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GENERALIZED MIDCONVEXITY

JACEK TABOR

Institute of Computer Science, Jagiellonian University
Łojasiewicza 6, 30-348 Kraków, Poland
E-mail: tabor@ii.uj.edu.pl

JÓZEF TABOR

Institute of Mathematics, University of Rzeszów
Rejtana 16A, 35-310 Rzeszów, Poland
E-mail: tabor@univ.rzeszow.pl

KRZYSZTOF MISZTAL

Institute of Computer Science, Jagiellonian University
Łojasiewicza 6, 30-348 Kraków, Poland
E-mail: krzysztof.misztal@ii.uj.edu.pl

Abstract. There are many types of midconvexities, for example Jensen convexity, t-convexity, (s,t)-convexity. We provide a uniform framework for all the above mentioned midconvexities by considering a generalized middle-point map on an abstract space X.

We show that we can define and study the basic convexity properties in this setting.

1. Introduction. Convexity has found applications in many parts of science. The simplest version of convexity is Jensen convexity (called also midconvexity). One can enumerate the following types of midconvexities:

(i) classical convexity in a linear space [9],
(ii) t-convexity [7],
(iii) (s,t)-midconvexities [7, 8],
(iv) (M,N)-convexity [6],
(v) metric convexity [1, 2],
(vi) convexity in Abelian groups (in particular in $\mathbb{Z}^N$) [3].

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We are going to unify and study the above mentioned midconvexities in one abstract theory. From now we assume that X is a nonempty set with multivalued operation

$$m_X : X \times X \multimap X,$$

with possibly empty values, which satisfies

$$x \in m_X(x,x) \text{ for } x \in X. \tag{1}$$

We interpret $m_X(x,y)$ as the set of middle-points between x and y. If there is no doubt which middle-point function we have in mind we will write m instead of m_X.

EXAMPLE 1.1. Commonly used middle-point functions:

(i) X is a real vector space

- $m(x,y) = \{tx + (1-t)y\}$, where $t \in (0,1)$ is fixed;
- $m(x,y) = [x,y] = \{tx + (1-t)y : t \in [0,1]\}$;
- $m(x,y) = \{t^s x + (1-t)^s y\}$, where $t \in (0,1)$ and $s \in [1,\infty)$ are fixed;
- $m(x,y) = \{t^s x + (1-t)^s y : t \in [0,1]\}$, where $s \in [1,\infty)$ is fixed;

(ii) (X,d) is a metric space

- $m(x,y) = \{z \in X : d(x,z) = td(x,y);\ d(z,y) = (1-t)d(x,y)\}$, where $t \in [0,1]$ is fixed;
- $m(x,y) = \{z \in X : d(x,z) + d(z,y) = d(x,y)\}$;

(iii) $(X,+)$ is an Abelian group

- $m(x,y) = \{z \in X : 2z = x + y\}$;

(iv) $(X,\le)$ is a linearly ordered set

- $m(x,y) = \{\min(x,y)\}$;
- $m(x,y) = \{\max(x,y)\}$;
- $m(x,y) = \{z : \min(x,y) \le z \le \max(x,y)\}$;

(v) $X = I$ where I is a subinterval of $\mathbb{R}$ and $m(x,y) = \{M_I(x,y)\}$, where M_I is a mean on I.

Using middle-point function we will define midconvex set and prove some its characteristic properties. Our aim is to show that one can build a natural convexity theory based on middle-point functions. In particular we will present the method of determining midconvex hull of a given set. In the last section we will define midconvex functions.

2. Midconvex sets. In this section we show how to define the natural convexity notions based on the middle-point functions. We begin with the definition of convex set.

DEFINITION 2.1. We say that a set $W \subset X$ is m_X-*convex* if

$$m_X(a,b) \subset W \quad \text{for } a,b \in W. \tag{2}$$

Condition (2) can be rewritten as follows

$$m_X(W \times W) \subset W.$$

Evidently the set X is m_X-convex.

EXAMPLE 2.2.

(i) $X = \mathbb{R}$, $m(x,y) = \{\frac{x+y}{2}\}$. Then the set $\mathcal{D}$ of dyadic numbers is m-convex.
(ii) $X = (0,\infty)$, $m(x,y) = \{\sqrt{xy}\}$. Then the set of positive algebraic numbers is m-convex.

Now we present some direct consequences of Definition 2.1.

PROPOSITION 2.3. *Intersection of a nonempty family of m-convex sets is m-convex.*

Proof. Let $\{W_i\}_{i\in I}$ be a nonempty family of m-convex sets. Consider arbitrary $a, b \in \bigcap_{i\in I} W_i$. Then

$$m(a,b) \subset W_i \quad \text{for } i \in I,$$

and hence

$$m(a,b) \subset \bigcap_{i\in I} W_i. \qquad \blacksquare$$

PROPOSITION 2.4. *Let $\{W_i\}_{i\in I}$ be a nonempty family of m-convex sets directed with respect to inclusion, i.e. such that for each $i_1, i_2 \in I$ there exists $i_3 \in I$ with the property $W_{i_1} \subset W_{i_3}$, $W_{i_2} \subset W_{i_3}$. Then*

$$W := \bigcup_{i\in I} W_i$$

is m-convex.

Proof. Consider arbitrary $a, b \in W$ and choose $i_1, i_2 \in I$ such that $a \in W_{i_1}$, $b \in W_{i_2}$. Then we can find $i_3 \in I$ such that

$$a \in W_{i_1} \subset W_{i_3}, \qquad b \in W_{i_2} \subset W_{i_3},$$

whence we obtain

$$m_X(a,b) \subset W_{i_3} \subset W. \qquad \blacksquare$$

Suppose that we are given a set $Y \subset X$. In a natural way we define the restriction $m_{X|Y} : Y \times Y \multimap Y$ of m_X to Y by the formula

$$m_{X|Y}(y_1, y_2) := m_X(y_1, y_2) \cap Y \quad \text{for } y_1, y_2 \in Y.$$

One can easily observe that $m_{X|Y}$ is a middle-point map on Y.

PROPOSITION 2.5. *Let $W \subset Y \subset X$. If Y is an m_X-convex subset of X and W is an $m_{X|Y}$-convex subset of Y then W is m_X-convex.*

Proof. Consider arbitrary $w_1, w_2 \in W$. Since W is $m_{X|Y}$-convex

$$m_{X|Y}(w_1, w_2) = m_X(w_1, w_2) \cap Y \subset W.$$

However, by the m_X-convexity of Y and the fact that $w_1, w_2 \in W \subset Y$ we obtain that

$$m_X(w_1, w_2) \subset Y.$$

Consequently we obtain that $m_X(w_1, w_2) = m_X(w_1, w_2) \cap Y = m_{X|Y}(w_1, w_2) \subset W$. $\blacksquare$

Assume that we are given nonempty sets X, Y and m_X and m_Y satisfying (1). In the Cartesian product $X \times Y$ we define the middle-point operation $m_{X\times Y}$ by

$$m_{X\times Y}\big((x_1, y_1), (x_2, y_2)\big) := m_X(x_1, x_2) \times m_Y(y_1, y_2) \quad \text{for } x_1, x_2 \in X,\ y_1, y_2 \in Y.$$

PROPOSITION 2.6. *We assume that we are given nonempty sets X, Y and operations m_X, m_Y satisfying* (1). *Let $\emptyset \neq V \subset X$ and $\emptyset \neq W \subset Y$. Then $V \times W$ is $m_{X\times Y}$-convex if and only if V is m_X-convex and W is m_Y-convex.*

Proof. Assume that V is m_X-convex and W is m_Y-convex. Consider arbitrary (v_1, w_1), $(v_2, w_2) \in V \times W$. Then $m_X(v_1, v_2) \subset V$ and $m_Y(w_1, w_2) \subset W$, and consequently

$$m_{X\times Y}\big((v_1, w_1), (v_2, w_2)\big) = m_X(v_1, v_2) \times m_Y(w_1, w_2) \subset V \times W,$$

which proves that $V \times W$ is $m_{X\times Y}$-convex.

Assume now that $V \times W$ is $m_{X\times Y}$-convex. Consider arbitrary $v_1, v_2 \in V$, $w \in W$. Then

$$m_{X\times Y}\big((v_1, w), (v_2, w)\big) = m_X(v_1, v_2) \times m_Y(w, w) \subset V \times W,$$

and hence $m_X(v_1, v_2) \subset V$, which proves that V is m_X-convex. By the similar argumentation we obtain that W is m_Y-convex. ∎

DEFINITION 2.7. Let $W \subset X$. Intersection of all m_X-convex sets containing W is called *m_X-convex hull* of W and denoted by $m_X\mathrm{conv}(W)$.

Since X is m-convex it follows from Proposition 2.3 that the definition of $m_X\mathrm{conv}(W)$ is well-posed. It is the smallest m_X-convex set containing W.

To characterize $\mathrm{conv}(W)$ we define the sequence of sets. We put

$$\begin{aligned} m_X\mathrm{conv}_0(W) &= W, \\ m_X\mathrm{conv}_1(W) &= m_X(W \times W), \\ m_X\mathrm{conv}_{n+1}(W) &= m_X\mathrm{conv}_1(m_X\mathrm{conv}_n(W)) \quad \text{for } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \end{aligned}$$

THEOREM 2.8. *Let $W \subset X$. Then*

$$m_X\mathrm{conv}(W) = \bigcup_{n\in\mathbb{N}_0} m_X\mathrm{conv}_n(W).$$

Proof. We first show that $\bigcup_{n\in\mathbb{N}_0} m_X\mathrm{conv}_n(W)$ is m_X-convex. Since it clearly contains W, this would prove by definition the inclusion $m_X\mathrm{conv}(W) \subset \bigcup_{n\in\mathbb{N}_0} m_X\mathrm{conv}_n(W)$.

Consider arbitrary $x, y \in \bigcup_{n\in\mathbb{N}_0} m_X\mathrm{conv}_n(W)$. Since the sequence $(m_X\mathrm{conv}_n(W))_{n\in\mathbb{N}_0}$ is ascending, there exists $n \in \mathbb{N}_0$ such that $x, y \in m_X\mathrm{conv}_n(W)$, and consequently $m_X(x, y) \subset m_X(m_X\mathrm{conv}_n(W)) = m_X\mathrm{conv}_{n+1}(W)$.

Now we prove the converse inclusion. Since $m_X\mathrm{conv}(W)$ is m-convex and $W \subset m_X\mathrm{conv}(W)$ we obtain that

$$m_X\mathrm{conv}_1(W) \subset m_X\mathrm{conv}(W),$$

and consequently that

$$m_X\mathrm{conv}_n(W) \subset m_X\mathrm{conv}(W) \quad \text{for } n \in \mathbb{N}_0.$$

The last inclusion implies that $\bigcup\limits_{n\in\mathbb{N}_0} \mathrm{conv}_n(W) \subset m_X\mathrm{conv}(W)$. ∎

As a direct corollary from Theorem 2.8 we obtain the following result.

COROLLARY 2.9. *If X is finite then there exists an $n_0 \in \mathbb{N}_0$ such that*

$$m_X\mathrm{conv}_{n_0+1}(W) = m_X\mathrm{conv}_{n_0}(W).$$

Then

$$m_X\mathrm{conv}(W) = m_X\mathrm{conv}_{n_0}(W).$$

We illustrate the above considerations.

EXAMPLE 2.10. Let $X = \mathbb{R}^2$ with the norm $\|(x,y)\| = \max(|x|,|y|)$, let m be defined as follows

$$m((x_1,y_1),(x_2,y_2)) = \big\{(x,y) \in \mathbb{R}^2 : \\ \|(x,y)-(x_1,y_1)\| = \|(x,y)-(x_2,y_2)\| = \tfrac{1}{2}\|(x_1-x_2,y_1-y_2)\|\big\},$$

and let $W = \{(0,0),(1,0)\}$. Then $\mathrm{conv}_1(W)$ and $\mathrm{conv}_2(W)$ are presented in Figure 1(a) and 1(b).

(a) (b)

Fig. 1. $\mathrm{conv}_1(W)$ and $\mathrm{conv}_2(W)$

Moreover, we have

$$m\mathrm{conv}(W) = \big\{(x,y) \in \mathbb{R}^2 : \\ (x \in [0,\tfrac{1}{2}] \cap \mathcal{D}, y \in [-x,x]) \vee (x \in [\tfrac{1}{2},1] \cap \mathcal{D}, y \in [x-1,1-x])\big\},$$

where $\mathcal{D}$ denotes the set of dyadic numbers.

EXAMPLE 2.11. Let $X = (\mathbb{R}^2, \prec)$, where α is the lexicographic order, i.e.

$$(x_1,y_1) \prec (x_2,y_2) \iff x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 \le y_2),$$

let m be defined as follows

$$m\big((x_1,y_1),(x_2,y_2)\big) = \big\{(x,y) \in \mathbb{R}^2 : (x_1,y_1) \prec (x,y) \prec (x_2,y_2)\big\},$$

and let $W = \{(0,0),(1,0)\}$. Then $\mathrm{conv}(W) = \mathrm{conv}_1(W)$ and it is presented at Figure 2.

Now we define the notion of extreme point.

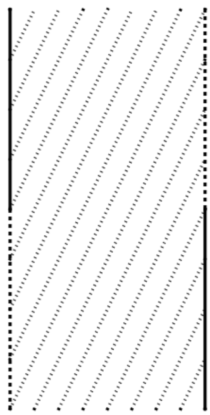

Fig. 2.

Definition 2.12. Let $W \subset X$ be an m-convex set. We say that $a \in W$ is an *extreme point* of W if

$$\forall x, y \in W \quad a \in m(x, y) \Rightarrow a = x \text{ or } a = y.$$

The set of extreme points of m-convex set W will be denoted be $\operatorname{ext} W$.

Remark 2.13. Let $W \subset X$ be an m-convex set. One could consider another definition of extreme point by considering elements $a \in W$ for which

$$\forall x, y \in W \quad a \in m(x, y) \Rightarrow a = x = y. \tag{3}$$

However, then for the m defined in Example 1.1 (first point) the extreme points would not coincide with the classical case.

More precisely, as was mentioned by the referee, for $X = \mathbb{R}$, $m(x, y) = [x, y]$ and $W = [0, 1]$, we obtain that 0 is an extreme point of $[0, 1]$, but the condition (3) is not satisfied: for $x = 0$, $y = 1$ we have $0 \in m(0, 1)$ but $0 \neq 1$.

It occurs that contrary to our intuition even finite sets may have no extreme points.

Example 2.14. Consider the set $\mathbb{Z}_3 = \{0, 1, 2\}$ with addition modulo 3 and the middle-point function defined by

$$m(k, l) = \{j : 2j = k + l\}.$$

Then $0 \in m(1, 2), 1 \in m(0, 2), 2 \in m(0, 1)$ and hence $\mathbb{Z}_3$ has no extreme points.

3. Midconvex functions. We are going to define the notion of midconvex function. For this purpose we need a specific class of means in $\mathbb{R}$. By $\mathcal{M}$ we denote the set of all functions $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$M(x, x) = x \quad \text{for } x \in X$$

and the condition

$$\forall x_1, x_2, y_1, y_2 \in \mathbb{R} \quad x_1 \leq x_2, y_1 \leq y_2 \Rightarrow M(x_1, y_1) \leq M(x_2, y_2). \tag{4}$$

For a function $f : W \to \mathbb{R}$ by $\operatorname{epi} f$ we denote its epigraph, i.e. we put

$$\operatorname{epi} f := \{(x, y) \in W \times \mathbb{R} : y \geq f(x)\}.$$

DEFINITION 3.1. Let $W \subset X$, let m_X be a middle-point function in X, and let $M \in \mathcal{M}$. We say that a function $f : W \to \mathbb{R}$ is *(m_X, M)-convex* if $\operatorname{epi} f$ is $m_X \times \{M\}$-convex.

PROPOSITION 3.2. *Let $W \subset X$, let m_X be a mean in X, and let $M \in \mathcal{M}$. A function $f : W \to \mathbb{R}$ is (m_X, M)-convex if and only if the following conditions hold*:

(i) *W is m_X-convex*;
(ii) *$f(w) \leq M(f(x_1), f(x_2))$ for $x_1, x_2, w \in W$, $w \in M(x_1, x_2)$.*

Proof. Let $f : W \to \mathbb{R}$ be an arbitrary function. Assume that $\operatorname{epi} f$ is (m_X, M)-convex. Consider arbitrary $x_1, x_2 \in W$. Then $(x_1, f(x_1)), (x_2, f(x_2)) \in \operatorname{epi} f$ and consequently

$$\big(m_X(x_1, x_2), M(f(x_1), f(x_2))\big) \subset \operatorname{epi} f. \tag{5}$$

Inclusion (5) implies that $m_X(x_1, x_2) \subset W$. We have proved (i).

Furthermore it results from (5) that

$$\big(w, M(f(x_1), f(x_2))\big) \in \operatorname{epi} f \quad \text{for } w \in m_X(x_1, x_2).$$

But it means that

$$M(f(x_1), f(x_2)) \geq f(w) \quad \text{for } w \in m_X(x_1, x_2).$$

Thus we have proved that condition (ii) is valid.

Assume now that conditions (i) and (ii) are satisfied. Consider arbitrary (x_1, y_1), $(x_2, y_2) \in \operatorname{epi} f$. Then $x_1, x_2 \in W$ and hence by (i)

$$m_X(x_1, x_2) \subset W.$$

We have to prove that

$$(w, M(y_1, y_2)) \in \operatorname{epi} f \quad \text{for } w \in m_X(x_1, x_2),$$

i.e.

$$M(y_1, y_2) \geq f(w) \quad \text{for } w \in m_X(x_1, x_2).$$

Consider an arbitrary $w \in m_X(x_1, x_2)$. We have

$$y_1 \geq f(x_1), \qquad y_2 \geq f(x_2),$$

and hence by (4)

$$M(y_1, y_2) \geq M(f(x_1), f(x_2)),$$

whence by (ii) we obtain

$$M(y_1, y_2) \geq f(w),$$

which completes the proof. ∎

EXAMPLE 3.3. Let X be a real vector space and let $t \in (0, 1)$ be arbitrarily fixed. Taking $m_X : X \times X \to X$ defined by

$$m_X(x_1, x_2) = \{tx_1 + (1 - t)x_2\}$$

and $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$M(y_1, y_2) = ty_1 + (1 - t)y_2,$$

by Proposition 3.2, we obtain that (m_X, M)-convexity of a function $f : W \to \mathbb{R}$, $W \subset X$ is equivalent to its t-convexity (in the classical sense). In particular for $t = \frac{1}{2}$ we obtain definition of the Jensen convex function.

If we take m_X defined by

$$m_X(x_1, x_2) = [x_1, x_2] = \{sx_1 + (1-s)x_2 : s \in [0,1]\},$$

and M defined by

$$M(y_1, y_2) = \max(y_1, y_2)$$

we obtain that (m_X, M)-convexity of a function is equivalent to its quasiconvexity (in the classical sense).

Convex function (in the classical sense) can be defined as (m_t, m_t)-convex for every $t \in [0,1]$, where

$$m_t(x_1, x_2) = \{tx_1 + (1-t)x_2\}.$$

The next result shows that (m_X, M)-convex functions have similar properties as convex (midconvex) ones.

Proposition 3.4. *Let $W \subset X$ be an m_X-convex set and let $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be of the form*

$$M(x_1, x_2) = tx_1 + (1-t)x_2$$

for some $t \in [0,1]$. Then the family of all (m_X, M)-convex functions $f : W \to \mathbb{R}$ is closed under addition, multiplication by positive number and operation of supremum.

Proof. The proofs rely on direct applications of condition (ii) from Proposition 3.2. We present the proof for supremum.

Let $f_i : w \to \mathbb{R}$, $i \in I$, be any family of (m_X, M)-convex functions. By Proposition 3.2 we have for $i \in I$, $x_1, x_2 \in W$, $w \in m_X(x_1, x_2)$

$$f_i(w) \le tf_i(x_1) + (1-t)f_i(x_2).$$

Hence we obtain for $x_1, x_2 \in W$, $w \in m_X(x_1, x_2)$

$$\begin{aligned} \sup_{i \in I} f_i(w) &\le \sup_{i \in I}\big(tf_i(x_1) + (1-t)f_i(x_2)\big) \\ &\le t \sup_{i \in I} f_i(x_1) + (1-t) \sup_{i \in I} f_i(x_2). \end{aligned}$$

It means that the function $\sup_{i \in I} f_i$ is (m_X, M)-convex. ∎

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