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A TOPOLOGICAL CHARACTERIZATION OF THE PRODUCT OF TWO CLOSED OPERATORS

BY

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Abstract. The purpose of this work is to give a topological condition for the usual product of two closed operators acting in a Hilbert space to be closed.

1. Introduction. We begin by recalling some known definitions and results.

Notations and definitions. Let H be a complex Hilbert space. All operators are assumed to be linear and defined from H into H. The domain of an operator A is denoted by $D(A)$, which we assume to be dense. The null space and range of A will be denoted by $N(A)$ and $R(A)$ respectively. The operator A is said to be *closed* if its graph $G(A) = \{(x, Ax) : x \in D(A)\}$ is closed in $H \times H$. The adjoint of A is denoted by A^*. The identity operator is denoted by I.

The set of closed linear operators in H is denoted by $C(H)$ while $B(H)$ denotes the set of all bounded elements of $C(H)$.

Any other result or notion about unbounded operators that has not been mentioned and which will be used is assumed to be known to the reader. The literature on this subject is vast. We cite [2, 6, 16, 18] among others.

The notion of the product of two closed operators in a Hilbert space, introduced several decades ago, is now broadly used in different areas of mathematics, both applied and pure. In particular, it is used widely in functional analysis. We may cite the dominated convergence theorem of Lebesgue, the differentiation of integral under the integral sign, etc.

The composition law constitutes a topological test for $C(H)$ because it is not generally a closure law. We know, for instance, that if two operators A and B are bounded, then their commutator $[A, B] = AB - BA$ is different from the identity operator on H.

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However, things get more complicated for unbounded operators. This is essentially due to the domains of the operators.

We now recall some deficiencies of the product of two closed operators. If A and B are two unbounded operators with domains $D(A)$ and $D(B)$ respectively, then their product is defined by $(AB)f := A(Bf)$ for $f \in D(AB) = B^{-1}(D(A))$. But AB may just not have any sense if for example $R(B) \cap D(A) = \{0\}$.

If $A, B \in C(H)$, then AB need not be closed. In fact, the product of a closed symmetric operator A with itself may have a domain that reduces to $\{0\}$. This was first shown by Naimark [12] who gave a non-explicit way of constructing such operators. Then Chernoff [1] gave simpler and more explicit operators A satisfying $D(A^2) = \{0\}$ (they are also semibounded). Also J. Dixmier [4] gave a method of constructing symmetric operators whose squares (and even their adjoint's squares) have trivial domain.

Nelson [13], Fuglede [5] and more recently Nussbaum [15] studied the link between the closedness and the commutator of unbounded symmetric operators.

The product AB (in this order) of two closed operators A and B is closed if one of the following occurs:

(1) B is bounded on H;
(2) A is invertible with a bounded inverse on H;
(3) A and B are Fredholm;
(4) A and B are paracomplete such that $N(AB)$ and $R(AB)$ are closed in H;
(5) A and B are paracomplete and AB is quasi-Fredholm with index 0;
(6) B is a generalized inverse of A and $R(A)$ is closed in H and conversely;
(7) B is a generalized inverse of A such that $R(A) \oplus N(B) = H$ and conversely.

For proofs of (3) to (7) see [7, 14].

An example with A bounded in H and B closed, but AB not closed is given in [8]. Stronger conditions on A and B do not help much. While an example of an unbounded self-adjoint operator A such that $D(A^2) = \{0\}$ does not exist thanks to the spectral theorem, the self-adjointness is still not sufficient. For instance, the operators

$$A = -i\frac{d}{dx}, \qquad B = |x|$$

are known to be self-adjoint (hence closed!) on their respective domains, $D(B) = \{f \in L^2(\mathbb{R}) : |x|f \in L^2(\mathbb{R})\}$ and $D(A) = H^1(\mathbb{R})$ is the Sobolev space $\{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\}$. Then AB is defined on its domain

$$D(AB) = \{f \in L^2(\mathbb{R}) : |x|f, -i(|x|f)' \in L^2(\mathbb{R})\}$$

by

$$ABf = -i(|x|f)'$$

where the derivative is a distributional one. $D(AB)$ is dense in $L^2(\mathbb{R})$ since it contains $C_0^\infty(\mathbb{R})$. Thus AB is certainly not closed (more details can be found in [10]). So self-adjointness is not sufficient to make the product closed. Accordingly, as soon as we leave the bounded operators we loose many important properties concerning sums, products and adjoints among others. To get round these problems J. Dixmier [4] defined a new product of two closed operators A and B which he denoted by $A \cdot B$. For the convenience of the reader we recall his definition.

DEFINITION. The product $A \cdot B$ of two operators A and B is defined in the following way. We say that $f \in D(A \cdot B)$ and $g := A \cdot Bf$ if there exist two sequences, $(f_n)_n$ in $D(B)$ and $(g_n)_n$ in $R(A)$, such that $f_n \to f$, $g_n \to g$ and $A^{-1}g_n - Bf_n \to 0$ (for some well-chosen $A^{-1}g_n$ and Bf_n as A^{-1} and B may be multifunctions).

Among the noteworthy results we recall the following. For all A and B in $C(H)$,

(1) $A \cdot B \in C(H)$;
(2) $A \cdot B = AB$ (the usual product) if either $B \in B(H)$ or $A^{-1} \in B(H)$;
(3) $A \cdot B = \overline{AB}$ (the closure of AB) if either $B^{-1} \in B(H)$ or $A \in B(H)$.

Dixmier did not get the well known formula for adjoints, that is, $(AB)^* = B^*A^*$. He "only" got $(AB)^* = B^* \cdot A^*$ and $(A \cdot B)^* = B^*A^*$.

B. Messirdi and M. H. Mortad proposed recently (cf. [8]) a new product of closed operators. The idea was based upon the bisecting $F(A)$ of an operator A in $C(H)$. If A and B are in $C(H)$, then this product is defined as

$$A \bullet B = F^{-1}(F(A)F(B))$$

where the function $A \mapsto F(A) = AS_A(I + S_A)^{-1}$ sends elements of $C(H)$ to contractions T such that $\|T\| \le 1$ and $N(I - T^*T) = \{0\}$ where $S_A = \sqrt{R_A}$ if $R_A = (I + A^*A)^{-1}$. The known properties of R_A and S_A (see [3]) are: $R_A, S_A \in B(H)$, $\|R_A\| \le 1$ and $\|S_A\| \le 1$, $\|AR_A\| \le 1$ and $\|AS_A\| \le 1$.

The important results obtained in [8] are summarized in

THEOREM 1. *For all A, B in $C(H)$,*

(1) $(A \bullet B) \in C(H)$;
(2) $(A \bullet B)^* = B^* \bullet A^*$.

An important property is that if A and B in $C(H)$ are close to each other with respect to the gap metric g (or equivalently p, f or h, all defined below), then their product remains in $C(H)$ (see [7]).

It seems desirable to obtain a topological sufficient condition for the usual product to be closed. The condition that we obtain uses a new metric d, strictly coarser than g, that characterizes with more precision the stability of the product in $C(H)$. This condition is for instance imposed in the spectral analysis of the elements of $C(H)$ (see the last Remark in this paper). So we recall some results and definitions concerning $C(H)$ and its metrics.

Finally, in Section 3 we show that if the distance between two closed operators with respect to some metric in $C(H)$ is fairly small, then their usual product is closed.

2. Topologies on $C(H)$

Definition. Let $A, B \in C(H)$. Denote their graphs by $G(A)$ and $G(B)$. Let $P_{G(A)}$ and $P_{G(B)}$ denote the orthogonal projections on $G(A)$ and $G(B)$ respectively. Now set

$$\delta(A,B) = \|(1 - P_{G(B)})P_{G(A)}\|_{B(H\times H)},$$
$$g(A,B) = \|P_{G(B)} - P_{G(A)}\|_{B(H\times H)}.$$

Then $g(A,B)$ is a metric on $C(H)$, usually called the metric gap, while $\delta(A,B)$ is not a distance. Moreover, the topology induced by g on $B(H)$ is equivalent to the usual uniform topology, and $B(H)$ is open in $C(H)$ (cf. [3]).

Furthermore ([3, 6]),

$$g(A,B) = \max(\delta(A,B), \delta(B,A)), \qquad g(A,B) = g(A^*, B^*),$$

and if A and B are invertible in $C(H)$, then

$$g(A^{-1}, B^{-1}) = g(A,B).$$

However, for applications other metrics on $C(H)$ may be more practical (see [3]). An example is

$$p(A,B) = [\|R_A - R_B\|^2_{B(H)} + \|R_{A^*} - R_{B^*}\|^2_{B(H)} + 2\|AR_A - BR_B\|^2_{B(H)}]^{1/2}.$$

This is a metric equivalent to g and one has

$$g(A,B) \le \sqrt{2}\, p(A,B) \le 2g(A,B), \qquad \forall A, B \in C(H),$$
$$p(A,B) \le 4\|A - B\|, \qquad \forall A, B \in B(H).$$

Proposition 1. *The map F defined in the introduction is open and bijective from $(C(H), g)$ into $C_0(H)$ (the space of bounded linear operators T with norm less than one and $N(I - T^*T) = \{0\}$) with the topology induced by $B(H)$.*

Proof. That F is bijective may be found in [8].

Now for all A and B in $C(H)$ we have on the one hand

$$g^2(A,B) \le 2[\|R_A - R_B\|^2_{B(H)} + \|R_{A^*} - R_{B^*}\|^2_{B(H)} + 2\|AR_A - BR_B\|^2_{B(H)}],$$

and on the other hand

$$\begin{aligned} \|R_A - R_B\|_{B(H)} &\le (\|S_A\|_{B(H)} + \|S_B\|_{B(H)})\|S_A - S_B\|_{B(H)} \\ &\le 2\|S_A - S_B\|_{B(H)}, \\ \|R_{A^*} - R_{B^*}\|_{B(H)} &\le 2\|S_{A^*} - S_{B^*}\|_{B(H)} \end{aligned}$$

and

$$\begin{aligned} &\|AR_A - BR_B\|^2_{B(H)} \\ &\quad \le [\|AS_A - BS_B\|_{B(H)}\|S_A\|_{B(H)} + \|BS_B\|_{B(H)}\|S_A - S_B\|_{B(H)}]^2, \end{aligned}$$

which, in turn, is smaller than

$$2[\|AS_A - BS_B\|^2_{B(H)} + \|S_A - S_B\|^2_{B(H)}].$$

Hence

$$g^2(A,B) \le 4[2\|S_A - S_B\|^2_{B(H)} + \|S_{A^*} - S_{B^*}\|^2_{B(H)} + \|AS_A - BS_B\|^2_{B(H)}].$$

Moreover, we have ([3])

$$\begin{aligned} \|S_A - S_B\|_{B(H)} &\le 2g(F(A),F(B)), \\ \|S_{A^*} - S_{B^*}\|_{B(H)} &\le 2g(F(A),F(B)), \\ \|AS_A - BS_B\|_{B(H)} &\le 2g(F(A),F(B)). \end{aligned}$$

Thus

$$g^2(A,B) \le 64g^2(F(A),F(B))$$

and

$$\begin{aligned} g(A,B) &\le 8g(F(A),F(B)) \le 8\sqrt{2}\,p(F(A),F(B)) \\ &\le 32\sqrt{2}\|F(A) - F(B)\|_{B(H)}. \ \blacksquare \end{aligned}$$

Remark. Let $A \in C(H)$. Set

$$F_A = \begin{pmatrix} R_A - P_{N(A)} & A^*R_{A^*} \\ AR_A & I - R_{A^*} \end{pmatrix}, \quad H_A = \begin{pmatrix} R_A & A^*R_{A^*} \\ AR_A & I - R_{A^*} + P_{N(A^*)} \end{pmatrix}$$

where $P_{N(A)}$ and $P_{N(A^*)}$ denote the orthogonal projections on $N(A)$ and $N(A^*)$ respectively. Then F_A and H_A are two orthogonal projections in $B(H \times H)$.

Among the metrics found in the literature we may cite f and h (cf. [17]) defined as follows. If A, B are in $C(H)$ then set

$$f(A,B) = \|F_A - F_B\|_{B(H\times H)} \quad \text{and} \quad h(A,B) = \|H_A - H_B\|_{B(H\times H)}.$$

Proposition 2. *The functions f and h are metrics on $C(H)$ satisfying*

$$f(A^*,B^*) = h(A,B) \quad \text{and} \quad h(A^*,B^*) = f(A,B).$$

Proof. Using the following properties of R_A:

$$AR_A = R_{A^*}A \text{ on } D(A), \quad (AR_A)^* = A^*R_{A^*}, \quad N(AR_A) = N(A),$$

we see easily that $F_A^2 = F_A$, $H_A^2 = H_A$, $F_A^* = F_A$ and $H_A^* = H_A$, i.e., both F_A and H_A are orthogonal projections.

Let us check that f is a metric. We trivially have

$$f(A,B) = f(B,A), \quad f(A,C) \le f(A,B) + f(B,C), \quad f(A,A) = 0$$

for all A, B, C in $C(H)$. If $f(A,B) = 0$ then $R_{A^*} = R_{B^*}$ and hence $S_A = S_{A^*}$. Thus $D(A^*) = D(B^*)$, and $A^*R_{A^*} = B^*R_{B^*}$ shows that $A^*S_{A^*} = AS_A$. So A^* and B^* coincide on their equal domains and hence $A = B$.

We proceed analogously for $h(A,B)$.

Finally, we have

$$F_A - F_B = -\mathcal{M}(H_{B^*} - H_{A^*})\mathcal{M} \quad \text{where} \quad \mathcal{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and since $\|\mathcal{M}\| = 1$, $f(A,B) = h(A^*,B^*)$. Since A and B are closed the other equality is merely a consequence of this one. ∎

Remark. Since (see [19])

$$P_{G(A)} = \begin{pmatrix} R_A & A^*R_{A^*} \\ AR_A & I - R_{A^*} \end{pmatrix},$$

one has

$$|f(A,B) - g(A,B)| \le \|P_{N(A)} - P_{N(B)}\|_{B(H)}, \quad \forall A,B \in C(H).$$

3. The inverse image topology on $C(H)$. The set

$$\tau = \{F^{-1}(W) : W \text{ open in } C_0(H)\}$$

is obviously a topology on $C(H)$. It is the coarsest (in the terminology of [11]) topology on $C(H)$ making the mapping F continuous.

Moreover, τ is metrizable since for $A,B \in C(H)$,

$$d(A,B) = \|F(A) - F(B)\|_{B(H)}$$

is a bounded metric in $C(H)$ making both F and F^{-1} continuous. This actually means that F is an isometry between $C(H)$ and $C_0(H)$. In particular, the topology induced by d on $C(H)$ coincides with τ.

For $A,B \in C(H)$ one also has

$$d(A,B) = d(A^*,B^*) \quad \text{and} \quad g(A,B) \le 32\sqrt{2}\, d(A,B).$$

Nevertheless, the topology induced by g (and hence by p) on $C(H)$ is strictly finer than τ, and g and d are not equivalent since otherwise F would be continuous on $(C(H),g)$, which is not the case (since $(C(H),g)$ would then be complete and this is not true, see [6]).

REMARK (cf. [8]). For A and B in $C(H)$, if $d(A,B) < 1$, then there exists an operator C such that

$$C = 2(F(A) - F(B))(I - (F(A^*) - F(B^*))(F(A) - F(B)))^{-1}.$$

This operator is bounded on H and satisfies $F(C) = F(B) - F(A)$ and

$$d(A,B) = \|F(C)\|_{B(H)} = d(C,0) = \frac{\|C\|_{B(H)}}{1 + \sqrt{1 + \|C\|^2_{B(H)}}},$$

or, equivalently,

$$\|C\|_{B(H)} = \frac{2d(A,B)}{1 - d^2(A,B)}.$$

LEMMA 1 (cf. [6]). *Let A and B be two closed operators in H.*

(1) *If $A \in B(H)$, then $A + B \in C(H)$.*
(2) $A^* + B^* \subseteq (A+B)^*$, *and if $A \in B(H)$, then $A^* + B^* = (A+B)^*$.*
(3) $B^*A^* \subseteq (AB)^*$.

REMARK. For more details on when (1) and (2) hold for other operators see [9]. Also there is a generalization of (3) for a different product (cf. [8]).

LEMMA 2. *Let $A, B \in C(H)$ and $0 < \varepsilon < 1/32\sqrt{2}$. Then*

$$d(A, B^*) < \varepsilon \ \Rightarrow\ G(A) \oplus G(B^*)^\perp = H \oplus H.$$

Proof. If $d(A,B^*) < \varepsilon$, then $\|P_{G(A)} - P_{G(B^*)}\|_{B(H)} < 1$. Let $X \in G(A) \cap G(B^*)^\perp$. Then $(P_{G(A)} - P_{G(B^*)})X = X$. Hence

$$\|X\| \le 32\sqrt{2}\, d(A,B^*)\|X\| < \|X\|.$$

Consequently, $X = 0$.

Similarly we obtain $G(B^*) \cap G(A)^\perp = \{0\}$. Hence

$$(G(B^*) \cap G(A)^\perp)^\perp = \overline{G(A) + G(B^*)^\perp} = H \oplus H.$$

We need only show that $G(A) + G(B^*)^\perp$ is closed in $H \oplus H$. Let $Y \in G(A) + G(B^*)^\perp$. Then $Y = Y_1 + Y_2$ with $Y_1 \in G(A)$ and $Y_2 \in G(B^*)^\perp$. We also have

$$\|Y\|^2_{H\oplus H} = \|Y_1\|^2 + \|Y_2\|^2 + 2\,\mathrm{Re}\langle Y_1, Y_2\rangle.$$

Thus

$$\begin{aligned}
\|Y_1\|^2 + \|Y_2\|^2 &= \|Y\|^2_{H\oplus H} - 2\,\mathrm{Re}\langle P_{G(A)}Y_1, (I - P_{G(B^*)})Y_2\rangle \\
&= \|Y\|^2_{H\oplus H} - 2\,\mathrm{Re}\langle (I - P_{G(B^*)})P_{G(A)}Y_1, Y_2\rangle \\
&\le \|Y\|^2_{H\oplus H} + 2\|(P_{G(A)} - P_{G(B^*)})P_{G(A)}Y_1\| \cdot \|Y_2\| \\
&\le \|Y\|^2_{H\oplus H} + 64\sqrt{2}\, d(A,B^*)\|Y_1\| \cdot \|Y_2\|.
\end{aligned}$$

This gives

$$(*) \qquad \|Y_1\| \le \frac{\|Y\|}{1-32\sqrt{2}d(A,B^*)},$$

$$(**) \qquad \|Y_2\| \le \frac{\|Y\|}{1-32\sqrt{2}d(A,B^*)}.$$

Let $Y \in H \oplus H$. Then there exists a sequence $Y_n = Y_{1,n} + Y_{2,n}$ that converges in $H \oplus H$ to Y where $Y_{1,n} \in G(A)$ and $Y_{2,n} \in G(B^*)^\perp$.

Now estimates $(*)$ and $(**)$ show that $(Y_{1,n})_n$ and $(Y_{2,n})_n$ are Cauchy sequences in $H\oplus H$. Therefore they converge to $Y_1 \in G(A)$ and $Y_2 \in G(B^*)^\perp$ respectively since both $G(A)$ and $G(B^*)^\perp$ are closed in $H \oplus H$. Thus $Y = Y_1 + Y_2 \in G(A) + G(B^*)^\perp$. ■

The main result in this paper is the following theorem.

THEOREM 2. *Let A and B be two closed operators in a Hilbert space H and let $0 < \varepsilon < 1/32\sqrt{2}$. If $d(A,B^*) < \varepsilon$, then AB and BA are both closed with dense domains in H.*

Proof. Assume that $d(A,B^*) < \varepsilon$. Then by the previous lemma $G(A) \oplus G(B^*)^\perp = H \oplus H$. Since $G(B^*) = V(G(B))^\perp$ we have $G(B^*)^\perp = V(G(B))$ where $V(x,y) = (-y,x)$ is a surjective isometry on $H \oplus H$ such that $V^2 = -I_{H\oplus H}$ and $V(E^\perp) = V(E)^\perp$ for any linear subspace E of $H \oplus H$.

If $f \in H$, then there exist unique $x \in D(A)$ and $y \in D(B)$ for which

$$(f,0) = (x,Ax) + (-By,y).$$

If we set $x = Pf$ and $y = Qf$ then we obtain

$$\begin{cases} f = Pf - BQf, \\ 0 = APf + Qf. \end{cases}$$

Hence $f = (I+BA)Pf$. This shows that P maps H into $D(I+BA)$ and hence $R(P) \subseteq D(I+BA)$.

On the other hand, since $d(A,B^*) = d(B^*,A)$ and using the same method, we obtain an operator, P^* say, defined in the domain $D(I+A^*B^*)$ of the operator $I + A^*B^*$ by

$$(I+A^*B^*)P^*g = g \quad \text{for all } g \in H.$$

In particular, for $g \in R(P)^\perp$, we have, for all f in H,

$$\begin{aligned} 0 = \langle g, Pf\rangle_H &= \langle (I+A^*B^*)P^*g, Pf\rangle_H \\ &= \langle P^*g, (I+BA)Pf\rangle_H = \langle P^*g, f\rangle_H. \end{aligned}$$

Hence $P^*g = 0$ and $g = (I+A^*B^*)P^*g = 0$ and as a consequence $R(P)^\perp = \{0\}$. It follows that $R(P)$ and $D(I+BA)$ are dense in H and so are $R(P^*)$ and $D(I+A^*B^*)$ (this is done in exactly the same way).

Now let $y \in D((I+BA)^*)$. Using Lemma 1 and $(I + A^*B^*)P^* = I$ one has

$$\forall x \in D(I+BA), \quad \langle (I+BA)x, y\rangle_H = \langle x, (I+BA)^* y\rangle_H$$
$$= \langle x, (I+A^*B^*)P^*(I+BA)^* y\rangle_H = \langle (I+BA)x, P^*(I+BA)^* y\rangle_H,$$

which gives $y = P^*(I+BA)^*y$.

Thus $y \in D(I + A^*B^*)$ and $(I+BA)^*y = (I+A^*B^*)y$. Accordingly,

$$D((I+BA)^*) \subseteq D(I+A^*B^*).$$

So from Lemma 1 we deduce that

$$(I+BA)^* = I + A^*B^* \quad \text{and} \quad (I+A^*B^*)^* = I + BA.$$

So $I + BA$ and hence BA are closed in H with dense domains.

Inverting the roles of A and B and using $d(A, B^*) = d(A^*, B)$ yield the closedness of AB and hence establishes the result. ∎

Corollary 1. *If* $A, B \in C(H)$ *are such that* $d(A, B^*) < \varepsilon$ *where* $0 < \varepsilon < 1/32\sqrt{2}$, *then*

$$(I+AB)^{-1}, (I+BA)^{-1} \in C(H).$$

Proof. Since $(I+BA)P = I$ and $(I+AB)P' = I$ where $R(P)$ and $R(P')$ are dense in H and P and P' are two bounded operators, it follows that $I + BA$ and $I + AB$ are both invertible in their respective domains, take their values in H and

$$(I+BA)^{-1} = P \quad \text{and} \quad (I+AB)^{-1} = P'. \ ∎$$

In fact, if AB and BA are closed, then $G(A) \oplus G(B^*)^{\perp} = H \oplus H$ and hence $g(A, B^*) < 1$. Nevertheless, we loose the optimality in Theorem 2.

Remarks. (1) If $d(A, B^*) < \varepsilon < 1/32\sqrt{2}$, then -1 is in the resolvent sets of AB and BA. Hence $(\lambda + AB)^{-1}$ and $(\lambda + BA)^{-1}$ are two analytical families of operators from the disc $\{\lambda \in \mathbb{C} : |\lambda + 1| < \varepsilon\}$ into $B(H)$ for sufficiently small ε.

(2) These results are easily generalized to the case when the resolvent sets are replaced with the Fredholm resolvent sets (cf. [7]).

(3) The stability for the class $C(H)$ with the metric d also follows from the behavior of the unitary groups associated with operators in $C(H)$ (see the notes in [16]).

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