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FUNCTIONAL ANALYSIS

An Isomorphic Classification of $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ Spaces

by

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Summary. We present an extension of the classical isomorphic classification of the Banach spaces $C([0,\alpha])$ of all real continuous functions defined on the nondenumerable intervals of ordinals $[0,\alpha]$. As an application, we establish the isomorphic classification of the Banach spaces $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ of all real continuous functions defined on the compact spaces $\mathbf{2}^{\mathfrak{m}} \times [0,\alpha]$, the topological product of the Cantor cubes $\mathbf{2}^{\mathfrak{m}}$ with $\mathfrak{m}$ smaller than the first sequential cardinal, and intervals of ordinal numbers $[0,\alpha]$. Consequently, it is relatively consistent with ZFC that this yields a complete isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ spaces.

1. Introduction and statement of the main result. Throughout the paper, we use standard notation and basic concepts in set theory [11] and theory of Banach spaces [12]. However, we want to explain some frequently used terms and fix some notations. For a compact Hausdorff topological space K and X a Banach space, let $C(K,X)$ denote the Banach space of all continuous X-valued functions defined on K, equipped with the usual supremum norm. When $X=\mathbb{R}$, the set of real numbers, this space will be denote by $C(K)$. As usual, if K_1 and K_2 are compact spaces, we denote by $K_1 \oplus K_2$ and $K_1 \times K_2$ respectively the topological sum and the topological product of K_1 and K_2. For a fixed cardinal $\mathfrak{m}$, $\mathbf{2}^{\mathfrak{m}}$ denotes the product of $\mathfrak{m}$ family of copies of the two-point space $\mathbf{2}$, provided with the product topology. For α an ordinal number, $[0,\alpha]$ denotes the interval of ordinals $\{\xi : 0 \le \xi \le \alpha\}$ endowed with the order topology. If X and Y are Banach spaces, then $X \sim Y$ means that X is isomorphic to Y. Finally, the symbol $X \oplus Y$ denotes the Cartesian product of X and Y.

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Very recently [8], it has been shown that is relatively consistent with Zermelo–Fraenkel set theory plus the axiom of choice (ZFC) that for any infinite cardinals $\mathfrak{m}$ and $\mathfrak{n}$ and nondenumerable ordinals ξ and η we have

$$C(\mathbf{2}^{\mathfrak{m}}\oplus[0,\xi])\sim C(\mathbf{2}^{\mathfrak{n}}\oplus[0,\eta]) \quad \text{if and only if} \quad \mathfrak{m}=\mathfrak{n} \text{ and } C([0,\xi])\sim C([0,\eta]).$$

In other words, the isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\alpha])$ spaces is reduced to the isomorphic classification of $C([0,\alpha])$ spaces, in the case where $\alpha \geq \omega_1$. Recall that the isomorphic classification of $C([0,\alpha])$ spaces is due to Bessaga and Pełczyński [2] in the case where $\omega \leq \alpha < \omega_1$; Semadeni [26] in the case where $\omega_1 < \alpha \leq \omega_1\omega$; Labbé [15] in the case where $\omega_1\omega < \alpha < \omega_1^{\omega}$; and independently Kislyakov [14] and Gul'ko and Os'kin [10] in the general case.

In the present paper, we turn our attention to $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ spaces. In contrast with the isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}}\oplus[0,\alpha])$ spaces mentioned above, the situation becomes quite different when we consider this new family of $C(K)$ spaces. This happens even in the isometric case. Indeed, it is well known that for every infinite cardinal $\mathfrak{m}$, $\mathbf{2}^{\mathfrak{m}}$ is homeomorphic to $\mathbf{2}^{\mathfrak{m}} \times \mathbf{2}^{\mathfrak{m}}$. Consequently, $\mathbf{2}^{\mathfrak{m}} \times [0,\omega_1]$ is homeomorphic to $\mathbf{2}^{\mathfrak{m}} \times [0,\omega_1 2]$. Therefore by [14, Theorem 1] we deduce that

$$C(\mathbf{2}^{\mathfrak{m}}\times[0,\omega_1]) \text{ is isometric to } C(\mathbf{2}^{\mathfrak{m}}\times[0,\omega_1 2]) \quad \text{but} \quad C([0,\omega_1])\nsim C([0,\omega_1 2]).$$

This motivates us to study the isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ spaces. In order to do this, we will prove Theorem 1.7, which is an extension of the isomorphic classification of $C([0,\alpha])$ spaces, with $\alpha \geq \omega_1$.

Remark 1.1. First notice that if $0 \leq \mathfrak{m} < \aleph_0$, then $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha]) \sim C([0,\alpha\,\mathbf{2}^{\mathfrak{m}}])$. Assume now that $\mathfrak{m} \geq \aleph_0$ and $\alpha < \omega_1$. According to the classical Milyutin theorem [27, Theorem 21.5.10] about the isomorphic classification of $C(K)$ spaces with K compact metric nondenumerable, $C(\mathbf{2}^{\aleph_0} \times [0,\alpha]) \sim C(\mathbf{2}^{\aleph_0})$. Consequently, by [27, Theorem 20.5.6] we deduce $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha]) \sim C(\mathbf{2}^{\mathfrak{m}} \times \mathbf{2}^{\aleph_0} \times [0,\alpha]) \sim C(\mathbf{2}^{\mathfrak{m}}, C(\mathbf{2}^{\aleph_0} \times [0,\alpha])) \sim C(\mathbf{2}^{\mathfrak{m}} \times \mathbf{2}^{\aleph_0}) \sim C(\mathbf{2}^{\mathfrak{m}})$.

Observe also that for every $\mathfrak{m} \geq \aleph_0$, $\mathfrak{n} \geq \aleph_0$ and $\alpha \geq \omega_1$, we have $C(\mathbf{2}^{\mathfrak{m}}) \nsim C(\mathbf{2}^{\mathfrak{n}} \times [0,\alpha])$. Indeed, otherwise $C([0,\alpha])$ would be isomorphic to a subspace of $C(\mathbf{2}^{\mathfrak{m}})$, which is absurd by [23, Theorem 4.5] and [5, Theorem 2.3.17].

So it remains to consider the cases $\mathfrak{m} \geq \aleph_0$ and $\alpha \geq \omega_1$. In order to describe our results, we recall that a cardinal $\mathfrak{m}$ is called *sequential* if there exists a sequentially continuous but not continuous real-valued function on $\mathbf{2}^{\mathfrak{m}}$. We also recall that a function $f : \mathbf{2}^{\mathfrak{m}} \to \mathbb{R}$ is said to be *sequentially continuous* when $f(k_n)$ converges to $f(k)$ whenever the sequence $(k_n)_{n<\omega}$ converges to k in $\mathbf{2}^{\mathfrak{m}}$ (see [1] and [19]). The cardinality of the ordinal ξ will be denoted by $\bar{\xi}$. Our first result is as follows.

THEOREM 1.2. *Suppose that $\mathfrak{m}$ and $\mathfrak{n}$ are nonsequential infinite cardinals and ξ and η are nondenumerable ordinals. Then*

$$C(\mathbf{2}^{\mathfrak{m}} \times [0,\xi]) \sim C(\mathbf{2}^{\mathfrak{n}} \times [0,\eta]) \quad \textit{implies that} \quad \mathfrak{m} = \mathfrak{n} \textit{ and } \bar{\xi} = \bar{\eta}.$$

The following is an analogue of the above mentioned result of [8].

THEOREM 1.3. *Let $\mathfrak{m}$ be a nonsequential infinite cardinal, α a nondenumerable initial ordinal and $\xi \leq \eta$ ordinals with $\bar{\xi} = \bar{\eta} = \bar{\alpha}$. If α is singular or $\alpha^2 \leq \xi$, then*

$$C(\mathbf{2}^{\mathfrak{m}} \times [0,\xi]) \sim C(\mathbf{2}^{\mathfrak{m}} \times [0,\eta]) \quad \textit{if and only if} \quad C([0,\xi]) \sim C([0,\eta]).$$

The next theorems complete the isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}}\times[0,\alpha])$ spaces with $\mathfrak{m}$ a nonsequential cardinal.

THEOREM 1.4. *Let $\mathfrak{m}$ be a nonsequential infinite cardinal, α a nondenumerable regular ordinal and ξ and η in $[\alpha,\alpha^2]$. Let ξ', η', γ and δ be ordinals such that $\xi = \alpha\xi' + \gamma$, $\eta = \alpha\eta' + \delta$, $\xi', \eta' \leq \alpha$ and $\gamma, \delta < \alpha$. Then*

$$C(\mathbf{2}^{\mathfrak{m}} \times [0,\xi]) \sim C(\mathbf{2}^{\mathfrak{m}} \times [0,\eta]) \quad \textit{if and only if} \quad \bar{\xi'}\,\bar{\eta'} \leq \aleph_0 \textit{ or } \bar{\xi'} = \bar{\eta'}.$$

THEOREM 1.5. *Let $\mathfrak{m}$ be a nonsequential infinite cardinal, α a nondenumerable regular ordinal and ξ and η with $\bar{\xi} = \bar{\eta}$ and $\alpha \leq \xi < \alpha^2 \leq \eta$. Then*

$$C(\mathbf{2}^{\mathfrak{m}} \times [0,\xi]) \nsim C(\mathbf{2}^{\mathfrak{m}} \times [0,\eta]).$$

REMARK 1.6. It is well known that it is relatively consistent with ZFC that there exist no sequential cardinals (see [20]). So it is relatively consistent with ZFC that Theorems 1.2–1.5 provide a complete isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ spaces. Furthermore, since $\aleph_0$ is not sequential [17], the above theorems give a complete isomorphic classification of $C(\mathbf{2}^{\aleph_0} \times [0,\alpha])$ spaces, without using the continuum hypothesis. Thus, we have got an answer to Question 3.5 raised in [7].

Although our work is motivated by the search for the isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ spaces, our main result holds for a more general setting. Indeed, from now on, our task is to prove Theorem 1.7. The preceding theorems are immediate consequences of Theorem 1.7 and Lemma 2.5.

Henceforth following [2], the $C([0,\alpha],X)$ spaces will also be denoted by X^α. Theorem 1.7 states that the isomorphic classification of X^α spaces, with $\alpha \geq \omega_1$ and X having the Mazur property and containing no subspace isomorphic to c_0, obtained recently in [9] is also true under the weaker hypothesis that X has the Mazur property and contains no subspace isomorphic to $c_0(\Gamma)$, where Γ is a set of cardinality $\aleph_1$.

We recall that a Banach space X is said to have the *Mazur property* if every element of X^{**}, the bidual space of X, which is sequentially weak* continuous is weak* continuous and thus is an element of X. Such spaces were investigated in [4], [16] and also in [13] and [28] where they were called d-complete and μB-spaces, respectively. The class of Banach spaces having the Mazur property includes the $C(\mathbf{2}^{\mathfrak{m}})$ spaces for every nonsequential cardinal $\mathfrak{m}$ [20] (see also [21, Theorem 5.2.c]). Given a set Γ, we denote by $|\Gamma|$ the cardinality of Γ.

THEOREM 1.7. *Let X be a Banach space having the Mazur property and containing no subspace isomorphic to $c_0(\Gamma)$, where $|\Gamma| = \aleph_1$, let α be an initial ordinal and $\xi \le \eta$ two infinite ordinals.*

(1) *If $X^\xi \sim X^\eta$ then $\bar{\xi} = \bar{\eta}$.*
(2) *Suppose $\bar{\xi} = \bar{\eta} = \bar{\alpha}$ and assume that α is a singular ordinal, or α is a nondenumerable regular ordinal with $\alpha^2 \le \xi$. Then $X^\xi \sim X^\eta$ if and only if $\eta < \xi^\omega$.*
(3) *Suppose that α is a nondenumerable regular ordinal, $\xi, \eta \in [\alpha, \alpha^2]$ and let ξ', η', γ and δ be ordinals such that $\xi = \alpha\xi' + \gamma$, $\eta = \alpha\eta' + \delta$, $\xi', \eta' \le \alpha$ and $\gamma, \delta < \alpha$. Then $X^\xi \sim X^\eta$ if and only if either $\bar{\xi'}\,\bar{\eta'} \le \aleph_0$ and $c_0(I, X) \sim c_0(J, X)$ where I and J are sets with $|I| = \bar{\xi'}$ and $|J| = \bar{\eta'}$, or $\bar{\xi'} = \bar{\eta'}$.*
(4) *Suppose that α is a nondenumerable regular ordinal and $\alpha \le \xi < \alpha^2 \le \eta$. Then $X^\xi \nsim X^\eta$.*

2. Preliminary lemmas. In this section we state and prove several lemmas from which Theorem 1.7 follows easily. The first three lemmas provide sufficient conditions for a Banach space X to contain a subspace isomorphic to $c_0(\Gamma)$, where $|\Gamma| = \aleph_1$. If X and Y are Banach spaces, then $X \hookrightarrow Y$ means that X is isomorphic to a subspace of Y.

LEMMA 2.1. *Let X be a Banach space and α an nondenumerable infinite initial ordinal. Suppose that $\mathbb{R}^\alpha \hookrightarrow X^\eta$ for some $\eta < \alpha$. Then $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$.*

Proof. Assume first that α is a regular ordinal. Let I be the set of isolated points of $[0, \alpha]$. Then $c_0(I) \hookrightarrow \mathbb{R}^\alpha$. So there exists an isomorphism T from $c_0(I)$ onto a subspace of X^η. Let $M \in \,]0, +\infty[$ be such that $M \le \|T(x)\|$ for all $x \in c_0(I)$, $\|x\| = 1$. Denote by $(e_i)_{i \in I}$ the unit-vectors basis of $c_0(I)$, that is, $e_i(j) = 1$ if $i = j$, $e_i(j) = 0$ if $i \ne j$, for all $i, j \in I$. Let K be the set of isolated points of $[0, \eta]$. Thus, by hypothesis $|K| < \bar{\alpha}$. For fixed $k \in K$, we define $I_k = \{i \in I : M/2 \le \|T(e_i)(k)\|\}$. Therefore $I = \bigcup_{k \in K} I_k$. Hence there is a $k \in K$ satisfying $|I_k| = |I|$. We identify $c_0(I_k)$ with the subspace of $c_0(I)$ consisting of those elements f such that $f(\gamma) = 0$ for every $\gamma \notin I_k$. Let $P_k : X^\eta \to X$ be the natural projection, that is, $P_k(f) = f(k)$ for all $f \in X^\eta$. Next, consider the operator $L = P_k T_{|c_0(I_k)} : c_0(I_k) \to X$. Then $\inf \{\|L(e_i)\| : i \in I_k\} > 0$. So, according to Remark 1 which follows

[22, Theorem 3.4], there exists $\Gamma \subset I_k$ with $|\Gamma| = |I_k|$ such that $L_{|c_0(\Gamma)}$ is an isomorphism onto its image. So we are done.

Let us now suppose that α is a singular ordinal. Then there exists an ordinal limit λ such that $\alpha = \omega_\lambda$. Let γ be an ordinal satisfying $\eta < \omega_{\gamma+1} < \omega_\lambda$. It is known that $\omega_{\gamma+1}$ is regular. Moreover, by hypothesis $\mathbb{R}^{\omega_{\gamma+1}} \hookrightarrow \mathbb{R}^{\omega_\lambda} \hookrightarrow X^\eta$. Hence by what we have just proved, we conclude that $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$. ∎

In the same fashion we can prove:

Lemma 2.2. *Let X be a Banach space such that $c_0(I) \hookrightarrow c_0(J, X)$ for some sets I and J with $|J| < |I|$ and $|I| \geq \aleph_1$. Then $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$.*

Before stating the next lemma, we recall some definitions from [6] and [9]. Let γ be an ordinal. A *γ-sequence* in a set A is a function $f : [1, \gamma[\to A$ and will be denoted by $(x_\theta)_{\theta<\gamma}$. If A is a topological space and β is an ordinal, we will say that the γ-sequence $(x_\theta)_{\theta<\gamma}$ is *β-continuous* if for every β-sequence of ordinals $(\theta_\xi)_{\xi<\beta}$ of $[0, \gamma]$ which converges to θ_β when ξ converges to β, the β-sequence x_{θ_ξ} converges to x_{θ_β}.

Let X be a Banach space, α an ordinal number and φ a cardinal number. By X_α^φ we will denote the space of all $x^{**} \in X^{**}$ having the following property: for every set B with $|B| = \varphi$, $\beta < \alpha$ and B-family $x^b = (x_\xi^*(b))_{\xi<\beta}$, $b \in B$, of β-sequences of X^* such that there exists $M \in \mathbb{R}$ with $\|x_\xi^*(b)\| \leq M$ for every $b \in B$ and $\xi < \beta$ and such that $x_\xi^*(b)(x) \xrightarrow{\xi\to\beta} 0$ for all $x \in X$, uniformly in b, we have $x^{**}(x_\xi^*(b)) \xrightarrow{\xi\to\beta} 0$ uniformly in b.

Clearly X_α^φ is a closed subspace of X^{**} and $cX \subset X_\alpha^\varphi$, where cX is the canonical image of X in X^{**}. Observe that if X has the Mazur property, then $X_\alpha^\varphi = cX$.

Let X be a Banach space and α a nondenumerable regular ordinal. Following [9, Definition 2.2], we set $[X]_\alpha = \bigcap_{\varphi<\alpha} X_\alpha^\varphi$.

We are ready to generalize [9, Lemma 2.8].

Lemma 2.3. *Let X be a Banach space having the Mazur property and α a nondenumerable regular ordinal. If $\mathbb{R}^{\alpha^2} \hookrightarrow X^\eta$ for some $\eta < \alpha^2$, then $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$.*

Proof. We distinguish two cases:

Case 1: $\eta < \alpha$. In this case, $\mathbb{R}^\alpha \hookrightarrow X^\eta$. Hence by Lemma 2.1, $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$, and we are done.

Case 2: $\alpha \leq \eta < \alpha^2$. Thus $\eta = \alpha\xi + \theta$ for some ordinals $\xi < \alpha$ and $\theta < \alpha$. Since then $\mathbb{R}^\eta \sim \mathbb{R}^{\alpha\xi}$ [14, Theorem 2], we have

$$\mathbb{R}^{\alpha^2} \hookrightarrow \mathbb{R}^\eta \hookrightarrow C([0,\eta], X) \sim C([0,\alpha\xi], X).$$

Let I and J be two sets with $|I| = \bar{\alpha}$ and $|J| = \bar{\xi}$. According to [9, Lemma 2.4 and Proposition 2.8], we have

$$c_0(I) \sim \frac{[\mathbb{R}^{\alpha^2}]_\alpha}{c\mathbb{R}^{\alpha^2}} \hookrightarrow \frac{[X^{\alpha\xi}]_\alpha}{cX^{\alpha\xi}} \sim c_0(J, X).$$

Thus by Lemma 2.2 we infer that $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$. So we are also done. ■

The main step in proving Theorem 1.7 is the following result. It is a generalization of part of [9, Lemma 2.10] (see also [25, Theorem 3.2]).

Lemma 2.4. *Let α be a nondenumerable initial ordinal and $\xi \leq \eta$ ordinals with $\bar{\xi} = \bar{\eta} = \bar{\alpha}$. Put $\alpha_0 = \alpha$ if α is a singular ordinal and $\alpha_0 = \alpha^2$ if α is a regular ordinal. Suppose that X is a Banach space having the Mazur property and containing no subspace isomorphic to $c_0(\Gamma)$, where $|\Gamma| = \aleph_1$. If $\mathbb{R}^\eta \hookrightarrow X^\xi$ with $\alpha_0 \leq \xi$, then $\mathbb{R}^\eta \hookrightarrow \mathbb{R}^\xi$.*

Proof. We introduce two sets of ordinals

$$I_1 = \{\theta : \bar{\theta} = \bar{\alpha},\ \alpha_0 \leq \theta,\ \mathbb{R}^\theta \not\hookrightarrow \mathbb{R}^\gamma,\ \forall \gamma < \theta\},$$
$$I_2 = \{\theta : \bar{\theta} = \bar{\alpha},\ \alpha_0 \leq \theta,\ \mathbb{R}^\theta \not\hookrightarrow X^\gamma,\ \forall \gamma < \theta\}.$$

First of all we will prove that $I_1 = I_2$. Clearly $I_2 \subset I_1$. Observe that by Lemmas 2.1 and 2.3 we deduce that $\alpha_0 \in I_2$. Now, assume that I_2 is a proper subset of I_1. Let α_1 be the least element of $I_1 \setminus I_2$. We have $\alpha_0 < \alpha_1$. Since $\alpha_1 \notin I_2$, there exists an ordinal $\gamma_1 < \alpha_1$ such that $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\gamma_1}$.

Let $\alpha_2 = \min\{\gamma : \alpha_0 \leq \gamma < \alpha_1,\ \mathbb{R}^{\alpha_1} \hookrightarrow X^\gamma\}$. We have $\alpha_2 \leq \gamma_1$. Now, we will show that $\alpha_2 \in I_1$. If this is not the case, there exists an ordinal $\gamma_2 < \alpha_2$ such that $\mathbb{R}^{\alpha_2} \hookrightarrow \mathbb{R}^{\gamma_2}$. Therefore $C([0, \alpha_2], X) \hookrightarrow C([0, \gamma_2], X)$. Consequently, $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\gamma_2}$, in contradiction with the definition of α_2.

So $\alpha_2 \in I_1$ and since $\alpha_2 < \alpha_1$, it follows from the definition of α_1 that $\alpha_2 \in I_2$. That is, $\mathbb{R}^{\alpha_2} \not\hookrightarrow X^\gamma$ for all $\gamma < \alpha_2$. Thus by [7, Lemma 3.3], we conclude that $\mathbb{R}^{\alpha_2^\omega} \not\hookrightarrow X^{\alpha_2}$.

On the other hand, note that if $\alpha_1 < \alpha_2^\omega$, then by [14, Theorems 1 and 2], $\mathbb{R}^{\alpha_1} \sim \mathbb{R}^{\alpha_2}$, which is absurd by the definition of α_1. Consequently, $\alpha_2^\omega \leq \alpha_1$ and $\mathbb{R}^{\alpha_2^\omega} \hookrightarrow \mathbb{R}^{\alpha_1}$. Furthermore, by the definition of α_2, $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\alpha_2}$. Therefore $\mathbb{R}^{\alpha_2^\omega} \hookrightarrow X^{\alpha_2}$, in contradiction with what we have just proved above. Hence $I_1 = I_2$.

Next, to complete the proof of the lemma, suppose that $\mathbb{R}^\eta \not\hookrightarrow \mathbb{R}^\xi$ and let $\xi_1 = \min\{\theta : \mathbb{R}^\eta \hookrightarrow \mathbb{R}^\theta\}$. Hence $\xi < \xi_1 \leq \eta$ and $\mathbb{R}^{\xi_1} \not\hookrightarrow \mathbb{R}^\gamma$ for all $\gamma < \xi_1$. In particular, $\xi_1 \in I_1 = I_2$, which is absurd, because $\mathbb{R}^{\xi_1} \hookrightarrow \mathbb{R}^\eta \hookrightarrow X^\xi$. ■

We conclude this section by proving part of Theorem 1.2.

Lemma 2.5. *If $C(\mathbf{2}^{\mathfrak{m}} \times [0, \xi]) \sim C(\mathbf{2}^{\mathfrak{n}} \times [0, \eta])$ for some infinite cardinals $\mathfrak{m}$ and $\mathfrak{n}$ and ordinals ξ and η, then $\mathfrak{m} = \mathfrak{n}$.*

Proof. Assume that $\mathfrak{m} < \mathfrak{n}$ and let Γ and Λ be two sets of the same cardinality of ξ and η, respectively. Therefore $l_1(\Gamma, C(\mathbf{2}^{\mathfrak{m}})^*) \sim l_1(\Lambda, C(\mathbf{2}^{\mathfrak{n}})^*)$. According to [23, Proposition 5.2] we infer that

$$l_1\Big(\Gamma, \Big(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}}\Big)_1\Big) \sim l_1\Big(\Lambda, \Big(\sum_{2^{\mathfrak{n}}} \oplus L^1[0,1]^{\mathfrak{n}}\Big)_1\Big).$$

Recall that given a Banach space X, the *dimension* of X is the smallest cardinal δ for which there exists a subset of cardinality δ with linear span norm-dense in X. Pick a subspace H of $L^1[0,1]^{\mathfrak{n}}$ which is isomorphic to a Hilbert space of dimension $\mathfrak{n}$ [24, Proposition 1.5]. Hence

$$H \hookrightarrow l_1\Big(\Gamma, \Big(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}}\Big)_1\Big).$$

Since H contains no subspace isomorphic to l_1, by a standard gliding hump argument (see [3]), we infer that there exist a finite sum of $L^1[0,1]^{\mathfrak{m}}$ and $1 \le p < \omega$ such that

$$H \hookrightarrow L^1[0,1]^{\mathfrak{m}} \oplus L^1[0,1]^{\mathfrak{m}} \oplus \cdots \oplus L^1[0,1]^{\mathfrak{m}} \oplus \mathbb{R}^p,$$

which is absurd, because it is easy to see that the dimension of $L^1[0,1]^{\mathfrak{m}}$ is $\mathfrak{m}$. ∎

3. Proof of Theorem 1.7. (1) Assume that $X^{\xi} \sim X^{\eta}$ and $\bar{\xi} < \bar{\eta}$. Let α be the initial ordinal of cardinality $\bar{\eta}$. Then $\mathbb{R}^{\alpha} \hookrightarrow X^{\eta} \sim X^{\xi}$ and by Lemma 2.1, $c_0(I) \hookrightarrow X$, where $|I| = \aleph_1$, which is absurd.

To prove the sufficiency of the statements (2) and (3) it is enough to keep in mind [14, Theorems 1 and 2] and observe that if $\mathbb{R}^{\xi} \sim \mathbb{R}^{\eta}$ then $X^{\xi} \sim X^{\eta}$.

Next we prove the necessity of the statements (2) and (3).

(2) Suppose that $X^{\eta} \sim X^{\xi}$. If $\eta > \xi^{\omega}$, then $\mathbb{R}^{\eta} \hookrightarrow X^{\eta} \sim X^{\xi}$. According to Lemma 2.4 we obtain $\mathbb{R}^{\eta} \hookrightarrow \mathbb{R}^{\xi}$, which is absurd by [14, Theorem 1].

(3) Let I and J be two sets with $|I| = \bar{\xi'}$ and $|J| = \bar{\eta'}$. Since X has the Mazur property, by [9, Remark 2.3] and [9, Proposition 2.8], we infer that

$$c_0(I, X) \sim \frac{[X^{\alpha\xi'}]_{\alpha}}{cX^{\alpha\xi'}} \sim \frac{[X^{\alpha\eta'}]_{\alpha}}{cX^{\alpha\eta'}} \sim c_0(J, X).$$

Therefore if $\bar{\xi'}\bar{\eta'} \le \aleph_0$ we are done. Suppose now that $\bar{\xi'}\bar{\eta'} > \aleph_0$ and $\bar{\xi'} \ne \bar{\eta'}$. We can assume without loss of generality that $\bar{\xi'} > \bar{\eta'}$ and $\bar{\xi'} \ge \aleph_1$. Since $c_0(I) \hookrightarrow c_0(J, X)$, it follows by Lemma 2.2 that $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$, a contradiction.

(4) Suppose that $X^{\xi} \sim X^{\eta}$ with $\alpha \le \xi < \alpha^2 \le \eta$. Then $\mathbb{R}^{\alpha^2} \hookrightarrow X^{\xi}$. Hence by Lemma 2.3, $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$. This contradiction finishes the proof.

4. Some questions. If we are working only in ZFC theory, then the following question arises naturally.

QUESTION 4.1. *Is the assumption that the cardinal* $\mathfrak{m}$ *is not sequential in Theorems* 1.2–1.5 *necessary?*

We close the paper by recalling that the isometric classification of $C([0,\alpha])$ spaces is a direct consequence of the homeomorphic classification of $[0,\alpha]$ spaces accomplished by Mazurkiewicz and Sierpiński [18] and the classical Banach–Stone Theorem [27, Theorem 7.8.4]. Therefore our result also leads naturally to the following question:

QUESTION 4.2. *Give an isometric classification of* $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ *spaces.*

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