} f(\sup_{m \geq l \geq k} \psi_l) d\mu_u, \end{aligned}$$

where the last integral tends to $\int_{\partial \Omega} f(\sup_{l \geq k} \psi_l) d\mu_u$ as $m \rightarrow \infty$. It follows that $\lim_{j \rightarrow \infty} (\sup_{l \geq k} \psi_l) (dd^c u^j)^n \geq (\sup_{l \geq k} \psi_l) d\mu_u$. On the other hand, by (3.1) and (3.2),

$$\int_{\Omega} (\sup_{m \geq l \geq k} \psi_l) (dd^c u^j)^n \leq \int_{\partial \Omega} (\sup_{m \geq l \geq k} \psi_l) d\mu_u$$

for each m and j . So by letting $m \rightarrow \infty$ we obtain

$$\int_{\Omega} (\sup_{l \geq k} \psi_l) (dd^c u^j)^n \leq \int_{\partial \Omega} (\sup_{l \geq k} \psi_l) d\mu_u,$$

which proves the claim.

Now, let $\{\varphi_{j_m} d\mu_u\}$ be any weak*-convergent subsequence of $\{\varphi_j d\mu_u\}$. (Such a sequence exists by the same reasoning as in the proof of Theorem 4.1.) Then, by standard measure theory, the limit measure is equal to $\varphi_0 d\mu_u$ for some $\varphi_0 \in L^\infty(\mu_u)$. We will show that $\varphi_0 = \varphi^u$ a.e. (μ_u) . It then follows that the original sequence itself converges to $\varphi^u d\mu_u$, and the proof will be complete.

From L^2 -theory it follows that we may choose $\psi_k = (1/M_k) \sum_{l=1}^{M_k} \varphi_{j_{m_l}}$ such that $\psi_k \rightarrow \varphi_0$ in $L^2(\mu_u)$ and then a subsequence converging to φ_0 a.e. (μ_u) , for simplicity call it $\{\psi_k\}$. Since by assumption the original sequence $\{\varphi_j\}$ tends to φ in the sense of distributions, the same holds for $\{\psi_k\}$. Now, for $f \in C(\bar{\Omega})$, $f \geq 0$, with the definition of φ^u , (5.1) and monotone convergence theorem yield

$$\begin{aligned} \int_{\partial \Omega} f \varphi^u d\mu_u &= \lim_{j \rightarrow \infty} \int_{\Omega} f \varphi (dd^c u^j)^n \\ &= \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{\Omega} f \psi_k (dd^c u^j)^n \quad (\text{Lemma 1.4 in [13]}) \\ &\leq \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_{\Omega} f(\sup_{l \geq k} \psi_l) (dd^c u^j)^n \\ &= \lim_{k \rightarrow \infty} \int_{\partial \Omega} f(\sup_{l \geq k} \psi_l) d\mu_u = \int_{\partial \Omega} f(\limsup_{k \rightarrow \infty} \psi_k) d\mu_u. \end{aligned}$$

Hence $\varphi^u \leq \limsup_{k \rightarrow \infty} \psi_k$ a.e. (μ_u) , which implies that $\varphi^u \leq \varphi_0$ a.e. (μ_u) . Furthermore, $\int_{\partial\Omega} \varphi_0 d\mu_u = \lim_{m \rightarrow \infty} \int_{\partial\Omega} \varphi_{j_m} d\mu_u = \int_{\partial\Omega} \varphi^u d\mu_u$, by assumption, so $\varphi^u = \varphi_0$ a.e. (μ_u) . Thus the theorem is proved. ■

6. More boundary measures. Let ν be a positive measure on Ω with finite total mass. Then there is a positive measure $\mu \neq 0$ which is supported by $\partial\Omega$, vanishes on pluripolar sets and satisfies

$$(6.1) \quad \int_{\Omega} \varphi d\nu \leq \int_{\partial\Omega} \varphi d\mu, \quad \forall \varphi \in \text{PSH}^-(\bar{\Omega}),$$

where $\text{PSH}^-(\bar{\Omega}) = \{\varphi : \varphi \in \text{PSH}^-(\Omega'), \Omega' \supset \bar{\Omega}\}$. To see this, let P_ν denote the pluricomplex potential of ν relative to Ω , i.e. $P_\nu(z) = \int_{\Omega} g(z, w) d\nu(w)$, where $g(z, w)$ is the pluricomplex Green's function for Ω with pole at w . Then Theorem 1.1 in [9] says that $P_\nu \in \mathcal{F}(\Omega)$ and that

$$\int_{\Omega} -\varphi (dd^c P_\nu)^n \leq (\nu(\Omega))^{n-1} \int_{\Omega} -\varphi d\nu, \quad \forall \varphi \in \text{PSH}^-(\Omega).$$

Moreover, $\int_{\Omega} \varphi (dd^c P_\nu)^n \leq \int_{\partial\Omega} \varphi d\mu_{P_\nu}$ for each $\varphi \in \text{PSH}^-(\bar{\Omega})$, by Remark 1 at the end of Section 3. Hence, the claim follows if we take $\mu = (\nu(\Omega))^{-n+1} \mu_{P_\nu}$.

Conversely, if a positive measure μ on $\partial\Omega$ is such that (6.1) holds for some finite measure ν on Ω , we would like to find an approximation procedure, similar to the one in Section 3. A motivation is that we are interested in boundary values of plurisubharmonic functions with respect to μ .

We will study the case when ν vanishes on all pluripolar subsets of Ω and Ω belongs to a more restricted class of hyperconvex domains:

- (6a) Ω and $\{\Omega_k\}$ are hyperconvex domains with $\Omega \subset\subset \Omega_{k+1} \subset\subset \Omega_k$, such that for each $t \in \mathcal{F}(\Omega)$ there is a sequence $\{t_k\}$, where $t_k \in \mathcal{F}(\Omega_k)$ and $t_k \nearrow t$ a.e. on Ω .
- (6b) Ω is not thin at any of its boundary points, so that $\limsup_{\Omega \ni z \rightarrow \xi} v(z) = v(\xi)$ for each $\xi \in \partial\Omega$ if $v \in \text{PSH}^-(\bar{\Omega})$.

Conditions for the approximation property in (6a) to hold true have been studied in for example [4] and [12]. Examples of domains satisfying (6a) and (6b) are polydiscs and strictly pseudoconvex domains. Note that if t is bounded, we may assume that each t_k is bounded.

THEOREM 6.1. *Let Ω be a domain satisfying (6a) and (6b). Assume that μ is a positive measure on $\partial\Omega$, vanishing on pluripolar sets. Then there is a sequence $\{w_k\}$ in $\mathcal{F}^a(\bar{\Omega}) = \{u : u \in \mathcal{F}^a(\Omega'), \Omega' \supset \bar{\Omega}\}$ such that $\text{supp}(dd^c w_k)^n \subset\subset \Omega$, $\int_{\Omega} (dd^c w_k)^n \leq \int_{\partial\Omega} d\mu$, and $(dd^c w_k)^n$ tends weak* to μ as $k \rightarrow \infty$.*

Furthermore, if there is a finite positive measure ν on Ω , vanishing on pluripolar sets, such that (6.1) holds, then $\lim_{k \rightarrow \infty} \int_{\Omega} t (dd^c w_k)^n = 0$ for each $t \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$. Hence $t (dd^c w_k)^n$ tends weak* to 0 for each such t .

If we compare this theorem with the results in the previous sections we see the following. In the setting of Section 3 we know that if $u \in \mathcal{F}^a(\Omega)$ and $\varphi \in \text{PSH}(W) \cap L^\infty(W)$, $W \supset \bar{\Omega}$, then $\int_{\Omega} \varphi (dd^c u)^n \leq \int_{\Omega} \varphi (dd^c u^j)^n$ which increases to $\int_{\partial\Omega} \varphi d\mu_u$ (see Theorem 5.3). In particular, if $\mu = \mu_u$ for some $u \in \mathcal{F}^a(\Omega)$, then (6.1) is satisfied if we take $\nu = (dd^c u)^n$. Moreover, $\int_{\Omega} (dd^c u^j)^n = \int_{\partial\Omega} d\mu_u$ and $\lim_{j \rightarrow \infty} \int_{\Omega} t (dd^c u^j)^n = 0$ for each $t \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ (see Remark 5). Hence, the approximation procedure in Theorem 6.1 is similar to the one in the previous sections, and it applies to a larger class of boundary measures (see also Example 6.3).

LEMMA 6.2. *Let $\{\mu_k^j\}_{j,k}$ be a sequence of positive measures on $\bar{\Omega}$ with uniformly bounded mass. Suppose that, for each fixed k , μ_k^j tends weak* to μ as $j \rightarrow \infty$. Then there is a subsequence $\{\mu_k^{j_k}\}_k$ such that $\mu_k^{j_k}$ tends weak* to μ as $k \rightarrow \infty$.*

Proof. Let $\{t_l\}$ be a dense sequence in $C(\bar{\Omega})$. For each k we choose j_k such that

$$\left| \int_{\bar{\Omega}} t_l d\mu - \int_{\bar{\Omega}} t_l d\mu_k^{j_k} \right| < \frac{1}{k}, \quad 1 \leq l \leq k.$$

It follows that $\mu_k^{j_k}$ tends weak* to μ as $k \rightarrow \infty$, since $\{t_l\}$ is dense and the measures have uniformly bounded total mass. ■

Proof of Theorem 6.1. For each k , the measure μ can be regarded as a finite measure on Ω_k which vanishes on pluripolar sets. Hence there is $u_k \in \mathcal{F}^a(\Omega_k)$ such that $(dd^c u_k)^n = \mu$ (see Lemma 5.14 in [8]). Choose a fundamental sequence $\{\omega_j\}$ of Ω , i.e. $\omega_j \subset\subset \omega_{j+1} \subset\subset \Omega$ and $\bigcup_{j=1}^\infty \omega_j = \Omega$. For each k and j , define $u_k^j = \sup \{\varphi \in \text{PSH}^-(\Omega_k) : \varphi|_{\omega_j} \leq u_k|_{\omega_j}\}$. Then $u_k^j \in \mathcal{F}^a(\Omega_k)$ (note that $(u_k^j)^* = u_k^j$ since ω_j is open, so u_k^j is plurisubharmonic) and we have the following:

- (i) $\text{supp} (dd^c u_k^j)^n \subset \partial\omega_j$, $u_k^j \geq u_k$ on Ω_k , $\int_{\Omega_k} (dd^c u_k^j)^n \leq \int_{\Omega_k} (dd^c u_k)^n = \int_{\partial\Omega} d\mu$.
- (ii) If $j_1 \leq j_2$ then $u_k^{j_1} \geq u_k^{j_2}$ on Ω_k .
- (iii) $\lim_{j \rightarrow \infty} u_k^j = u_k$ on Ω_k .

The first two statements are obvious. To prove the third, let $v_k = \lim_j u_k^j$. Then $v_k \in \mathcal{F}(\Omega_k)$, $v_k \geq u_k$ on Ω_k and $v_k = u_k$ on Ω . Thus $v_k(\xi) = u_k(\xi)$ for $\xi \in \partial\Omega$, using the assumption (6b), so $v_k \leq u_k$ on Ω_k by Lemma 4.9, and the statement follows. Now, (ii) and (iii) imply that $(dd^c u_k^j)^n$ tends weak* to

$(dd^c u_k)^n = \mu$ as $j \rightarrow \infty$, for each fixed k . Hence, by (i) we can use Lemma 6.2 to pick $\{j_k\}$ such that $(dd^c u_k^{j_k})^n$ tends weak* to μ as $k \rightarrow \infty$. This proves the first part of the theorem if we let $w_k = u_k^{j_k}$.

It remains to show that $\lim_{k \rightarrow \infty} \int t (dd^c u_k^{j_k})^n = 0$ for all $t \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$, assuming that (6.1) holds. Given $t \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ there is by (6a) a sequence $\{t_k\}$ with $t_k \in \mathcal{F}(\Omega_k) \cap L^\infty(\Omega_k)$ such that t_k increases a.e. to t on Ω . Now,

$$\begin{aligned} \int_{\Omega_k} t (dd^c u_k^{j_k})^n &\geq \int_{\Omega_k} t_k (dd^c u_k^{j_k})^n \geq \int_{\Omega_k} t_k (dd^c u_k)^n \\ &= \int_{\partial\Omega} t_k d\mu \geq \int_{\Omega} t_k d\nu > -\infty, \end{aligned}$$

so $\liminf_{k \rightarrow \infty} \int_{\Omega_k} t (dd^c u_k^{j_k})^n \geq \int_{\Omega} t d\nu$. Define $t^i = \sup \{\varphi \in \text{PSH}(\Omega) : \varphi|_{\Omega \setminus \omega_i} \leq t|_{\Omega \setminus \omega_i}\}$. Then $t^i \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ and $t^i = t$ on $\Omega \setminus \omega_i$, so

$$\liminf_{k \rightarrow \infty} \int_{\Omega_k} t (dd^c u_k^{j_k})^n = \liminf_{k \rightarrow \infty} \int_{\Omega_k} t^i (dd^c u_k^{j_k})^n \geq \int_{\Omega} t^i d\nu,$$

by the above calculations. Now, the left hand side is independent of i , while the right hand side tends to 0 as i tends to ∞ , since ν vanishes on pluripolar sets. This completes the proof. ■

The reason not to keep k fixed in the proof above is to be able to prove the second part of the theorem. Also, one can prove that $\lim_{k \rightarrow \infty} u_k^j = 0$ a.e. on Ω , for each fixed j .

REMARK 6. Suppose that $v \in \text{PSH}^-(\Omega)$ satisfies $\tilde{v} \geq v \geq \tilde{v} + \psi$ for some $\psi \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ and that $\tilde{v} \in C(\bar{\Omega})$. (Thus, v is a function in $\mathcal{F}(\Omega, \tilde{v})$ with some additional properties, see Section 2.) Then the preceding theorem implies that

$$(6.2) \quad \lim_{k \rightarrow \infty} \int_{\partial\Omega} v (dd^c u_k^{j_k})^n = \int_{\partial\Omega} \tilde{v} d\mu,$$

where the limit is in weak* sense. To see this, take $f \in C(\bar{\Omega})$ with $f \geq 0$. Then by the theorem we have $\lim_{k \rightarrow \infty} \int_{\Omega} f \tilde{v} (dd^c u_k^{j_k})^n = \int_{\partial\Omega} f \tilde{v} d\mu$ and $0 \geq \int_{\Omega} f \psi (dd^c u_k^{j_k})^n \geq \max f \cdot \int_{\Omega} \psi (dd^c u_k^{j_k})^n$, where the last integral tends to 0 as $k \rightarrow \infty$. Hence the inequality $f \tilde{v} \geq f v \geq f \tilde{v} + f \psi$ implies that $\lim_{k \rightarrow \infty} \int_{\Omega} f v (dd^c u_k^{j_k})^n = \int_{\partial\Omega} f \tilde{v} d\mu$, and (6.2) follows.

Furthermore, if we assume that $\int_{\Omega} \varphi d\nu > -\infty$ for all $\varphi \in \mathcal{F}(\bar{\Omega})$, then (6.2) holds for all $v \in \mathcal{F}(\Omega, \tilde{v})$ where $\tilde{v} \in C(\bar{\Omega})$. This is due to the fact that the boundedness of t in the second part of Theorem 6.1 is used only to ensure that $\int_{\Omega} t_k d\nu > -\infty$ (because if t is bounded then t_k is bounded). Hence the assumption that $t \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ can be replaced by the assumption that $t \in \mathcal{F}(\Omega)$ and $\int_{\Omega} \varphi d\nu > -\infty$ for all $\varphi \in \mathcal{F}(\bar{\Omega})$.

EXAMPLE 6.3. Let Ω be the unit bidisc $\mathbb{D} \times \mathbb{D}$ in \mathbb{C}^2 . Let μ and ν be defined by

$$\mu = \sigma_1 \times dV_{1/2} \quad \text{and} \quad \nu = \sigma_{1/2} \times dV_{1/2},$$

where σ_r denotes the normalized Lebesgue measure on the circle $\partial\mathbb{D}(0, r)$ and $dV_{1/2}$ the normalized Lebesgue measure on the disc $\mathbb{D}(0, 1/2)$. Then μ and ν satisfy (6.1), so Theorem 6.1 tells us that we can approximate μ from the inside of Ω by our procedure. Moreover, by Example 4.11 we see that μ is not in the weak* closure of $\{\mu_u : u \in \mathcal{F}(\Omega)\}$. Hence, we do reach more measures by the method in this section than we could before.

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References

- [1] P. Åhag, U. Cegrell, R. Czyż and H. H. Phạm, *Monge–Ampère measures on pluripolar sets*, Institut Mittag-Leffler Report No. 21, 2007/2008.
- [2] P. Åhag, U. Cegrell, S. Kołodziej, H. H. Phạm and A. Zeriahi, *Partial pluricomplex energy and integrability exponents of plurisubharmonic functions*, Institut Mittag-Leffler Report No. 24, 2007/2008.
- [3] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.
- [4] S. Benelkourchi, *A note on the approximation of plurisubharmonic functions*, C. R. Math. Acad. Sci. Paris 342 (2006), 647–650.
- [5] Z. Błocki, *On the definition of the complex Monge–Ampère operator in \mathbb{C}^2* , Math. Ann. 328 (2004), 415–423.
- [6] U. Cegrell, *Pluricomplex energy*, Acta Math. 180 (1998), 187–217.
- [7] —, *Maximal plurisubharmonic functions*, Uzbek. Math. J. 11 (2009), 10–16.
- [8] —, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.
- [9] —, *Potentials with respect to the pluricomplex Green function*, Research Reports 3, Mid Sweden Univ., 2006.
- [10] —, *A general Dirichlet problem for the complex Monge–Ampère operator*, Ann. Polon. Math. 94 (2008), 131–147.
- [11] —, *Approximation of plurisubharmonic functions in hyperconvex domains*, in: Complex Analysis and Digital Geometry, Proceedings of the Kiselmanfest, Acta Universitatis Upsaliensis, to appear.
- [12] U. Cegrell and L. Hed, *Subextension and approximation of negative plurisubharmonic functions*, Michigan Math. J. 56 (2008), 593–601.
- [13] U. Cegrell and S. Kołodziej, *The equation of complex Monge–Ampère type and stability of solutions*, Math. Ann. 334 (2006), 713–729.
- [14] J.-P. Demailly, *Mesures de Monge–Ampère et mesures pluriharmoniques*, Math. Z. 194 (1987), 519–564.
- [15] G. M. Henkin, *The Banach space of analytic functions on the ball and the bicylinder are not isomorphic*, Funct. Anal. Appl. 2 (1968), 334–341.
- [16] P. Lelong, *Fonctions entières de type exponentiel dans \mathbb{C}^n* , Ann. Inst. Fourier (Grenoble) 16 (1966), 269–318.

- [17] P. Lelong, *Fonction de Green pluricomplexe et lemmes de Schwarz dans les espaces de Banach*, J. Math. Pures Appl. 68 (1989), 319–347.
- [18] E. A. Poletsky, *Approximation of plurisubharmonic functions by multipole Green functions*, Trans. Amer. Math. Soc. 355 (2003), 1579–1591.
- [19] N. Sibony, *Une classe de domaines pseudoconvexes*, Duke Math. J. 55 (1987), 299–319.

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