COLLOQUIUM MATHEMATICUM

VOL. 113 2008 NO. 1

SUM THEOREMS FOR OHIO COMPLETENESS

BY

D. BASILE, J. VAN MILL and G. J. RIDDERBOS (Amsterdam)

Abstract. We present several sum theorems for Ohio completeness. We prove that Ohio completeness is preserved by taking σ-locally finite closed sums and also by taking point-finite open sums. We provide counterexamples to show that Ohio completeness is preserved neither by taking locally countable closed sums nor by taking countable open sums.

1. Introduction. *All spaces under consideration are Tikhonov.* A topological space X is called *Ohio complete* if for every compactification γX of X there is a G_δ-subset S of γX such that $X \subseteq S$ and, for every $y \in S \setminus X$, there is a G_δ-subset of γX which contains y and misses X.

Ohio completeness was introduced by Arhangel′skiĭ in [1] to study generalized metrizability properties of remainders of compactifications. It was shown in [1] that among the Ohio complete spaces are all Čech-complete spaces, Lindelöf spaces, p-spaces and spaces with a G_δ-diagonal. D. Basile and J. van Mill [2] have studied the behaviour of Ohio completeness with respect to taking products and closed subspaces.

In [2] it is shown that the disjoint sum of Ohio complete spaces is again Ohio complete. In this paper we prove more sum theorems for Ohio completeness. Below we prove that Ohio completeness is preserved by taking σ-locally finite closed sums. This generalizes the disjoint sum theorem from [2], and it also follows that a countable closed sum of Ohio complete subspaces is Ohio complete. Moreover, we also prove that Ohio completeness is preserved by taking point-finite open sums.

In the final section of this paper, we provide several examples of non-Ohio complete spaces to show the sharpness of our results. We provide an example of a space which is not Ohio complete but which is covered by a locally countable family of closed subspaces all of which are Ohio complete. This shows that there is no locally countable closed sum theorem for Ohio completeness. We also provide an example of a first countable homogeneous space which is not Ohio complete. This shows that the statement "Every

2000 *Mathematics Subject Classification*: 54D35, 54B05, 54B25, 54G20.

Key words and phrases: Ohio complete, sum theorems, compactification.

first countable topological group is Ohio complete" cannot be generalized to homogeneous spaces. Since every first countable space is an open image of a metrizable space, it follows that Ohio completeness is not preserved by taking open images. Finally, we present an example of a non-Ohio complete space which is covered by a countable collection of open and Ohio complete subspaces.

Our examples indicate that there is an essential difference between sum theorems for open and for closed subspaces. Note that any space X is the union of a point-finite family of closed and Ohio complete subspaces, namely the family $\{\{x\} : x \in X\}$. So any example of a non-Ohio complete space shows that there is no point-finite closed sum theorem for Ohio completeness. This contrasts with the point-finite open sum theorem for Ohio completeness which we will prove below. Furthermore, as mentioned before, we shall provide a counterexample to a countable open sum theorem for Ohio completeness. So although Ohio completeness is preserved by taking countable closed sums, this property is not preserved by taking countable open sums.

We would like to thank the referee for some helpful comments and remarks.

2. Preliminaries. Ohio completeness was introduced by Arhangel′skiĭ in [1] as a property of remainders of compactifications of spaces. In this preliminary section we show that one may also study the Ohio completeness property in a much wider setting. This leads to several characterizations of Ohio completeness. We also prove that the Ohio completeness property is transitive.

We say that a subspace X of a space Z is *Ohio embedded* in Z if there is a G_δ-subset S of Z such that $X \subseteq S$ and, for every $y \in S \setminus X$, there is a G_δ-subset of Z which contains y and misses X. So a space X is Ohio complete if and only if X is Ohio embedded in γX for every compactification γX of X.

As in [2], we call a compactification γX a *good compactification* of X if X is Ohio embedded in γX. Given spaces X and Z such that X is Ohio embedded in Z, we say that a G_δ-subset S of Z is *good* with respect to X if S contains X and every point in $S \setminus X$ can be separated from X by a G_δ-subset of Z.

For a space Z we shall study the collection of all Ohio embedded subspaces of Z. Of course, this collection contains all Ohio complete subspaces of Z. The following propositions provide some more properties of the collection of Ohio embedded subspaces. We omit the simple proofs.

PROPOSITION 2.1. *If X is either a G_δ- or an F_σ-subset of Z, then X is Ohio embedded in Z.*

Proposition 2.2. *If $X \subseteq Y \subseteq Z$ and X is Ohio embedded in Z, then X is Ohio embedded in Y.*

We now prove that the Ohio completeness property is transitive.

Proposition 2.3. *If X is Ohio embedded in Y and Y is Ohio embedded in Z, then X is Ohio embedded in Z.*

Proof. By hypothesis, we may fix a G_δ-subset R of Y and a G_δ-subset S of Z such that R is good with respect to X and S is good with respect to Y.

We may fix a G_δ-subset $\widetilde{R}$ of Z such that $R = Y \cap \widetilde{R}$. We claim that the G_δ-subset $\widetilde{R} \cap S$ of Z is good with respect to X. So pick an arbitrary point $p \in (\widetilde{R} \cap S) \setminus X$. There are two cases to consider. First assume that $p \in Y$. In this case, $p \in R$ and therefore p is separated from X by a G_δ-subset T of Y. We may fix a G_δ-subset $\widetilde{T}$ of Z such that $T = Y \cap \widetilde{T}$. But then $\widetilde{T}$ separates the point p from X.

Secondly, suppose that $p \notin Y$. Then $p \in S \setminus Y$, hence by the choice of S, the point p can be separated from Y by a G_δ-subset T of Z. Since $X \subseteq Y$, the set T also separates p from X. This completes the proof. ∎

Proposition 2.4. *Let X be a space. The following are equivalent*:

(1) *X is Ohio complete*,
(2) *X is Ohio embedded in Z whenever X is a dense subspace of Z*,
(3) *X is Ohio embedded in Z whenever X is a subspace of Z.*

Proof. The implication (3)⇒(1) is obvious. We first prove (1)⇒(2). So let X be a dense subspace of Z. The Čech–Stone compactification βZ of Z is also a compactification of X (since X is dense in Z). But then X is Ohio embedded in βZ by (1). Since $X \subseteq Z \subseteq \beta Z$, it follows from Proposition 2.2 that X is Ohio embedded in Z.

Finally, we prove (2)⇒(3). So let X be a subspace of Z. Closures are taken in Z. The set $\overline{X}$ is a closed subspace of Z and X is dense in $\overline{X}$. By (2) it follows that X is Ohio embedded in $\overline{X}$. Since $\overline{X}$ is closed in Z, it follows from Proposition 2.1 that $\overline{X}$ is Ohio embedded in Z.

So we see that X is Ohio embedded in $\overline{X}$ and $\overline{X}$ is Ohio embedded in Z. By Proposition 2.3, it follows that X is Ohio embedded in Z. ∎

Corollary 2.5. *If X is Ohio embedded in Y, $Y \subseteq Z$ and Y is Ohio complete, then X is Ohio embedded in Z.*

Proof. This follows from Propositions 2.3 and 2.4. ∎

It was asked in [2, Question 3.3] whether a closed subspace of an Ohio complete space is again Ohio complete, and it was proved there (see [2, Theorem 3.1]) that this is the case for C^*-embedded subspaces. We do not know the answer to [2, Question 3.3], but the following proposition provides some good compactifications of F_σ- and G_δ-subsets of Ohio complete spaces.

PROPOSITION 2.6. *Let $X \subseteq Y \subseteq Z$ and suppose that Y is Ohio complete. If γZ is any compactification of Z, then $\overline{X}^{\gamma Z}$ is a good compactification of X in each of the following cases*:

(1) *X is an F_σ-subset of Z*,
(2) *X is a G_δ-subset of Z.*

Proof. If X is either an F_σ- or a G_δ-subset of Z, then it is a subset of similar kind of Y. So in either case it follows from Proposition 2.1 that X is Ohio embedded in Y. By Corollary 2.5, we also know that X is Ohio embedded in γZ. Since $X \subseteq \overline{X}^{\gamma Z} \subseteq \gamma Z$, it follows from Proposition 2.2 that X is Ohio embedded in $\overline{X}^{\gamma Z}$. ∎

We do not know whether a G_δ-subspace of an Ohio complete space is again Ohio complete, but we shall prove below (see Corollary 3.4) that the assertion "Every closed subset of an Ohio complete space is again Ohio complete" is equivalent to the assertion "Every F_σ-subspace of an Ohio complete space is again Ohio complete".

QUESTION 2.7. *Is a G_δ-subspace of an Ohio complete space again Ohio complete?*

3. Closed sum theorems for Ohio completeness. A family $\mathcal{A}$ of subsets of a space X is called *locally finite* provided that for every $x \in X$ there is a neighbourhood U of x such that the set $\{A \in \mathcal{A} : A \cap U \neq \emptyset\}$ is finite. A family $\mathcal{A}$ of subsets of X is called *σ-locally finite* if $\mathcal{A}$ is the countable union of locally finite families, i.e. $\mathcal{A} = \bigcup_{n\in\omega} \mathcal{A}_n$ where each $\mathcal{A}_n$ is locally finite.

For a subspace Y of Z, showing that Y is Ohio embedded in Z usually consists of two tasks: the first is to find a special G_δ-subset S of Z which contains Y, and secondly one needs to prove that S is good with respect to Y. The following lemma provides G_δ-subsets containing Y that are only good with respect to certain subsets of Y.

LEMMA 3.1. *Let $X \subseteq Y \subseteq Z$ and suppose that X is the union of a family $\mathcal{F}$ of closed subspaces of Y. Suppose moreover that every element of $\mathcal{F}$ is Ohio embedded in Z, and that the family $\mathcal{F}$ is locally finite in Y. Then there is a G_δ-subset S of Z which contains Y and such that every point of $S \setminus Y$ can be separated from X by a G_δ-subset of Z.*

Proof. Let $\mathcal{F} = \{X_i : i \in I\}$ be the locally finite family consisting of closed subspaces of Y such that $X = \bigcup \mathcal{F}$ and such that X_i is Ohio embedded in Z, for every $i \in I$. For every $y \in Y$, fix an open neighbourhood $\widetilde{U}_y$ of y in Z such that if $U_y = Y \cap \widetilde{U}_y$, then $\{i \in I : X_i \cap U_y \neq \emptyset\}$ is finite.

We will first find a G_δ-subset of Z containing Y and then we will prove that this G_δ-subset has the required properties. All closures are taken in Z. Note that by Proposition 2.2, X_i is Ohio embedded in $\overline{X}_i$ for every $i \in I$, so we may fix a G_δ-subset S_i of $\overline{X}_i$ which is good with respect to X_i. The set $\overline{X}_i \setminus S_i$ is an F_σ-subset of $\overline{X}_i$ and hence of Z, so we may fix a collection $\{F_{i,n} : n \in \omega\}$ of closed subsets of Z such that $\overline{X}_i \setminus S_i = \bigcup_{n\in\omega} F_{i,n}$. Note that since each X_i is a closed subset of Y, it follows that for every $i \in I$ and $n \in \omega$, $Y \cap F_{i,n} = \emptyset$. For every $n \in \omega$, we define

$$G_n = \overline{\bigcup_{i\in I} F_{i,n}}.$$

The set $G = \bigcup_{n\in\omega} G_n$ is an F_σ-subset of Z, so its complement is a G_δ-subset of Z. We claim that $G \cap Y = \emptyset$. To see this, note that if $y \in Y$ and $y \in G_n$ for some $n \in \omega$, then the set $\{i \in I : F_{i,n} \cap \widetilde{U}_y \neq \emptyset\}$ is infinite. Since $F_{i,n} \subseteq \overline{X}_i$, it then follows that the collection $\{i \in I : X_i \cap U_y \neq \emptyset\}$ is infinite, and this is impossible.

We now define our G_δ-subset of Z as follows: let $U = \bigcup\{\widetilde{U}_y : y \in Y\}$ and let $S = U \setminus G$. Since $G \cap Y = \emptyset$ and U is an open subset of Z containing Y, it follows that S is a G_δ-subset of Z containing Y. We will now prove that this G_δ-subset has the required properties.

So fix $z \in S \setminus Y$. Then $z \in \widetilde{U}_y$ for some $y \in Y$. Let $J = \{i \in I : z \in \overline{X}_i\}$ and $K = \{i \in I : \overline{X}_i \cap \widetilde{U}_y \neq \emptyset\}$. Note that $J \subseteq K$ and K is finite by the choice of $\widetilde{U}_y$. Since $z \notin G$, it follows that $z \in S_j$ for all $j \in J$. Since S_j is a G_δ-subset of $\overline{X}_j$ which is good for X_i, for every $j \in J$ we may fix a G_δ-subset T_j of Z such that $z \in T_j$ and $T_j \cap X_j = \emptyset$. Now let

$$T = \widetilde{U}_y \cap \bigcap_{j\in J} T_j \cap \bigcap_{k\in K\setminus J} (Z \setminus \overline{X}_k).$$

It is not hard to verify that T is a G_δ-subset of Z which contains z and misses X. Since $z \in S$ was arbitrary, this shows that the G_δ-set S has the required properties, and this completes the proof. ∎

THEOREM 3.2. *Let X be the union of a σ-locally finite family $\mathcal{F}$ of closed subspaces. If $X \subseteq Z$ and every element of $\mathcal{F}$ is Ohio embedded in Z, then X is Ohio embedded in Z.*

Proof. Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, where each $\mathcal{F}_n$ is locally finite in X. For $n \in \omega$, let $X_n = \bigcup \mathcal{F}_n$. Applying Lemma 3.1, for every $n \in \omega$ we find a G_δ-subset S_n of Z containing X such that every point in $S_n \setminus X$ can be separated from X_n by a G_δ-subset of Z.

Now let $S = \bigcap_{n\in\omega} S_n$. It is clear that S is a G_δ-subset of Z which contains X. We will show that S is good with respect to X. So let $z \in S\setminus X$. For every $n \in \omega$ we may find a G_δ-subset T_n of Z such that $z \in T_n$ and $X_n \cap T_n = \emptyset$. But then $T = \bigcap_{n\in\omega} T_n$ is a G_δ-subset of Z which separates z from X. ∎

Corollary 3.3. *Let X be the union of a σ-locally finite family of closed subspaces. If every element of the family is contained in an Ohio complete subspace of X, then X is itself Ohio complete.*

Proof. Fix a σ-locally finite family $\mathcal{F}$ of closed subspaces of X such that $X = \bigcup \mathcal{F}$, and such that every element of $\mathcal{F}$ is contained in an Ohio complete subspace of X. To show that X is Ohio complete, we fix an arbitrary compactification γX of X. First note that by Proposition 2.1 and Corollary 2.5, every element of $\mathcal{F}$ is Ohio embedded in γX. So it follows from the previous theorem that X is Ohio embedded in γX. Since γX was an arbitrary compactification of X, this completes the proof. ∎

It follows that if X is the union of a locally finite family of closed and Ohio complete subspaces, then X is itself Ohio complete. This generalizes the disjoint sum theorem proved in [2]. We also see that if X is the countable union of closed and Ohio complete subspaces, then X is Ohio complete. So Ohio completeness is also preserved by taking countable closed sums. This yields the following equivalence.

Corollary 3.4. *Let X be Ohio complete. Then the following statements are equivalent*:

(1) *every closed subset of X is Ohio complete*,
(2) *every F_σ-subset of X is Ohio complete.*

4. Open sum theorems for Ohio completeness. It is a well known fact that a finite union of G_δ-subsets is again a G_δ-subset ([5, p. 26]). This fact yields the following observation.

Proposition 4.1. *Suppose $\mathcal{G}$ is a finite cover of X consisting of G_δ-subsets of X. If $X \subseteq Z$ and every element of $\mathcal{G}$ is Ohio embedded in Z, then X is Ohio embedded in Z.*

Proof. Let $\mathcal{G} = \{G_i : i \in I\}$, where I is finite. For every $i \in I$, we may fix a G_δ-subset S_i of Z which is good with respect to G_i. Note that since G_i is a G_δ-subset of X, we may assume without loss of generality that $S_i \cap X = G_i$. Then $S = \bigcup_{i\in I} S_i$ is a G_δ-subset of Z since it is a finite union of G_δ-subsets of Z. We claim that S is good with respect to X. First of all, note that $X \subseteq S$, since $G_i \subseteq S_i$ for $i \in I$. So it remains to verify that every point in $S \setminus X$ can be separated from X by a G_δ-subset of Z.

So fix an arbitrary point $z \in S \setminus X$. Then $z \in S_i \setminus G_i$ for some $i \in I$. So by construction, there is a G_δ-subset T of Z such that $z \in T$ and $T \cap G_i = \emptyset$. But then, since $S_i \cap X = G_i$, the set $S_i \cap T$ is a G_δ-subset of Z which separates z from X. ∎

COROLLARY 4.2. *Let $\mathcal{G}$ be a finite cover of X whose members are G_δ-subsets. If every element of $\mathcal{G}$ is contained in an Ohio complete subspace of X, then X is itself Ohio complete.*

Proof. Fix an arbitrary compactification γX of X. Since every element of $\mathcal{G}$ is contained in an Ohio complete subspace of X, it follows from Proposition 2.1 and Corollary 2.5 that every element of $\mathcal{G}$ is Ohio embedded in γX. So by the previous proposition, X is Ohio embedded in γX. Since γX was an arbitrary compactification of X, this shows that X is Ohio complete. ∎

So in particular, if X is covered by a finite family of open and Ohio complete subspaces, then X is also Ohio complete.

The following lemma is well known (see [10, Lemma 3]). For completeness we include the simple proof. Recall that a family $\mathcal{A}$ of subsets of X is called *point-finite* if for every $x \in X$, the set $\{A \in \mathcal{A} : x \in A\}$ is finite.

LEMMA 4.3. *Let $\mathcal{G}$ be a family of G_δ-subsets of a space X. If there is a point-finite family $\mathcal{U} = \{U(G) : G \in \mathcal{G}\}$ of open subsets of X such that $G \subseteq U(G)$ for all $G \in \mathcal{G}$, then $\bigcup \mathcal{G}$ is also a G_δ-subset of X.*

Proof. Fix the point-finite family $\mathcal{U} = \{U(G) : G \in \mathcal{G}\}$ of open subsets of X such that $G \subseteq U(G)$ for all $G \in \mathcal{G}$. For every $G \in \mathcal{G}$, we fix a decreasing sequence $(G_n)_{n\in\omega}$ of open subsets of $U(G)$ (and hence of X) such that $G = \bigcap_{n\in\omega} G_n$. We now let $W_n = \bigcup\{G_n : G \in \mathcal{G}\}$ and $W = \bigcap_{n\in\omega} W_n$. Note that W is a G_δ-subset of X.

We will prove that $\bigcup \mathcal{G} = W$. Clearly, $\bigcup \mathcal{G} \subseteq W$. For the reverse inclusion, let $x \in W$ be arbitrary. By hypothesis, the set $\mathcal{F} = \{G \in \mathcal{G} : x \in U(G)\}$ is finite. Suppose, aiming at a contradiction, that $x \notin \bigcup \mathcal{F}$. Since $\mathcal{F}$ is finite, we may find an index $n \in \omega$ so large that $x \notin G_n$ for all $G \in \mathcal{F}$. Then $x \notin W_n$, which is a contradiction, since $x \in W$. So it follows that $x \in \bigcup \mathcal{F}$ and hence $x \in \bigcup \mathcal{G}$. ∎

We now come to the main result of this section.

THEOREM 4.4. *Let $\mathcal{U}$ be a point-finite open cover of X. If $X \subseteq Z$ and every element of $\mathcal{U}$ is Ohio embedded in Z, then X is Ohio embedded in Z.*

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$, and for every $i \in I$, fix an open set $\widetilde{U}_i$ of Z such that $U_i = X \cap \widetilde{U}_i$. We let Y be the subspace of Z given by

$$Y = \{z \in Z : \{i \in I : z \in \widetilde{U}_i\} \text{ is finite}\}.$$

Note that since $\mathcal{U}$ is a point-finite cover of X, we have $X \subseteq Y$. We now prove that X is Ohio embedded in Z in two steps; first we show that X is Ohio embedded in Y and then we show that Y is Ohio embedded in Z.

Claim 1. *X is Ohio embedded in Y.*

Proof. For $U \in \mathcal{U}$ we have $U \subseteq Y \subseteq Z$, so it follows from Proposition 2.2 that every element of $\mathcal{U}$ is Ohio embedded in Y. So for every $i \in I$, we may fix a G_δ-subset S_i of Y which is good with respect to U_i. Without loss of generality, we may assume that $S_i \subseteq \widetilde{U}_i \cap Y$. Let $S = \bigcup\{S_i : i \in I\}$. By definition of Y, the family $\{\widetilde{U}_i \cap Y : i \in I\}$ is a point-finite family of open subsets of Y. So it follows from Lemma 4.3 that S is a G_δ-subset of Y. We leave it to the reader to verify that S is good with respect to X. ◂

Claim 2. *Y is Ohio embedded in Z.*

Proof. We will show that Z is good with respect to Y. So suppose that $z \in Z \setminus Y$. By definition, the set $\{i \in I : z \in \widetilde{U}_i\}$ is infinite. So we may fix a countably infinite subset J of I such that $z \in \widetilde{U}_j$ for every $j \in J$. But then $T = \bigcap_{j \in J} \widetilde{U}_j$ is a G_δ-subset of Z which separates z from Y. ◂

It now follows from Proposition 2.3 that X is Ohio embedded in Z. ■

Corollary 4.5. *Let $\mathcal{U}$ be a point-finite open cover of X. If every element of $\mathcal{U}$ is contained in an Ohio complete subspace of X, then X is itself Ohio complete.*

Proof. Fix an arbitrary compactification γX of X. By Proposition 2.1 and Corollary 2.5, the assumptions imply that every element of $\mathcal{U}$ is Ohio embedded in γX. So it follows from the previous theorem that X is Ohio embedded in γX. Since γX was an arbitrary compactification of X, this proves that X is Ohio complete. ■

So in particular it follows from the previous result that if X is covered by a locally finite family of open and Ohio complete subspaces, then X is also Ohio complete. Recall that a space X is (*countably*) *metacompact* if every (countable) open cover of X has a point-finite open refinement. A space X is (*countably*) *submetacompact* if for every (countable) open cover $\mathcal{U}$ of X, there is a countable collection $\mathcal{E}$ of closed subsets of X such that for every $E \in \mathcal{E}$, there exists an open cover $\mathcal{U}_E$ of X refining $\mathcal{U}$ and point-finite on E. Gittings proved in [7] that countable submetacompactness is equivalent to countable metacompactness.

Corollary 4.6. *Let X be a* (*countable*) *submetacompact space and $\mathcal{U}$ a* (*countable*) *open cover of X. If every element of $\mathcal{U}$ is contained in an Ohio complete subspace of X, then X is itself Ohio complete.*

Proof. We may fix a countable collection $\mathcal{E}$ of closed subsets of X and for every $E \in \mathcal{E}$ an open cover $\mathcal{U}_E$ of X refining $\mathcal{U}$ and point-finite on E. Now let γX be an arbitrary compactification of X. By Theorem 3.2 it suffices to prove that each $E \in \mathcal{E}$ is Ohio embedded in γX. Fix $E \in \mathcal{E}$. It follows from Propositions 2.1 and 2.3 that for every $U \in \mathcal{U}_E$, the set $U \cap E$ is Ohio embedded in γX. Since $\{U \cap E : U \in \mathcal{U}_E\}$ is a point-finite open cover of E, it follows from Theorem 4.4 that E is Ohio embedded in γX. This completes the proof. ∎

With respect to countable open covers, the following corollary is the most general result of this section. Note that if Question 2.7 has a positive answer, then the countable case of the previous result follows from the next.

Corollary 4.7. *Let X be a countably submetacompact space and let $\mathcal{U}$ be a σ-point-finite open cover of X. If every element of $\mathcal{U}$ is Ohio complete, then X is also Ohio complete.*

Proof. Note that X is countably metacompact by [7, Theorem 2.2]. Let $X = \bigcup_{n\in\omega} U_n$, where every U_n is the union of a point-finite family of open and Ohio complete subspaces of X. By Corollary 4.5, every U_n is Ohio complete. Since X is countably metacompact, there exists a point-finite open refinement $\mathcal{V}$ of the cover $\{U_n : n \in \omega\}$. Applying Corollary 4.5 for a second time shows that X is Ohio complete. ∎

Let $\mathcal{U}$ be a countable open cover of the space X. It follows from the previous result that if X is countably submetacompact and every member of $\mathcal{U}$ is Ohio complete, then X is also Ohio complete. Note that among the countably submetacompact spaces are all countably compact, countably paracompact and countably subparacompact spaces. In general, there is no countable open sum theorem for Ohio completeness: in the final section of this paper we present an example of a non-Ohio complete space which is covered by countably many open and Ohio complete subspaces.

5. Examples. In this section we provide some simple examples of spaces that are not Ohio complete. The following simple observation is used in every example.

Lemma 5.1. *Let X be an Ohio complete subspace of Z. If X is not a G_δ-subset of Z, then $Z \setminus X$ contains a non-empty G_δ-subset of Z.*

Proof. Since X is Ohio complete, it follows from Proposition 2.4 that X is Ohio embedded in Z. So we may fix a G_δ-subset S of Z which is good with respect to X. Since X is not a G_δ-subset of Z, the set $S \setminus X$ is non-empty. Since every point of $S \setminus X$ can be separated from X by a G_δ-subset of Z, it follows that $Z \setminus X$ contains a non-empty G_δ-subset of Z. ∎

The first simple example shows that a space which is the union of two Ohio complete subspace, one open and the other closed, need not be Ohio complete.

EXAMPLE 5.2. Let Y be an uncountable discrete space and let $\alpha Y = Y \cup \{\infty\}$ be its one-point compactification. The example is the subspace X of the product $Z = \alpha Y \times \alpha Y$ where $X = (Y \times Y) \cup \{(\infty, \infty)\}$. Note that both $Y \times Y$ and $\{(\infty, \infty)\}$ are Ohio complete.

If G is a G_δ-subset of Z which contains the point (∞, ∞), then $G \cap (Z \setminus X)$ is non-empty, so X is not a G_δ-subset of Z. Similarly, $Z \setminus X$ contains no non-empty G_δ-subset of Z, so it follows from Lemma 5.1 that X is not Ohio complete.

A family $\mathcal{A}$ of subsets of a space X is called *locally countable* provided that for every $x \in X$, there is a neighbourhood U of x such that the set $\{A \in \mathcal{A} : A \cap U \neq \emptyset\}$ is countable. Note that every σ-locally finite family is locally countable. In view of Corollary 3.3 it is natural to ask whether Ohio completeness is preserved by taking locally countable closed sums. We now provide an example to show that this is not the case.

The spaces ω_1 and $\omega_1{+}1$ carry the usual order topology. Whenever $\alpha < \beta \leq \omega_1$, then (α, β), $[\alpha, \beta]$ and $(\alpha, \beta]$ denote the usual intervals in $\omega_1{+}1$.

EXAMPLE 5.3. Let L be the set of all limit ordinals in ω_1. We let $Z = \omega_1 \times (\omega_1 + 1)$ and X be the subspace of Z given by

$$X = (\omega_1 \times \omega_1) \cup (L \times \{\omega_1\}).$$

To avoid confusion with intervals, we denote elements of Z by $\langle \alpha, \beta \rangle$. We use Lemma 5.1 to show that X is not Ohio complete. Since every closed subset of ω_1 that misses L is finite, it follows that L is not a G_δ-subset of ω_1. But then X is not a G_δ-subset of Z. To conclude that X is not Ohio complete, observe that $Z \setminus X$ contains no non-empty G_δ-subset of Z.

It remains to verify that X is the union of a locally countable family of closed and Ohio complete subspaces of X. Let π be the projection of X onto the first coordinate. That is, $\pi : X \to \omega_1$ is given by $\pi(\langle \alpha, \beta \rangle) = \alpha$. We let the closed cover $\mathcal{A}$ of X be given by

$$\mathcal{A} = \{\pi^{-1}(\alpha) : \alpha \in \omega_1\}.$$

Note that the fibers of π are homeomorphic to $\omega_1{+}1$ or to ω_1, which are both Ohio complete spaces since they are (locally) compact. Furthermore, since ω_1 is locally countable in itself and π is continuous, it follows that $\mathcal{A}$ is locally countable in X.

It follows from Corollary 3.3 that if the family $\{\{x\} : x \in X\}$ is σ-locally finite in X, then $X \times Y$ is Ohio complete whenever Y is Ohio complete. In particular, if X is either countable or discrete and Y is Ohio complete, then

$X \times Y$ is Ohio complete. In the previous example we have made good use of the fact that the family $\{\{\alpha\} : \alpha \in \omega_1\}$ is locally countable but not σ-locally finite in ω_1. This raises the following question:

QUESTION 5.4. *Suppose that X is Ohio complete. Is $\omega_1 \times X$ also Ohio complete?*

We shall now provide several examples of first countable spaces that are not Ohio complete. In the following theorem we use a modification of the well known Aleksandrov duplicate to obtain examples of first countable non-Ohio complete spaces.

THEOREM 5.5. *Suppose X is a dense Lindelöf subspace of Z such that every G_δ-subset of Z containing X is uncountable. Then there is a non-Ohio complete space Y which satisfies the following conditions*:

(1) *If X is first countable*, *then so is Y.*
(2) *If X is zero-dimensional*, *then so is Y.*

Proof. The space ω_1+2 carries the usual order topology, it is the disjoint sum of the space ω_1+1 and the point ω_1+1. The set W is given by $Z\times(\omega_1+2)$ and Y is given as the following subset of W:

$$Y = (Z \times \omega_1) \cup (X \times \{\omega_1+1\}).$$

We define a topology on W as follows: basic open neighbourhoods of points of the form $\langle z, \alpha\rangle$, where $z \in Z$ and $\alpha \in \omega_1+1$, are of the form $\{z\} \times U$ where U is an open subset of ω_1+1. Basic open neighbourhoods of points of the form $\langle z, \omega_1+1\rangle$ are given by

$$U(z) = (U \times (\omega_1+2)) \setminus (\{z\} \times (\omega_1+1)),$$

where U is an open neighbourhood of z in Z. We leave it to the reader to verify that these sets may serve as a basis for a Tikhonov topology on W. We point out that the topology on W may be viewed as a resolution topology (see [6] and [11] for details on resolutions). The subset Y is given the subspace topology inherited from W and it is not hard to verify that (2) holds. For (1), assume that X is first countable. Since X is dense in Z, it follows that Z is first countable at every point of X (see for example [9, 2.7]). Now it follows easily that Y is also first countable.

We shall now show that Y is not Ohio complete. Since $\{z\} \times (\omega_1+1)$ is homeomorphic to ω_1+1, no point of the set $Z \times \{\omega_1\}$ can be separated from Y by a G_δ-subset of W. Using the techniques of Lemma 5.1, non-Ohio completeness of Y follows from the following observation:

CLAIM 1. *Whenever G is a non-empty G_δ-subset of W containing Y then $G \cap (Z \times \{\omega_1\}) \neq \emptyset$.*

Proof. Let $G = \bigcap_{n\in\omega} G_n$ be a G_δ-subset containing Y, where each G_n is an open subset of W. Note that the closed subspace $X \times \{\omega_1{+}1\}$ of W is homeomorphic to X. Since X is Lindelöf, for every $n \in \omega$ we may find a countable subset F_n of X and for each $x \in F_n$ an open neighbourhood $U_{x,n}$ of x in Z such that

$$X \times \{\omega_1{+}1\} \subseteq \bigcup\{U_{x,n}(x) : x \in F_n\} \subseteq G_n.$$

For $n \in \omega$, we let $U_n = \bigcup\{U_{x,n} : x \in F_n\}$ and $U = \bigcap_{n\in\omega} U_n$. We also let $F = \bigcup_{n\in\omega} F_n$. Note that since each F_n is countable, the set F is also countable. Since U is a G_δ-subset of Z which contains X, we see that $U \setminus F$ is uncountable, so in particular it is non-empty. Now let $z \in U \setminus F$. It is not hard to realize that in this case we have

$$\{z\} \times (\omega_1{+}2) \subseteq G,$$

so that $\langle z, \omega_1\rangle \in G \cap (Z \times \{\omega_1\})$, and this proves the claim. ◂

As indicated before, this proves the theorem. ■

From this theorem follow many examples of first countable non-Ohio complete spaces. For instance, the set of rationals $\mathbb{Q}$ is a Lindelöf subspace of the reals. Since $\mathbb{Q}$ is not a G_δ-subset of the reals, it follows that every G_δ-set containing $\mathbb{Q}$ is uncountable. Secondly, in the previous theorem we may also take $X = Z = \mathsf{C}$, where C is the usual Cantor set. Since C is compact and uncountable, the previous theorem yields a first countable zero-dimensional non-Ohio complete space Y. It was proved by Dow and Pearl [4] that in this case Y^ω is homogeneous. One verifies easily that since Y is not Ohio complete, neither is Y^ω. So Y^ω is an example of a homogeneous first countable space which is not Ohio complete.

It is well known that every first countable topological group is metrizable (see for example [8, Theorem 8.3]). Since every metrizable space is Ohio complete, it follows that first countable topological groups are Ohio complete. The last example demonstrates that this is not true in general for first countable homogeneous spaces.

The fact that non-Ohio complete first countable spaces exist yields the following:

COROLLARY 5.6. *Ohio completeness is not preserved by open mappings.*

Proof. Every first countable space is the image of a metrizable space under an open mapping. This was proved by Ponomarev (see for example [5, Problem 4.2.D]). We have just provided an example of a first countable space which is not Ohio complete. Since every metrizable space is Ohio complete, the statement follows. ■

Our final example shows that there is no countable open sum theorem for Ohio completeness. We present an example of a non-Ohio complete, zero-dimensional and first countable space which is covered by countably many open and Ohio complete subspaces.

EXAMPLE 5.7. The example is a slight modification of [3, Example 2.4]. For each $q \in \mathbb{Q}$, let $A(q)$ be a maximal family of one-to-one functions from the set $\mathbb{N}$ of natural numbers into the set $\mathbb{P}$ of irrationals, such that

(1) If $a \in A(q)$, then $|a(n) - q| < 1/n$ for all $n \in \mathbb{N}$.
(2) If $a, b \in A(q)$ are different, then $a(\mathbb{N}) \cap b(\mathbb{N})$ is finite.

We fix an uncountable discrete space Y and let $\alpha Y = Y \cup \{\infty\}$ be its one-point compactification. Put $A = \bigcup\{A(q) : q \in \mathbb{Q}\}$ and let $Z = A \cup (\mathbb{P} \times \alpha Y)$. For $a \in A$ and $k \in \mathbb{N}$, we let

$$U(a, k) = \{a\} \cup \bigcup\{a(n) \times \alpha Y : n \geq k\}.$$

The collection $\mathcal{B}$, which serves as a base for a topology on Z, is given by

$$\mathcal{B} = \{U(a, k) : a \in A,\ k \in \mathbb{N}\} \cup \{R \times U : R \subseteq \mathbb{P},\ U \text{ is an open subset of } \alpha Y\}.$$

From now on we consider Z with the topology generated by $\mathcal{B}$. It is easily verified that Z is Hausdorff and locally compact and hence Tikhonov. We let X be the subspace of Z wich is given by $A \cup (\mathbb{P} \times Y)$. We shall show that X is the union of countably many open and Ohio complete subspaces but that X itself is not Ohio complete. Note that for $p \in \mathbb{P}$, the subspace $\{p\} \times \alpha Y$ of Z is homeomorphic to αY and therefore $Z \setminus X$ contains no non-empty G_δ-subset of Z. So by Lemma 5.1, to show that X is not Ohio complete it suffices to show that X is not a G_δ-subset of Z:

CLAIM 1. *X is not a G_δ-subset of Z.*

Proof. If X is a G_δ-subset of Z, then A is a G_δ-subset of the subspace $A \cup (\mathbb{P} \times \{\infty\})$ of Z. This subspace is just the space Z_4 in [3, Example 2.4] and it is proved there that A is not a G_δ-subset of Z_4. It follows that X is not a G_δ-subset of Z. ◂

We now show that X is the union of countably many open and Ohio complete subspaces. For each $q \in \mathbb{Q}$, we let $X_q = A(q) \cup (\mathbb{P} \times Y)$. It is not hard to verify that X_q is an open subspace of X and of course $X = \bigcup\{X_q : q \in \mathbb{Q}\}$. It remains to verify that each X_q is Ohio complete.

CLAIM 2. *For each $q \in \mathbb{Q}$, the space X_q is Ohio complete.*

Proof. Fix $q \in \mathbb{Q}$. Note that both $A(q)$ and $\mathbb{P} \times Y$ are discrete subspaces of X_q. Since a discrete space is Ohio complete, we see that X_q is the union of two Ohio complete subspaces. The space $\mathbb{P} \times Y$ is clearly an open subspace of X_q, and as in [3, Example 2.4], the set $A(q)$ is a G_δ-subset of X_q. It follows from Corollary 4.2 that X_q is Ohio complete. ◂

REFERENCES

[1] A. V. Arhangel′skiĭ, *Remainders in compactifications and generalized metrizability properties*, Topology Appl. 150 (2005), 79–90.
[2] D. Basile and J. van Mill, *Ohio completeness and products*, Topology Appl. 155 (2008), 180–189.
[3] J. Chaber, *Metacompactness and the class MOBI*, Fund. Math. 91 (1976), 211–217.
[4] A. Dow and E. Pearl, *Homogeneity in powers of zero-dimensional first-countable spaces*, Proc. Amer. Math. Soc. 125 (1997), 2503–2510.
[5] R. Engelking, *General Topology*, 2nd ed., Sigma Ser. Pure Math. 6, Heldermann, Berlin, 1989.
[6] V. Fedorčuk [V. Fedorchuk], *Bicompacta with noncoinciding dimensionalities*, Dokl. Akad. Nauk SSSR 182 (1968), 275–277 (in Russian); English transl.: Soviet Math. Dokl. 9 (1968), 1148–1150.
[7] R. F. Gittings, *Some results on weak covering conditions*, Canad. J. Math. 26 (1974), 1152–1156.
[8] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. 1, Academic Press, New York, 1963.
[9] I. Juhász, *Cardinal Functions in Topology—Ten Years Later*, 2nd ed., Math. Centre Tracts 123, Math. Centrum, Amsterdam, 1980.
[10] N. Kemoto, *Subnormality in* ω_1^2, Topology Appl. 122 (2002), 287–296.
[11] S. Watson, *The construction of topological spaces: planks and resolutions*, in: Recent Progress in General Topology (Prague, 1991), North-Holland, Amsterdam, 1992, 673–757.

Faculty of Sciences
Department of Mathematics
VU University Amsterdam
De Boelelaan 1081^A
1081 HV Amsterdam, the Netherlands
E-mail: basile@dmi.unict.it
vanmill@few.vu.nl
gfridder@few.vu.nl
http://www.math.vu.nl/~vanmill
http://www.math.vu.nl/~gfridder

Received 18 June 2007;
revised 13 November 2007 (4936)