COLLOQUIUM MATHEMATICUM

VOL. 144 2016 NO. 2

MELKERSSON CONDITION ON SERRE SUBCATEGORIES

BY

REZA SAZEEDEH (Urmia and Tehran) and RASUL RASULI (Tehran)

Abstract. Let R be a commutative noetherian ring, let $\mathfrak{a}$ be an ideal of R, and let $\mathcal{S}$ be a subcategory of the category of R-modules. The condition $C_{\mathfrak{a}}$, defined for R-modules, was introduced by Aghapournahr and Melkersson (2008) in order to study when the local cohomology modules relative to $\mathfrak{a}$ belong to $\mathcal{S}$. In this paper, we define and study the class $\mathcal{S}_{\mathfrak{a}}$ consisting of all modules satisfying $C_{\mathfrak{a}}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R, we get a necessary and sufficient condition for $\mathcal{S}$ to satisfy $C_{\mathfrak{a}}$ and $C_{\mathfrak{b}}$ simultaneously. We also find some sufficient conditions under which $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$. As an application, we investigate when local cohomology modules lie in a Serre subcategory.

1. Introduction. Throughout this paper, R is a commutative noetherian ring and $\mathfrak{a}$ is an arbitrary ideal of R. We denote by R-Mod the category of R-modules and R-homomorphisms, and by R-mod the full subcategory of finitely generated R-modules. All subcategories of R-Mod considered in this paper are full. A subcategory $\mathcal{S}$ of R-Mod is called *Serre* if it is closed under taking submodules, quotients and extensions of modules and every R-module isomorphic to an R-module in $\mathcal{S}$ is in $\mathcal{S}$. For every module M, we recall from [BS] the submodule $\Gamma_{\mathfrak{a}}(M)$ of M consisting of all elements of M annihilated by some powers of $\mathfrak{a}$.

We say that a class S satisfies the *condition* $C_{\mathfrak{a}}$ if for every module M, the following implication holds:

$$\text{If } \Gamma_{\mathfrak{a}}(M) = M \text{ and } (0 :_M \mathfrak{a}) \text{ is in } \mathcal{S}, \text{ then } M \text{ is in } \mathcal{S}.$$

The condition $C_{\mathfrak{a}}$ is called the *Melkersson condition* as it was first introduced by Melkersson [M] for the class $\mathcal{S}$ consisting of all artinian modules.

Let M be an R-module and fix $n \in \mathbb{N}$. It is a natural question to ask when the local cohomology modules $H^i_{\mathfrak{a}}(M)$ belong to $\mathcal{S}$ for all $i < n$ (or for all $i > n$). The same question can be asked for the graded local cohomology modules $H^i_{R_+}(M)$, where R is a graded ring, R_+ is the irrelevant ideal and M is a graded module. Some examples for $\mathcal{S}$ are R-mod and R-art, the subcategory of artinian R-modules. It is worth pointing out that in the

2010 *Mathematics Subject Classification*: 13C60, 13D45.
Key words and phrases: Serre subcategory, Melkersson condition, local cohomology.
Received 28 September 2014; revised 5 August 2015.
Published online 14 April 2016.

DOI: 10.4064/cm6384-9-2015

case of graded local cohomology, the affirmative solution for these questions allows us to assess the number of minimal generators of the components of graded local cohomology modules (cf. [BFT, BRS, S]).

An affirmative answer was presented by M. Aghapournahr and L. Melkersson [AM] when $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Many examples demonstrate that Serre subcategories do not satisfy $C_{\mathfrak{a}}$ in general. The aim of this paper is to define and study the class $\mathcal{S}_{\mathfrak{a}}$ consisting of all modules satisfying the above implication for a class $\mathcal{S}$ of R-modules. Clearly if $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$, then $\mathcal{S}_{\mathfrak{a}} = R$-Mod.

In Section 2, for any class $\mathcal{S}$ of modules, we introduce the class $\mathcal{S}_{\mathfrak{a}}$ of modules containing $\mathcal{S}$ and satisfying C_{α}. We show that if a subcategory $\mathcal{S}$ is closed under taking submodules, then $\mathcal{S}_{\sqrt{\mathfrak{a}}} \subseteq \mathcal{S}_{\mathfrak{a}}$. Moreover $\mathcal{S}_{\sqrt{\mathfrak{a}}} = \mathcal{S}_{\mathfrak{a}}$ if S is Serre. Let $\mathfrak{b}$ be another ideal of R. We show that if $\mathcal{S}_{\mathfrak{b}} \subseteq \mathcal{S}_{\mathfrak{a}}$, then $\mathcal{S}_{\mathfrak{b}} \subseteq \mathcal{S}_{\mathfrak{a}+\mathfrak{b}}$. When $\mathcal{S}$ is Serre we find a relation between $\mathcal{S}_{\mathfrak{a}}, \mathcal{S}_{\mathfrak{b}}, \mathcal{S}_{\mathfrak{a}+\mathfrak{b}}, \mathcal{S}_{\mathfrak{a}\mathfrak{b}}$. As a conclusion, $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ and $C_{\mathfrak{b}}$ if and only if it satisfies $C_{\mathfrak{a}+\mathfrak{b}}$ and $C_{\mathfrak{a}\mathfrak{b}}$. When R is artinian, we show that every Serre subcategory satisfies $C_{\mathfrak{a}}$. Also, $\mathcal{S}_{\mathfrak{a}}$ is closed under taking extensions of modules for any Serre subcategory $\mathcal{S}$. We prove that if $\mathcal{S}$ is closed under taking submodules and arbitrary direct sums, then $\mathcal{S}_{\mathfrak{a}}$ is closed under arbitrary direct sums. We find some sufficient conditions for $\mathcal{S}$ to satisfy $C_{\mathfrak{a}}$ (cf. Theorem 2.20). We also show that the condition $C_{\mathfrak{a}}$ can be transferred via ring homomorphisms (cf. Theorem 2.21). For a class $\mathcal{S}$ of R-modules, we define $\operatorname{Supp}_R(\mathcal{S})$, and we prove that if $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(\mathcal{S}_{\mathfrak{a}})$ for a finitely generated R-module M and a Serre subcategory $\mathcal{S}$ of R-modules, then $M \in \mathcal{S}_{\mathfrak{a}}$.

In Section 3, as an application of our results, we show when local cohomology modules can lie in a Serre subcategory.

2. Melkersson condition on subcategories. Throughout this section $\mathfrak{a}$ is an ideal of R.

DEFINITIONS 2.1. Let $\mathcal{S}$ be a class of R-modules and let M be an R-module. Then $\mathcal{S}$ is said to satisfy the *condition* $C_{\mathfrak{a}}$ *on* M if the following implication holds:

$$\text{If } \Gamma_{\mathfrak{a}}(M) = M \text{ and } (0:_M \mathfrak{a}) \in \mathcal{S}, \text{ then } M \in \mathcal{S}.$$

Let $\mathcal{D}$ be a class of R-modules. Then $\mathcal{S}$ is said to satisfy the *condition* $C_{\mathfrak{a}}$ *on* $\mathcal{D}$ if $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on M for every M in $\mathcal{D}$.

We denote by $\mathcal{S}_{\mathfrak{a}}$ the largest class of R-modules such that $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on $\mathcal{S}_{\mathfrak{a}}$. Clearly, $\mathcal{S} \subseteq \mathcal{S}_{\mathfrak{a}}$.

The class $\mathcal{S}$ is said to satisfy the *condition* $C_{\mathfrak{a}}$ whenever $\mathcal{S}_{\mathfrak{a}} = R$-Mod, and $\mathcal{S}$ is said to be *closed under the condition* $C_{\mathfrak{a}}$ whenever $\mathcal{S}_{\mathfrak{a}} = \mathcal{S}$.

In order to illustrate the above definitions, we give some examples.

EXAMPLES 2.2. (i) Let R be a domain and let $\mathcal{S}_{\mathrm{tf}}$ be the class of torsion-free modules. Then $\mathcal{S}_{\mathrm{tf}}$ satisfies $C_{\mathfrak{a}}$ for each ideal $\mathfrak{a}$ of R. Indeed, the case $\mathfrak{a}=0$ is clear. For each non-zero ideal $\mathfrak{a}$ of R, if $\Gamma_{\mathfrak{a}}(M) = M$ and $(0 :_M \mathfrak{a}) \in \mathcal{S}$, then $(0 :_M \mathfrak{a}) = \Gamma_{\mathfrak{a}}(M) = 0$. Furthermore, let $\mathcal{S}_{\mathrm{tors}}$ be the class of torsion modules. Then $\mathcal{S}_{\mathrm{tors}}$ satisfies $C_{\mathfrak{a}}$ for each ideal $\mathfrak{a}$ of R.

(ii) Let $\mathcal{S}$ be a Serre subcategory of R-mod. It follows from [Y, Proposition 4.3] that R-mod $\subseteq \mathcal{S}_{\mathfrak{a}}$ for every ideal $\mathfrak{a}$ of R.

(iii) Let $(R, \mathfrak{m})$ be a local ring and let $\mathcal{S} = R$-mod. Then $E(R/\mathfrak{m})$ is in $\mathcal{S}_{\mathfrak{m}}$ if and only if R is artinian. To be more precise, suppose $E(R/\mathfrak{m}) \in \mathcal{S}_{\mathfrak{m}}$. Since $\Gamma_{\mathfrak{m}}(E(R/\mathfrak{m})) = E(R/\mathfrak{m})$ and $\mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \cong R/\mathfrak{m} \in \mathcal{S}$, it follows that $E(R/\mathfrak{m})$ is finitely generated and so R is artinian. Conversely, if R is artinian, then $E(R/\mathfrak{m}) \in \mathcal{S} \subseteq \mathcal{S}_{\mathfrak{m}}$.

(iv) Let $(R, \mathfrak{m})$ be a local ring and let $\mathcal{S}$ be a Serre subcategory of R-Mod. Then R-art $\cap\, \mathcal{S}_{\mathfrak{m}}$ is a subclass of S where R-art is the subcategory of artinian modules. To be more precise, for every $M \in R$-art $\cap\, \mathcal{S}_{\mathfrak{m}}$, the module $(0 :_M \mathfrak{m})$ has finite length and so is in $\mathcal{S}$. Now, since M is in $\mathcal{S}_{\mathfrak{m}}$, it is in $\mathcal{S}$.

(v) For each class $\mathcal{S}$ of R-modules, all modules annihilated by an ideal $\mathfrak{a}$ belong to $\mathcal{S}_{\mathfrak{a}}$.

The following proposition provides some basic properties of the condition $C_{\mathfrak{a}}$ on classes of modules.

PROPOSITION 2.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be classes of R-modules. Then:*

(i) *If $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{S}_{\mathfrak{a}}$, then $\mathcal{T}_{\mathfrak{a}} \subseteq \mathcal{S}_{\mathfrak{a}}$.*
(ii) *$(\mathcal{S}_{\mathfrak{a}})_{\mathfrak{a}} = \mathcal{S}_{\mathfrak{a}}$.*

Proof. (i) Suppose that M is an R-module in $\mathcal{T}_{\mathfrak{a}}$ with $M = \Gamma_{\mathfrak{a}}(M)$ and $(0 :_M \mathfrak{a}) \in \mathcal{S}$. Then $(0 :_M \mathfrak{a}) \in \mathcal{T}$, and hence $M \in \mathcal{T}$ because $M \in \mathcal{T}_{\mathfrak{a}}$. Now, since $\mathcal{T} \subseteq \mathcal{S}_{\mathfrak{a}}$, we deduce that $M \in \mathcal{S}_{\mathfrak{a}}$.

(ii) The other inclusion follows from (i), by setting $T = S_{\mathfrak{a}}$. ∎

PROPOSITION 2.4. *Assume $\mathcal{S}$ is a subclass of* R-Mod *closed under taking submodules. Then $\mathcal{S}_{\sqrt{\mathfrak{a}}} \subseteq \mathcal{S}_{\mathfrak{a}}$. Furthermore, if $\mathcal{S}$ is a Serre subcategory, then $\mathcal{S}_{\sqrt{\mathfrak{a}}} = \mathcal{S}_{\mathfrak{a}}$.*

Proof. Assume that $M \in \mathcal{S}_{\sqrt{\mathfrak{a}}}$ with $\Gamma_a(M) = M$ and $(0 :_M \mathfrak{a}) \in \mathcal{S}$. Then $\Gamma_{\sqrt{a}}(M) = M$, and since $(0 :_M \sqrt{a}) \subset (0 :_M \mathfrak{a})$, by assumption $(0 :_M \sqrt{a}) \in \mathcal{S}$. Therefore the assumption on M forces that $M \in \mathcal{S}$. To prove the equality, for convenience, we set $\mathfrak{b} = \sqrt{\mathfrak{a}}$. As R is noetherian, there exists a non-negative integer n such that $\mathfrak{b}^n \subseteq \mathfrak{a}$. Assume that $M \in \mathcal{S}_{\mathfrak{a}}$ with $\Gamma_{\mathfrak{b}}(M) = M$ and $(0 :_M \mathfrak{b}) \in \mathcal{S}$. We notice that $\mathfrak{b}/\mathfrak{b}^2$ is a finitely generated $R/\mathfrak{b}$-module, and so for some $m \in \mathbb{N}$ there exists an exact sequence of R-modules

$$0 \to K \to (R/\mathfrak{b})^m \to \mathfrak{b}/\mathfrak{b}^2 \to 0.$$

Applying the functor $\operatorname{Hom}_R(-,M)$, we deduce that $\operatorname{Hom}_R(\mathfrak{b}/\mathfrak{b}^2,M)\in\mathcal{S}$. Moreover, taking $\operatorname{Hom}_R(-,M)$ of the exact sequence

$$0\to\mathfrak{b}/\mathfrak{b}^2\to R/\mathfrak{b}^2\to R/\mathfrak{b}\to 0$$

we find that $(0:_M\mathfrak{b}^2)\cong\operatorname{Hom}_R(R/\mathfrak{b}^2,M)\in\mathcal{S}$. Continuing this way and using an easy induction on n, we conclude that $(0:_M\mathfrak{b}^n)\in\mathcal{S}$. Application of $\operatorname{Hom}_R(-,M)$ to the exact sequence $0\to\mathfrak{a}/\mathfrak{b}^n\to R/\mathfrak{b}^n\to R/\mathfrak{a}\to 0$ implies that $(0:_M\mathfrak{a})\in\mathcal{S}$. Now, since $M\in\mathcal{S}_\mathfrak{a}$, we conclude that $M\in\mathcal{S}$. ∎

Proposition 2.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R and let $\mathcal{S}$ be a subclass of R-*Mod*. If $\mathcal{S}_\mathfrak{b}\subseteq\mathcal{S}_\mathfrak{a}$, then $\mathcal{S}_\mathfrak{b}\subseteq\mathcal{S}_{\mathfrak{a}+\mathfrak{b}}$.*

Proof. Assume that $M\in S_\mathfrak{b}$ with $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M)=M$ and $(0:_M\mathfrak{a}+\mathfrak{b})\in\mathcal{S}$. Clearly $\Gamma_\mathfrak{a}(M)=\Gamma_\mathfrak{b}(M)=M$ and the isomorphisms

$$\begin{aligned}(0:_M\mathfrak{a}+\mathfrak{b})&\cong\operatorname{Hom}(R/\mathfrak{a}+\mathfrak{b},M)\\&\cong\operatorname{Hom}(R/\mathfrak{a},\operatorname{Hom}(R/\mathfrak{b},M))\cong(0:_{(0:_M\mathfrak{b})}\mathfrak{a})\end{aligned}$$

imply that $(0:_{(0:_M\mathfrak{b})}\mathfrak{a})\in\mathcal{S}$. Moreover,

$$\Gamma_\mathfrak{a}((0:_M\mathfrak{b}))=(0:_M\mathfrak{b}).$$

In view of Example 2.2(v), the module $(0:_M\mathfrak{b})$ belongs to $\mathcal{S}_\mathfrak{b}$ and so by assumption it belongs to $\mathcal{S}_\mathfrak{a}$. Therefore the preceding argument implies that $(0:_M\mathfrak{b})\in\mathcal{S}$. Now, since $M\in S_\mathfrak{b}$, we deduce that $M\in\mathcal{S}$. ∎

Corollary 2.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R and let $\mathcal{S}$ be a subcategory of R-*Mod* satisfying the condition $C_\mathfrak{a}$. Then $\mathcal{S}_\mathfrak{b}$ is a subclass of $\mathcal{S}_{\mathfrak{a}+\mathfrak{b}}$. Moreover, if $\mathcal{S}$ satisfies $C_\mathfrak{b}$, then $\mathcal{S}$ satisfies $C_{\mathfrak{a}+\mathfrak{b}}$.*

The same proof as in Proposition 2.5 still works for the following result.

Proposition 2.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R and let $\mathcal{S}$ be a class of R-*Mod*. If $\mathcal{S}_\mathfrak{a}$ is closed under taking submodules, then $\mathcal{S}_\mathfrak{b}\cap\mathcal{S}_\mathfrak{a}\subseteq\mathcal{S}_{\mathfrak{a}+\mathfrak{b}}$.*

The following well-known fact is used in the proof of the next theorem.

Lemma 2.8. *If $\mathcal{S}$ is a Serre subcategory of R-*Mod* and M is in $\mathcal{S}$, then $\operatorname{Ext}^i_R(R/\mathfrak{a},M)$ is in $\mathcal{S}$ for each $i\ge 0$.*

Proof. Let $\cdots\to F_1\to F_0\to 0$ be a free resolution of $R/\mathfrak{a}$ such that each F_i is finitely generated. As $\mathcal{S}$ is Serre, $\operatorname{Hom}_R(F_i,M)\in\mathcal{S}$ for each i. Since $\operatorname{Ext}^i_R(R/\mathfrak{a},M)$ is a quotient of submodules of $\operatorname{Hom}_R(F_i,M)$, we deduce that $\operatorname{Ext}^i_R(R/\mathfrak{a},M)\in\mathcal{S}$. ∎

Now we are in a position to state one of the main results of this paper.

Theorem 2.9. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and let $\mathcal{S}$ be a Serre subcategory of R-*Mod*. Then:*

(i) $\mathcal{S}_{\mathfrak{a}\mathfrak{b}}=\mathcal{S}_{\mathfrak{a}\cap\mathfrak{b}}$.
(ii) *If $\mathcal{S}_{\mathfrak{a}+\mathfrak{b}}$ is closed under taking submodules, then $\mathcal{S}_{\mathfrak{a}+\mathfrak{b}}\cap\mathcal{S}_{\mathfrak{a}\mathfrak{b}}\subseteq\mathcal{S}_\mathfrak{a}\cap\mathcal{S}_\mathfrak{b}$.*

(iii) *If* $\mathcal{S}_{\mathfrak{a}}$ *is closed under taking submodules and* $\mathcal{S}_{\mathfrak{b}}$ *is closed under taking quotients, then* $\mathcal{S}_{\mathfrak{a}} \cap \mathcal{S}_{\mathfrak{b}} \subseteq \mathcal{S}_{\mathfrak{a}+\mathfrak{b}} \cap \mathcal{S}_{\mathfrak{ab}}$.

Proof. Part (i) follows from $\sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{ab}}$ and Proposition 2.4.

(ii) It suffices by symmetry to show that $\mathcal{S}_{\mathfrak{a}+\mathfrak{b}} \cap \mathcal{S}_{\mathfrak{a}\cap\mathfrak{b}} \subseteq \mathcal{S}_{\mathfrak{a}}$. Assume that $M \in \mathcal{S}_{\mathfrak{a}+\mathfrak{b}} \cap \mathcal{S}_{\mathfrak{a}\cap\mathfrak{b}}$ with $M = \Gamma_{\mathfrak{a}}(M)$ and that $(0 :_M \mathfrak{a}) \in \mathcal{S}$. The inclusion $(0 :_M \mathfrak{a} + \mathfrak{b}) \subseteq (0 :_M \mathfrak{a})$ implies that $(0 :_M \mathfrak{a} + \mathfrak{b}) \in \mathcal{S}$. Since $M \in \mathcal{S}_{\mathfrak{a}+\mathfrak{b}}$, it follows from the hypothesis that $\Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \Gamma_{\mathfrak{b}}(M) \in \mathcal{S}_{\mathfrak{a}+\mathfrak{b}}$. Therefore $\Gamma_{\mathfrak{b}}(M) \in \mathcal{S}$. We now consider the following exact sequence of modules:

$$(\dagger) \qquad 0 \to \Gamma_{\mathfrak{b}}(M) \to M \to M/\Gamma_{\mathfrak{b}}(M) \to 0.$$

Applying $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ and using Lemma 2.8, we conclude $(0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}) \in \mathcal{S}$. We now prove that $(0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}) = (0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{ab})$. The inclusion $(0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}) \subseteq (0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{ab})$ is obvious. Conversely, let $m + \Gamma_{\mathfrak{b}}(M) \in (0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{ab})$. Then $\mathfrak{ab}m \subseteq \Gamma_{\mathfrak{b}}(M)$ and so there exists $n \in \mathbb{N}$ such that $\mathfrak{b}^n(\mathfrak{ab}m) = 0$. This implies that $\mathfrak{a}m \subseteq \Gamma_{\mathfrak{b}}(M)$, and hence $m + \Gamma_{\mathfrak{b}}(M) \in (0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{a})$. Therefore $(0 :_{M/\Gamma_{\mathfrak{b}}(M)} \mathfrak{ab}) \in \mathcal{S}$. Application of $\mathrm{Hom}_R(R/\mathfrak{ab}, -)$ to $(\dagger)$ shows that $(0 :_M \mathfrak{ab}) \in \mathcal{S}$. Now, since $\Gamma_{\mathfrak{ab}}(M) = M$ and $M \in \mathcal{S}_{\mathfrak{ab}}$, we deduce that $M \in S$.

(iii) That $\mathcal{S}_{\mathfrak{a}} \cap \mathcal{S}_{\mathfrak{b}} \subseteq \mathcal{S}_{\mathfrak{a}+\mathfrak{b}}$ follows from Proposition 2.7. Assume that $M \in \mathcal{S}_{\mathfrak{a}} \cap \mathcal{S}_{\mathfrak{b}}$ with $\Gamma_{\mathfrak{ab}}(M) = M$ and that $(0 :_M \mathfrak{ab}) \in \mathcal{S}$. The inclusions $(0 :_{\Gamma_{\mathfrak{a}}(M)} \mathfrak{a}) \subseteq (0 :_M \mathfrak{a}) \subseteq (0 :_M \mathfrak{ab})$ force that $(0 :_{\Gamma_{\mathfrak{a}}(M)} \mathfrak{a}) \in \mathcal{S}$. Furthermore, since by assumption $\Gamma_{\mathfrak{a}}(M)$ is in $\mathcal{S}_{\mathfrak{a}}$, it lies in $\mathcal{S}$, and so in view of the exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$$

it suffices to show that $M/\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$. Application of $\mathrm{Hom}_R(R/\mathfrak{b}, -)$ induces the exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{b}, M) \to \mathrm{Hom}_R(R/\mathfrak{b}, M/\Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}^1_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M)).$$

As $(0 :_M \mathfrak{b}) \subseteq (0 :_M \mathfrak{ab})$, we deduce that $\mathrm{Hom}_R(R/\mathfrak{b}, M) \cong (0 :_M \mathfrak{b}) \in \mathcal{S}$; moreover, Lemma 2.8 implies that $\mathrm{Ext}^1_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$. Therefore, since $\mathcal{S}$ is Serre, $(0 :_{M/\Gamma_{\mathfrak{a}}(M)} \mathfrak{b}) \cong \mathrm{Hom}_R(R/\mathfrak{b}, M/\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$. We now show that $\Gamma_{\mathfrak{b}}(M/\Gamma_{\mathfrak{a}}(M)) = M/\Gamma_{\mathfrak{a}}(M)$. Let $m + \Gamma_{\mathfrak{a}}(M) \in M/\Gamma_{\mathfrak{a}}(M)$. Since $\Gamma_{\mathfrak{ab}}(M) = M$, there exists a positive integer n such that $(\mathfrak{ab})^n m = 0$. Thus $\mathfrak{b}^n m \subseteq \Gamma_{\mathfrak{a}}(M)$ so that $m + \Gamma_{\mathfrak{a}}(M) \in \Gamma_{\mathfrak{b}}(M/\Gamma_{\mathfrak{a}}(M))$. On the other hand, since $\mathcal{S}_{\mathfrak{b}}$ is closed under quotients, $M/\Gamma_{\mathfrak{a}}(M)$ is in $\mathcal{S}_{\mathfrak{b}}$ and hence in $\mathcal{S}$. ∎

The following corollary can be obtained immediately from the above theorem.

Corollary 2.10. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of* R *and let* $\mathcal{S}$ *be a Serre subcategory of* R-Mod. *Then* $\mathcal{S}$ *satisfies the conditions* $C_{\mathfrak{a}}$ *and* $C_{\mathfrak{b}}$ *if and only if it satisfies* $C_{\mathfrak{a}+\mathfrak{b}}$ *and* $C_{\mathfrak{a}\cap\mathfrak{b}}$.

Corollary 2.11. *Let $\mathcal{S}$ be a Serre subcategory of R-Mod. If $\mathcal{S}$ satisfies $C_{\mathfrak{p}}$ for every minimal prime ideal $\mathfrak{p}$ of $\mathfrak{a}$, then $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$.*

Proof. In view of Proposition 2.4, it suffices to show that $\mathcal{S}$ satisfies $C_{\sqrt{\mathfrak{a}}}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal prime ideals of $\mathfrak{a}$. Then $\sqrt{\mathfrak{a}} = \bigcap_{i=1}^n \mathfrak{p}_i$. As $\mathcal{S}$ satisfies $C_{\mathfrak{p}_i}$ for each i, using Corollary 2.10 and applying an easy induction, we deduce that $\mathcal{S}$ satisfies $C_{\sqrt{\mathfrak{a}}}$. ∎

Corollary 2.12. *Let $\mathcal{S}$ be a Serre subcategory of R-*Mod *and $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be maximal ideals. If $\mathcal{S}$ satisfies the condition $C_{\prod_{i=1}^n \mathfrak{m}_i}$, then it satisfies $C_{\mathfrak{m}_i}$ for each i.*

Proof. Clearly $\mathcal{S}$ satisfies C_R. For each i, we have $\prod_{j=1,\, j\neq i}^n \mathfrak{m}_j + \mathfrak{m}_i = R$. The assertion now follows from Corollary 2.10. ∎

The next corollary shows that over an artinian ring, every Serre subcategory of R-Mod satisfies $C_{\mathfrak{a}}$.

Corollary 2.13. *Let R be an artinian ring and let $\mathcal{S}$ be a Serre subcategory of R-*Mod*. Then $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$ for each ideal $\mathfrak{a}$ of R.*

Proof. Assuming $\operatorname{Max} R = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$, we have $\sqrt{0} = \prod_{i=1}^n \mathfrak{m}_i$. Obviously $\mathcal{S}$ satisfies C_0, and hence in view of Proposition 2.4 it satisfies $C_{\prod_{i=1}^n \mathfrak{m}_i}$. Thus Corollary 2.12 implies that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{m}_i}$ for each i. Consequently, according to Corollary 2.11, $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ for each ideal $\mathfrak{a}$ of R. ∎

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subcategories of R-Mod. Let $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ be the subclass of R-Mod consisting of all modules M such that there exists an exact sequence of modules $0 \to M_1 \to M \to M_2 \to 0$ with $M_i \in \mathcal{S}_i$ for $i = 1, 2$. We can also refer to $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ as the *class of extension modules of* $\mathcal{S}_1$ *by* $\mathcal{S}_2$. An example is the class of *minimax* modules $\mathcal{M} = \langle R\text{-mod}, R\text{-art} \rangle$.

Theorem 2.14. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-*Mod *and let $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ and $\mathcal{S}_1 \cap \mathcal{S}_2$ satisfy the condition $C_{\mathfrak{a}}$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy $C_{\mathfrak{a}}$.*

Proof. We prove the claim for $\mathcal{S}_1$; the proof for $\mathcal{S}_2$ is similar. Suppose that M is an R-module with $M = \Gamma_{\mathfrak{a}}(M)$ and $(0 :_M \mathfrak{a}) \in \mathcal{S}_1$. As $\mathcal{S}_1$ is a subclass of $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ and $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ satisfies $C_{\mathfrak{a}}$, we deduce that $M \in \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$. Then there is an exact sequence of R-modules $0 \to M_1 \to M \to M_2 \to 0$ such that $M_1 \in \mathcal{S}_1$ and $M_2 \in \mathcal{S}_2$. Since $\mathcal{S}_1$ is Serre, it suffices to verify that $M_2 \in \mathcal{S}_1$. Taking $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ of the above short exact sequence, we obtain the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, M) \to \operatorname{Hom}_R(R/\mathfrak{a}, M_2) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, M_1).$$

It follows from Lemma 2.8 that $\operatorname{Ext}^1_R(R/\mathfrak{a}, M_1) \in \mathcal{S}_1$, and since $\mathcal{S}_1$ and $\mathcal{S}_2$ are Serre, $(0 :_{M_2} \mathfrak{a}) \cong \operatorname{Hom}_R(R/\mathfrak{a}, M_2)$ is in $\mathcal{S}_1 \cap \mathcal{S}_2$. On the other hand, since $\Gamma_{\mathfrak{a}}(M_2) = M_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2$ satisfies $C_{\mathfrak{a}}$, we conclude that $M_2 \in \mathcal{S}_1$. ∎

COROLLARY 2.15. *Let $\mathcal{M}$ and $\mathcal{F}$ be the classes of all minimax modules and all modules of finite length, respectively. If $\mathcal{M}$ and $\mathcal{F}$ satisfy $C_{\mathfrak{a}}$, then so does R-mod.*

Proof. Set $\mathcal{S}_1 = R$-mod and $\mathcal{S}_2 = R$-art. Then it is evident that $\mathcal{S}_1$ and $\mathcal{S}_2$ are Serre, $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \mathcal{M}$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{F}$. Now, the result follows immediately from the previous theorem. ■

PROPOSITION 2.16. *Let $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ be subcategories of R-Mod such that $\mathcal{S}$ is Serre. If $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on $\mathcal{S}_1$ and $\mathcal{S}_2$, then it satisfies $C_{\mathfrak{a}}$ on $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$.*

Proof. Assume that $M \in \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ with $M = \Gamma_{\mathfrak{a}}(M)$ and $(0 :_M \mathfrak{a}) \in \mathcal{S}$. Then there is an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ such that $M_i \in \mathcal{S}_i$ for $i = 1, 2$. Since $\Gamma_{\mathfrak{a}}(M_i) = M_i$ for $i = 1, 2$ and $\mathcal{S}$ is Serre, $(0 :_{M_1} \mathfrak{a})$ is in $\mathcal{S}$. Now, since $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on $\mathcal{S}_1$, we deduce that $M_1 \in \mathcal{S}$. Applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ to the above exact sequence and using Lemma 2.8 we find that $(0 :_{M_2} \mathfrak{a}) \in \mathcal{S}$. Since $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$ on $\mathcal{S}_2$, we deduce that M_2 is in $\mathcal{S}$ and so, by the fact that $\mathcal{S}$ is Serre, M is in $\mathcal{S}$. ■

COROLLARY 2.17. *Let $\mathcal{S}$ be a Serre subcategory of R-Mod. If $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on R-art, then it satisfies $C_{\mathfrak{a}}$ on $\mathcal{M}$, where $\mathcal{M}$ is the class of all minimax modules.*

Proof. Observe that $\mathcal{S} \cap R$-mod is a Serre subcategory of R-mod, and it follows from [Y, Proposition 4.3] that $\mathcal{S} \cap R$-mod satisfies $C_{\mathfrak{a}}$ on R-mod. Thus $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on R-mod. Now the result is a consequence of Proposition 2.16 because $\mathcal{M} = \langle R\text{-mod}, R\text{-art} \rangle$. ■

For each subcategory $\mathcal{S}$ of R-Mod, we set $\mathcal{S}^0 = \{0\}$ and $\mathcal{S}^{n+1} = \langle \mathcal{S}^n, \mathcal{S} \rangle$ for $n \in \mathbb{N}$. Moreover, we set $\langle \mathcal{S} \rangle_{\mathrm{ext}} = \bigcup \mathcal{S}^n$. According to [K, Proposition 2.4] the subcategory $\langle \mathcal{S} \rangle_{\mathrm{ext}}$ is closed under taking extensions of modules.

THEOREM 2.18. *Let $\mathcal{S}$ be a Serre subcategory of R-Mod. Then $\mathcal{S}_{\mathfrak{a}}$ is closed under taking extensions of modules.*

Proof. As $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on $\mathcal{S}_{\mathfrak{a}}$, Proposition 2.16 shows that $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on $\mathcal{S}_{\mathfrak{a}}^2$. Repeating this argument, we deduce that $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on $\mathcal{S}_{\mathfrak{a}}^n$ for each $n \in \mathbb{N}$. Therefore $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$ on $\langle \mathcal{S}_{\mathfrak{a}} \rangle_{\mathrm{ext}}$. On the other hand, $\mathcal{S} \subseteq \mathcal{S}_{\mathfrak{a}} \subseteq \langle \mathcal{S}_{\mathfrak{a}} \rangle_{\mathrm{ext}}$, and by the definition $\mathcal{S}_{\mathfrak{a}}$ is the largest subcategory of R-Mod such that $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ on $\mathcal{S}_{\mathfrak{a}}$; hence $\mathcal{S}_{\mathfrak{a}} = \langle \mathcal{S}_{\mathfrak{a}} \rangle_{\mathrm{ext}}$. ■

We recall from [St] that a Serre subcategory $\mathcal{S}$ of R-Mod is a *torsion subcategory* if it is closed under taking arbitrary direct sums of modules. As the direct limit of a direct system of modules is a quotient of a direct sum of modules, every torsion subcategory is closed under taking direct limits. A well-known example of a torsion subcategory has been given in [AM, Example 2.4(e)]. Namely, let $Z \subseteq \operatorname{Spec} R$ be *closed under specialization*, that is, if $\mathfrak{q} \supseteq \mathfrak{p} \in Z$, then $\mathfrak{q} \in Z$. The class of all R-modules with $\mathrm{Ass}_R(M) \subseteq Z$ (equivalently, $\mathrm{Supp}_R(M) \subseteq Z$) is a torsion subcategory of R-Mod.

The following theorem shows that every torsion subcategory $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$ for each ideal $\mathfrak{a}$ of R.

THEOREM 2.19. *Assume $\mathcal{S}$ is a subcategory of R-Mod closed under taking submodules. Then:*

(i) *If $\mathcal{S}$ is closed under taking arbitrary direct sums, then so is $\mathcal{S}_{\mathfrak{a}}$.*
(ii) *If $\mathcal{S}$ is a torsion subcategory, then $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$.*

Proof. (i) Given $\{M_i\}$ a family of modules in $\mathcal{S}_{\mathfrak{a}}$, we prove that $\coprod M_i \in \mathcal{S}_{\mathfrak{a}}$. Suppose that $\coprod M_i = \Gamma_{\mathfrak{a}}(\coprod M_i)$ and $(0 :_{\coprod M_i} \mathfrak{a}) \in \mathcal{S}$. Since $\mathcal{S}$ is closed under taking submodules, $(0 :_{M_i} \mathfrak{a}) \in \mathcal{S}$ for each i; and moreover $M_i = \Gamma_{\mathfrak{a}}(M_i)$ for each i. Thus $M_i \in \mathcal{S}$ because $M_i \in \mathcal{S}_{\mathfrak{a}}$ for each i. Now, according to the hypothesis, $\coprod M_i \in \mathcal{S}$ so that $\coprod M_i \in \mathcal{S}_{\mathfrak{a}}$.

(ii) Suppose $M = \Gamma_{\mathfrak{a}}(M)$ and $(0 :_M \mathfrak{a}) \in \mathcal{S}$. For every finitely generated submodule N of M, we have $\Gamma_{\mathfrak{a}}(N) = N$ and $(0 :_N \mathfrak{a}) \in \mathcal{S} \cap R$-mod. Now, since $\mathcal{S} \cap R$-mod satisfies $C_{\mathfrak{a}}$ on R-mod by [Y, Proposition 4.3], we conclude that $N \in \mathcal{S}$. Finally, since M is the direct limit of its finitely generated submodules, the assumption implies that $M \in \mathcal{S}$. ∎

THEOREM 2.20. *Let $\mathcal{S}$ be a Serre subcategory of R-Mod such that $S_{\mathfrak{a}}$ is closed under taking submodules. Then $\mathcal{S}$ satisfies $C_{\mathfrak{a}}$ if one of the following conditions holds:*

(i) *$\mathcal{S}_{\mathfrak{a}}$ is closed under taking direct unions;*
(ii) *$\mathcal{S}_{\mathfrak{a}}$ is closed under taking injective hulls.*

Proof. (i) Let M be an R-module. If $\Gamma_{\mathfrak{a}}(M) = 0$, then it is evident that $M \in \mathcal{S}_{\mathfrak{a}}$. Now, suppose $\Gamma_{\mathfrak{a}}(M) \neq 0$, and so there is an exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0.$$

Using Theorem 2.18 and the first case, it suffices to prove that $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$. Since $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\in\mathbb{N}}(0 :_M \mathfrak{a}^n)$, by hypothesis we should prove that each $(0 :_M \mathfrak{a}^n)$ is in $\mathcal{S}_{\mathfrak{a}}$. We can proceed by induction on n. The case $n = 1$ is clear. Assume that $n > 1$ and that the result has been proved for all values smaller than n. Consider the exact sequence $0 \to \mathfrak{a}^{n-1}/\mathfrak{a}^n \to R/\mathfrak{a}^n \to R/\mathfrak{a}^{n-1} \to 0$. Using the induction hypothesis, the fact that $S_{\mathfrak{a}}$ is closed under taking submodules, and Theorem 2.18, it is enough to show that $\mathrm{Hom}_R(\mathfrak{a}^{n-1}/\mathfrak{a}^n, M)$ is in $\mathcal{S}_{\mathfrak{a}}$. As $\mathfrak{a}^{n-1}/\mathfrak{a}^n$ is an $R/\mathfrak{a}$-module, there exists a positive integer t and an exact sequence $0 \to X \to (R/\mathfrak{a})^t \to \mathfrak{a}^{n-1}/\mathfrak{a}^n \to 0$. Now the claim is obtained by applying $\mathrm{Hom}_R(-, M)$ and the fact that $\mathcal{S}$ is Serre.

(ii) Assume that M is a module with $\Gamma_{\mathfrak{a}}(M) = M$ and $(0 :_M \mathfrak{a}) \in \mathcal{S}$. Then M and $(0 :_M \mathfrak{a})$ have the same injective hull E, and so by hypothesis, $E \in \mathcal{S}_{\mathfrak{a}}$. Now the assumption implies that M is in $\mathcal{S}_{\mathfrak{a}}$, and hence in $\mathcal{S}$. ∎

Let $\phi : R \to S$ be a ring homomorphism. Let $\phi_\star : S\text{-Mod} \to R\text{-Mod}$ and $\phi^\star : R\text{-Mod} \to S\text{-Mod}$ be two functors defined as $\phi_\star(N) = N$ and $\phi^\star(M) = M \otimes_R S$ for every S-module N and R-module M. We notice that $\phi^\star$ is a left adjoint of $\phi_\star$. For any subcategory $\mathcal{S}$ of S-Mod, we set $\phi_\star(\mathcal{S}) = \{N = \phi_\star(N) \mid N \text{ is in } \mathcal{S}\}$. Clearly, if $\phi_\star(\mathcal{S})$ is a Serre subcategory of R-Mod, then $\mathcal{S}$ is a Serre subcategory of S-Mod. For any subcategory $\mathcal{T}$ of R-Mod, we set $\phi^\star(\mathcal{T}) = \{M \otimes_R S \mid M \in \mathcal{T}\}$. The next theorem shows that the condition $C_\mathfrak{a}$ can be transferred via ring homomorphisms.

Theorem 2.21. *Let $\phi : R \to S$ be a ring homomorphism, let $\mathfrak{a}$ be an ideal of R, let $\mathcal{S}$ be a subcategory of S-*Mod*, and let $\mathcal{T}$ be a subcategory of R-*mod *closed under isomorphisms. Then the following implications hold:*

(i) *$\phi_\star(\mathcal{S}_{\mathfrak{a}S}) \subseteq \phi_\star(\mathcal{S})_\mathfrak{a}$. Moreover, if $\phi_\star(\mathcal{S})$ satisfies the condition $C_\mathfrak{a}$, then $\mathcal{S}$ satisfies $C_{\mathfrak{a}S}$.*
(ii) *If ϕ is faithfully flat, then $\phi^\star(\mathcal{T}_\mathfrak{a}) \subseteq \phi^\star(\mathcal{T})_{\mathfrak{a}S}$. Moreover, if $\phi^\star(\mathcal{T})$ satisfies $C_{\mathfrak{a}S}$, then $\mathcal{T}$ satisfies $C_\mathfrak{a}$.*

Proof. (i) Assume $M \in \phi_\star(\mathcal{S}_{\mathfrak{a}S})$ with $\Gamma_\mathfrak{a}(M) = M$ and $(0 :_M \mathfrak{a}) \in \phi_\star(\mathcal{S})$. Clearly, $\Gamma_{\mathfrak{a}S}(M) = \Gamma_\mathfrak{a}(M) = M$ and $(0 :_M \mathfrak{a}) = (0 :_M \mathfrak{a}S) \in \mathcal{S}$. Now since $M \in \mathcal{S}_{\mathfrak{a}S}$, we see that M is in $\mathcal{S}$, hence in $\phi_\star(\mathcal{S})$. To prove the second claim, assume that M is an S-module with $M = \Gamma_{\mathfrak{a}S}(M)$ and $(0 :_M \mathfrak{a}S) \in \mathcal{S}$. Then $M = \Gamma_\mathfrak{a}(M)$ and $(0 :_M \mathfrak{a}S) = (0 :_M \mathfrak{a}) \in \phi_\star(\mathcal{S})$. Since $\phi_\star(\mathcal{S})$ satisfies $C_\mathfrak{a}$, we find that M is in $\phi_\star(\mathcal{S})$, hence in $\mathcal{S}$.

(ii) Assume that $M \otimes_R S \in \phi^\star(\mathcal{T}_\mathfrak{a})$ with $\Gamma_{\mathfrak{a}S}(M \otimes_R S) = M \otimes_R S$ and $(0 :_{M\otimes_R S} \mathfrak{a}S) \in \phi^\star(\mathcal{T})$. Then there exists an R-module N in $\mathcal{T}$ such that $(0 :_{M\otimes_R S} \mathfrak{a}S) = N \otimes_R S$. As S is a faithfully flat R-module, we have $\Gamma_\mathfrak{a}(M) = M$ and the isomorphism $\mathrm{Hom}_S((0 :_{M\otimes_R S} \mathfrak{a}S), N \otimes_R S) \cong \mathrm{Hom}_R((0 :_M \mathfrak{a}), N) \otimes_R S$ implies that $(0 :_M \mathfrak{a}) \cong N$. Therefore $(0 :_M \mathfrak{a}) \in \mathcal{T}$. Now since $M \in \mathcal{T}_\mathfrak{a}$, we deduce that $M \in \mathcal{T}$ so that $M \otimes_R S \in \phi^\star(\mathcal{T})$. To prove the second claim, assume that M is an R-module with $M = \Gamma_\mathfrak{a}(M)$ and $(0 :_M \mathfrak{a}) \in \mathcal{T}$. Thus

$$M \otimes_R S = \Gamma_{\mathfrak{a}S}(M \otimes_R S) \quad \text{and} \quad (0 :_{M\otimes_R S} \mathfrak{a}S) \in \phi^\star(\mathcal{T}).$$

Now, since $\phi^\star(\mathcal{T})$ satisfies $C_{\mathfrak{a}S}$, we see that $M \otimes_R S \in \phi^\star(\mathcal{T})$ and so there exists $N \in \mathcal{T}$ such that $M \otimes_R S = N \otimes_R S$. Using an analogous proof to the first part, we deduce $M \cong N$ and so $M \in \mathcal{T}$. ∎

Given a class $\mathcal{S}$ of R-modules, we define the *support* of $\mathcal{S}$ to be

$$\mathrm{Supp}_R(\mathcal{S}) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid R/\mathfrak{p} \text{ is in } \mathcal{S}\}.$$

Proposition 2.22. *Let $\mathcal{S}$ be a Serre subcategory of R-*Mod*. If M is a finitely generated R-module with $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(\mathcal{S}_\mathfrak{a})$, then $M \in \mathcal{S}_\mathfrak{a}$. In particular, if $V(\mathfrak{a}) \subseteq \mathrm{Supp}_R(\mathcal{S}_\mathfrak{a})$, then R-*mod *is a subclass of $S_\mathfrak{a}$.*

Proof. There exists a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ where $\mathfrak{p}_i \in \mathrm{Supp}_R(M)$ for $i = 1, \dots, n$. By hypothesis, each $R/\mathfrak{p}_i$ is in $\mathcal{S}_\mathfrak{a}$. Since, by Theorem 2.18, $\mathcal{S}_\mathfrak{a}$ is closed under extension of modules, M is in $\mathcal{S}_\mathfrak{a}$. In order to prove the second claim, suppose that M is a finitely generated R-module with $M = \Gamma_\mathfrak{a}(M)$ and $(0 :_M \mathfrak{a}) \in \mathcal{S}$. Since $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$, according to the first part, M is in $\mathcal{S}_\mathfrak{a}$. ∎

3. Applications to local cohomology modules. In [SR], we investigated when local cohomology modules lie in a Serre subcategory of R-modules. In this section we show that the Melkersson condition plays a key role in this material. Throughout this section $\mathcal{S}$ is a Serre subcategory of R-Mod containing a non-zero module, $\mathfrak{a}$ is an ideal of R and n is a non-negative integer.

THEOREM 3.1. *Let M be a finitely generated R-module and let $H^i_\mathfrak{a}(M)$ be in $\mathcal{S}_\mathfrak{a}$ with $\mathrm{Ass}_R(H^i_\mathfrak{a}(M)) \subseteq \mathrm{Supp}_R(\mathcal{S})$ for each $i \le n$. Then $H^i_\mathfrak{a}(M) \in \mathcal{S}$ for each $i \le n$.*

Proof. We proceed by induction on n. If $n = 0$, then $\mathrm{Ass}_R(\Gamma_\mathfrak{a}(M)) \subseteq \mathrm{Supp}_R(\mathcal{S})$. Hence $\mathrm{Supp}_R(\Gamma_\mathfrak{a}(M)) \subseteq \mathrm{Supp}_R(\mathcal{S})$, so $\Gamma_\mathfrak{a}(M)$ is in $\mathcal{S}$ by using a finite filtration of $\Gamma_\mathfrak{a}(M)$ as in the proof of Proposition 2.22. Let $n > 0$ and suppose inductively that the result has been proved for all values smaller than n and all finitely generated R-modules. As $H^i_\mathfrak{a}(M) \cong H^i_\mathfrak{a}(M/\Gamma_\mathfrak{a}(M))$ for each $i > 0$, without loss of generality we may assume that $\Gamma_\mathfrak{a}(M) = 0$. Then there exists $x \in \mathfrak{a} \setminus Z(M)$ and an exact sequence $0 \to M \xrightarrow{x\cdot} M \to M/xM \to 0$. Fix $i < n$. Applying $H^i_\mathfrak{a}(-)$ yields the exact sequence

$$H^i_\mathfrak{a}(M) \xrightarrow{x\cdot} H^i_\mathfrak{a}(M) \to H^i_\mathfrak{a}(M/xM) \to H^{i+1}_\mathfrak{a}(M) \xrightarrow{x\cdot} H^{i+1}_\mathfrak{a}(M).$$

By the induction hypothesis, $H^i_\mathfrak{a}(M)$ is in $\mathcal{S}$. Then the above exact sequence implies that $H^i_\mathfrak{a}(M)/xH^i_\mathfrak{a}(M)$ is in $\mathcal{S}$; therefore $\mathrm{Ass}_R(H^i_\mathfrak{a}(M)/xH^i_\mathfrak{a}(M)) \subseteq \mathrm{Supp}_R(\mathcal{S})$. Moreover, since

$$\mathrm{Ass}_R((0 :_{H^{i+1}_\mathfrak{a}(M)} x)) \subseteq \mathrm{Ass}_R(H^{i+1}_\mathfrak{a}(M)) \subseteq \mathrm{Supp}_R(\mathcal{S}),$$

the exact sequence $0 \to H^i_\mathfrak{a}(M)/xH^i_\mathfrak{a}(M) \to H^i_\mathfrak{a}(M/xM) \to (0 :_{H^{i+1}_\mathfrak{a}(M)} x) \to 0$ implies that $\mathrm{Ass}_R(H^i_\mathfrak{a}(M/xM)) \subseteq \mathrm{Supp}_R(\mathcal{S})$. On the other hand, since $(0 :_{(0 :_{H^{i+1}_\mathfrak{a}(M)} x)} \mathfrak{a}) = (0 :_{H^{i+1}_\mathfrak{a}(M)} \mathfrak{a})$ and $H^{i+1}_\mathfrak{a}(M) \in \mathcal{S}_\mathfrak{a}$, we deduce that $(0 :_{H^{i+1}_\mathfrak{a}(M)} x) \in \mathcal{S}_\mathfrak{a}$. Now, it follows from Theorem 2.18 that $H^i_\mathfrak{a}(M/xM) \in \mathcal{S}_\mathfrak{a}$. Thus the induction hypothesis implies that $H^i_\mathfrak{a}(M/xM) \in \mathcal{S}$ for each $i < n$, and so $(0 :_{H^i_\mathfrak{a}(M)} x) \in \mathcal{S}$ for each $i \le n$. Therefore $(0 :_{H^i_\mathfrak{a}(M)} \mathfrak{a}) \in \mathcal{S}$ for each $i \le n$, and since $\mathcal{S}$ satisfies $C_\mathfrak{a}$ on $H^i_\mathfrak{a}(M)$, the module $H^i_\mathfrak{a}(M)$ is in $\mathcal{S}$ for each $i \le n$. ∎

COROLLARY 3.2. *Let* $(R,\mathfrak{m})$ *be a local ring and let* M *be a finitely generated* R*-module. If* $H^i_{\mathfrak{m}}(M) \in \mathcal{S}_{\mathfrak{m}}$ *for each* $i \leq n$*, then* $H^i_{\mathfrak{m}}(M) \in \mathcal{S}$ *for each* $i \leq n$.

Proof. The result follows immediately by the previous theorem. ∎

THEOREM 3.3. *Let* $(R,\mathfrak{m})$ *be a local ring, let* M *be a finitely generated* R*-module and assume that* $\mathcal{S}$ *satisfies* $C_{\mathfrak{m}}$*. If* $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ *for all* $i < n$*, then* $\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M)) \in \mathcal{S}$.

Proof. The case $n = 0$ is clear, and so we assume that $n > 0$. Since $H^n_{\mathfrak{a}}(M) \cong H^n_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$, we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$ so that the ideal $\mathfrak{a}$ contains a non-zerodivisor x on M. We proceed by induction on n. If $n = 1$, then $\Gamma_{\mathfrak{m}}(\Gamma_{\mathfrak{a}}(M/xM)) = \Gamma_{\mathfrak{m}}(M/xM)$ is of finite length, and hence it lies in $\mathcal{S}$. Thus $\Gamma_{\mathfrak{m}}(M/xM) \cong (0 :_{\Gamma_{\mathfrak{m}}(H^1_{\mathfrak{a}}(M))} x) \in \mathcal{S}$ so that $(0 :_{\Gamma_{\mathfrak{m}}(H^1_{\mathfrak{a}}(M))} \mathfrak{m}) \in \mathcal{S}$. Now, since $\mathcal{S}$ satisfies $C_{\mathfrak{m}}$, we see that $\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M)) \in \mathcal{S}$. Let $n > 1$ and suppose that the result has been proved for all values smaller than n. Clearly $H^i_{\mathfrak{a}}(M/xM) \in \mathcal{S}$ for all $i < n-1$. For the convenience of the reader, we write $A = H^{n-1}_{\mathfrak{a}}(M)/xH^{n-1}_{\mathfrak{a}}(M)$ and $B = (0 :_{H^n_{\mathfrak{a}}(M)} x)$. Thus, the exact sequence

$$H^{n-1}_{\mathfrak{a}}(M) \xrightarrow{x} H^{n-1}_{\mathfrak{a}}(M) \to H^{n-1}_{\mathfrak{a}}(M/xM) \to H^n_{\mathfrak{a}}(M) \xrightarrow{x} H^n_{\mathfrak{a}}(M)$$

induces the exact sequence

$$0 \to A \to H^{n-1}_{\mathfrak{a}}(M/xM) \to B \to 0.$$

Since $H^{n-1}_{\mathfrak{a}}(M) \in \mathcal{S}$, the module A is in $\mathcal{S}$, and hence [AM, Theorem 2.9] shows that $H^i_{\mathfrak{m}}(A) \in \mathcal{S}$ for each i. We note that the induction hypothesis implies that $\Gamma_{\mathfrak{m}}(H^{n-1}_{\mathfrak{a}}(M/xM)) \in \mathcal{S}$. Now applying $\Gamma_{\mathfrak{m}}(-)$ gives the exact sequence

$$\Gamma_{\mathfrak{m}}(H^{n-1}_{\mathfrak{a}}(M/xM)) \to \Gamma_{\mathfrak{m}}(B) \to H^1_{\mathfrak{m}}(A),$$

which forces that $\Gamma_{\mathfrak{m}}(B) \in \mathcal{S}$. Therefore $\Gamma_{\mathfrak{m}}(B) = \Gamma_{\mathfrak{m}}((0 :_{H^n_{\mathfrak{a}}(M)} x)) = (0 :_{\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))} x) \in \mathcal{S}$, which in turn implies that $(0 :_{\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))} \mathfrak{m}) \in \mathcal{S}$. Consequently, since $\mathcal{S}$ satisfies $C_{\mathfrak{m}}$, we deduce that $H^n_{\mathfrak{m}}(M) \in \mathcal{S}$. ∎

PROPOSITION 3.4. *Let* $(R,\mathfrak{m})$ *be a local ring, and let* M *be a finitely generated* R*-module such that* $H^i_{\mathfrak{a}}(M)$ *is minimax for each* $i < n$*. If* $\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))$ *is in* $\mathcal{S}_{\mathfrak{a}}$*, then it is in* $\mathcal{S}$.

Proof. According to [BN, Theorem 2.3], the R-module $(0 :_{H^n_{\mathfrak{a}}(M)} \mathfrak{a})$ is finitely generated so that $\Gamma_{\mathfrak{m}}((0 :_{H^n_{\mathfrak{a}}(M)} \mathfrak{a})) = (0 :_{\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))} \mathfrak{a})$ has finite length. Then $(0 :_{\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))} \mathfrak{a}) \in \mathcal{S}$, and since $\Gamma_{\mathfrak{m}}(H^n_{\mathfrak{a}}(M))$ is in $\mathcal{S}_{\mathfrak{a}}$, it is in $\mathcal{S}$. ∎

PROPOSITION 3.5. *Let* $(R,\mathfrak{m})$ *be a local ring and let* M *be a finitely generated* R*-module of dimension* n*. If* $\mathcal{S}$ *satisfies* $C_{\mathfrak{a}}$*, then* $H^n_{\mathfrak{a}}(M) \in \mathcal{S}$.

Proof. The proof is similar to that of Theorem 3.3. ∎

Acknowledgments. The authors would like to thank the referee for her/his careful reading and many helpful comments and suggestions.

REFERENCES

[AM] M. Aghapournahr and L. Melkersson, *Local cohomology and Serre subcategories*, J. Algebra 320 (2008), 1275–1287.

[BN] K. Bahmanpour and R. Naghipour, *On the cofiniteness of local cohomology modules*, Proc. Amer. Math. Soc. 136 (2008), 2359–2363.

[BFT] M. Brodmann, S. Fumasoli and R. Tajarod, *Local cohomology over homogeneous rings with one-dimensional local base ring*, Proc. Amer. Math. Soc. 131 (2003), 2977–2985.

[BRS] M. Brodmann, F. Rohrer and R. Sazeedeh, *Multiplicities of graded components of local cohomology modules,* J. Pure Appl. Algebra 197 (2005), 249–278.

[BS] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge Stud. Adv. Math. 60, Cambridge Univ. Press, 1998.

[K] R. Kanda, *Classifying Serre subcategories via atom spectrum*, Adv. Math. 231 (2012), 1572–1588.

[M] L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal*, Math. Proc. Cambridge Philos. Soc. 107 (1990), 267–271.

[S] R. Sazeedeh, *Artinianess of graded local cohomology modules*, Proc. Amer. Math. Soc. 135 (2007), 2339–2345.

[SR] R. Sazeedeh and R. Rasuli, *Some results in local cohomology and Serre subcategory*, Rom. J. Math. Comput. Sci. 3 (2013), 185–190.

[St] B. Stenström, *Rings of Quotients*, Grundlehren Math. Wiss. 217, Springer, 1975.

[Y] T. Yoshizawa, *An example of Melkersson subcategory which is not closed under injective hulls*, arXiv:1011.1663v2 (2010).

Reza Sazeedeh
Department of Mathematics
Urmia University
P.O. Box 165, Urmia, Iran
and
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran
E-mail: rsazeedeh@ipm.ir

Rasul Rasuli
Mathematics Department
Faculty of Science
Payame Noor University (PNU)
Tehran, Iran
E-mail: rasulirasul@yahoo.com