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A SHARP BOUND FOR A SINE POLYNOMIAL

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Abstract. We prove that

$$\Big|\sum_{k=1}^{n}\frac{\sin((2k-1)x)}{k}\Big| < \mathrm{Si}(\pi) = 1.8519\ldots$$

for all integers $n \geq 1$ and real numbers x. The upper bound $\mathrm{Si}(\pi)$ is best possible. This result refines inequalities due to Fejér (1910) and Lenz (1951).

1. Introduction. In 1905, Kneser [7] established that there exists a constant M such that

$$\Big|\sum_{k=1}^{n}\frac{\sin(kx)}{k}\Big| < M \quad (x\in\mathbb{R};\ n=1,2,\ldots), \tag{1.1}$$

but he did not provide a numerical value for M. Fejér [4] showed that (1.1) holds with $M = 3.6$. A classical conjecture of Fejér states that the best possible value for M (which does not depend on x and n) is $\mathrm{Si}(\pi) = 1.8519\ldots$, where

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt \quad (x>0)$$

denotes the sine integral. This was proved in 1911 by Jackson [6] and in 1912 by Gronwall [5], who used (1.1) with $M = \mathrm{Si}(\pi)$ to study the well-known Gibbs phenomenon.

Fejér [4] also presented an estimate for the closely related sine polynomial $\sum_{k=1}^{n}\sin((2k-1)x)/k$. If M denotes the bound in (1.1), then

$$\Big|\sum_{k=1}^{n}\frac{\sin((2k-1)x)}{k}\Big| < 2+3M \quad (x\in\mathbb{R};\ n=1,2,\ldots). \tag{1.2}$$

Moreover, he applied (1.2) to get an upper bound for $|\theta(n,r,x)|$, where

$$\theta(n,r,x) = \sum_{k=1}^{n}\left(\frac{\cos((r+k)x)}{n+1-k} - \frac{\cos((r+n+k)x)}{k}\right) \quad (r,x\in\mathbb{R};\ n\in\mathbb{N}).$$

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Furthermore, Fejér used θ to construct an example of a function which is continuous on $[0, 2\pi]$ and has a Fourier series which is divergent at $x = 0$.

In 1951, Lenz [8] provided a refinement of (1.2):

$$\left|\sum_{k=1}^{n} \frac{\sin((2k-1)x)}{k}\right| \le \frac{1}{\sin(1/2)} = 2.0858\ldots \qquad (x \in \mathbb{R};\ n = 1, 2, \ldots). \tag{1.3}$$

It is natural to ask for a sharp bound for the sum given in (1.2) and (1.3). In this paper we solve the problem. It is our aim to prove that the best possible upper bound is equal to $\mathrm{Si}(\pi)$.

A collection of the most important inequalities for trigonometric sums and polynomials with interesting historical comments and a detailed list of references can be found in the monograph [9, Chapters 2.2 and 4]. New results and a compilation of recently published papers on this subject are given in [1] and [2].

2. Lemmas. In this section we present several lemmas which we need to prove our main result.

LEMMA 1. *The function*

$$f(x) = \frac{x\cot(x)}{\sin(x)} \tag{2.1}$$

is strictly decreasing on $(0, \pi/2)$ *and satisfies* $0 < f(x) < 1/x$ *for* $0 < x < \pi/2$.

Proof. Since $x \mapsto x\cot(x)$ and $x \mapsto 1/\sin(x)$ are strictly decreasing and positive on $(0, \pi/2)$, we conclude that f is also strictly decreasing on $(0, \pi/2)$. Let $x \in (0, \pi/2)$ and

$$g(x) = \frac{\sin^2(x)}{x^2} - \cos(x).$$

Differentiation gives

$$\frac{x^3}{2\sin(x)}\, g'(x) = x\cos(x) - \sin(x) + \frac{1}{2}x^3 = h(x), \qquad \text{say.}$$

We have $h(0) = 0$ and $h'(x) = x[3x/2 - \sin(x)] > 0$. This leads to $g'(x) > 0$ and $g(x) > g(0) = 0$, which is equivalent to $f(x) < 1/x$. ∎

LEMMA 2. *The function*

$$\Delta(x) = \log\left(\frac{4x+2}{3\pi}\left(\sin\frac{3\pi}{4x+2}\right)\right) + \log\left(\frac{x}{2x+1}\right) + \frac{1}{2}\log\left(1 + \frac{1}{x}\right) \tag{2.2}$$

is strictly increasing on $[1, \infty)$.

Proof. Let $x > 1$. Differentiation yields

$$2x(x+1)(2x+1)^2\Delta'(x) = \cot\left(\frac{3\pi}{4x+2}\right)\sigma(x),$$

where

$$\sigma(x) = [8x^3+12x^2+6x+1]\tan\left(\frac{3\pi}{4x+2}\right) - 6\pi x(x+1).$$

Since $\tan(y) > y$ for $0 < y < \pi/2$, we get

$$\sigma(x) > [8x^3+12x^2+6x+1]\,\frac{3\pi}{4x+2} - 6\pi x(x+1) = \frac{3}{2}\,\pi.$$

Hence, $\Delta'(x) > 0$. ∎

LEMMA 3. *Let* $T_n(y) = \sum_{k=1}^n \cos(ky)/k$.

(i) *If* $n \geq 2$ *and* $0 \leq y \leq \pi$, *then* $T_n(y) \geq -5/6$.
(ii) *If* $n \geq 6$ *and* $0 < y \leq 3\pi/(2n+1)$, *then* $T_n(y) > 0$.
(iii) *If* $n \geq 8$ *and* $3\pi/(2n+1) < y < 2\pi/n$, *then* $T_n(y) > 0$.

Proof. Part (i) is proved in [3]. Let $n \geq 6$ and $y \in (0, 3\pi/(2n+1)]$. Then

$$T_n'(y) = -\sum_{k=1}^n \sin(ky) = \frac{1}{2\sin(y/2)}\,[\cos((n+1/2)y) - \cos(y/2)] < 0.$$

This implies

$$T_n(y) \geq T_n(3\pi/(2n+1)). \tag{2.3}$$

In [3] it is shown that

$$T_n\left(\frac{3\pi}{2n+1}\right) > \sum_{k=1}^n \frac{1}{k} - \log\left(\frac{3\pi}{2}\right) + \mathrm{Ci}\left(\frac{3\pi}{2}\right) - \gamma, \tag{2.4}$$

where $\gamma = 0.57721\ldots$ is Euler's constant and

$$\mathrm{Ci}(x) = \gamma + \log x - \int_0^x \frac{1-\cos(t)}{t}\,dt$$

denotes the cosine integral. From (2.3) and (2.4) we obtain

$$T_n(y) > \sum_{k=1}^6 \frac{1}{k} - \log\left(\frac{3\pi}{2}\right) + \mathrm{Ci}\left(\frac{3\pi}{2}\right) - \gamma = 0.124\ldots$$

Let $n \geq 8$ and $y \in (3\pi/(2n+1), 2\pi/n)$. Further, let Δ be the function defined in (2.2). Applying

$$T_n(y) \geq \mathrm{Ci}\left(\frac{3\pi}{2}\right) + \Delta(n) - \log\left(\sin\left(\frac{y}{2}\right)\right) + \int_{3\pi/(2n+1)}^{y} \frac{\cos((n+1/2)t)}{2\sin(t/2)}\,dt,$$

which is proved in [3], and Lemma 2 we get

$$T_n(y) \geq \mathrm{Ci}\left(\frac{3\pi}{2}\right) + \Delta(8) - \log(\sin(\pi/8)) = 0.054\ldots \quad \blacksquare$$

LEMMA 4. *Let*

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{k}. \tag{2.5}$$

If $n \in \{5,6,7\}$ *and* $x \in (0, \pi/n)$, *then* $S_n(x) < \mathrm{Si}(\pi)$.

Proof. Let

$$p(a) = \tfrac{5}{2}\, a - 2a^3 \tag{2.6}$$

and $c = \sqrt{5/12} = 0.645\ldots$ The function p is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$ with $p(c) = 1.075\ldots$

Let $x \in (0, \pi/5)$. Then $p(\sin(x)) < p(\sin(\pi/5)) = 1.063\ldots$ Since $S_2(x) = p(\sin(x))$, we get

$$S_5(x) < 1.064 + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} = 1.847\ldots < \mathrm{Si}(\pi).$$

Let $x \in (0, 3\pi/26]$. Applying Lemma 3(ii) (with $n = 6$), inequality (1.1) (with $M = \mathrm{Si}(\pi)$), and

$$S_n(x) = \cos(x) \sum_{k=1}^{n} \frac{\sin(2kx)}{k} - \sin(x) \sum_{k=1}^{n} \frac{\cos(2kx)}{k}, \tag{2.7}$$

we obtain

$$S_6(x) < \cos(x) \sum_{k=1}^{6} \frac{\sin(2kx)}{k} < \mathrm{Si}(\pi).$$

If $x \in [3\pi/26, \pi/6)$, then

$$\begin{aligned} S_6(x) &< p(\sin(\pi/6)) + \tfrac{1}{3}\sin(15\pi/26) + \tfrac{1}{4}\sin(21\pi/26) + \tfrac{1}{5} + \tfrac{1}{6} \\ &= 1.832\ldots < \mathrm{Si}(\pi). \end{aligned}$$

Let $x \in (0, \pi/10]$. Lemma 3(ii) (with $n = 7$) implies that $\sum_{k=1}^{7} \cos(2kx)/k > 0$, so that (1.1) (with $M = \mathrm{Si}(\pi)$) and (2.7) yield $S_7(x) < \mathrm{Si}(\pi)$.

Finally, if $x \in [\pi/10, \pi/7)$, then

$$\begin{aligned} S_7(x) &< p(\sin(\pi/7)) + \tfrac{1}{3} + \tfrac{1}{4}\sin(7\pi/10) + \tfrac{1}{5}\sin(9\pi/10) + \tfrac{1}{6} + \tfrac{1}{7} \\ &= 1.828\ldots < \mathrm{Si}(\pi). \quad \blacksquare \end{aligned}$$

LEMMA 5. *Let*

$$D_n(t) = \frac{\sin((n+1/2)t)}{2\sin(t/2)}. \tag{2.8}$$

If $n \geq 5$ *and* $y \in [2\pi/n, \pi]$, *then*

$$\int_0^y D_n(t)\,dt \leq \int_0^{3\pi/(n+1/2)} D_n(t)\,dt < \mathrm{Si}(\pi). \tag{2.9}$$

Proof. Let $\delta = \pi/(n+1/2)$ and

$$F_n(y) = \int_0^y D_n(t)\,dt \qquad (2\pi/n \leq y \leq \pi).$$

If $n = 2N$ or $n = 2N+1$, then

$$[2\pi/n, \pi] \subset \bigcup_{j=1}^{N} [2j\delta, (2j+1)\delta] \cup \bigcup_{j=1}^{N} [(2j+1)\delta, (2j+2)\delta].$$

Since F_n is increasing on $[2j\delta, (2j+1)\delta]$ $(j = 1, \ldots, N)$ and decreasing on $[(2j+1)\delta, (2j+2)\delta]$ $(j = 1, \ldots, N)$, we obtain

$$F_n(y) \leq \max\{F_n(3\delta), F_n(5\delta), \ldots, F_n((2N+1)\delta)\} \quad \text{for } 2\pi/n \leq y \leq \pi. \tag{2.10}$$

Let $n = 2N$ or $n = 2N+1$. We show: if $j \in \{1, \ldots, N-1\}$ and $t \in ((2j+1)\delta, (2j+2)\delta)$, then

$$0 \leq \frac{1}{\sin(t/2)} - \frac{1}{\sin((t+\delta)/2)}. \tag{2.11}$$

First, we assume that $n = 2N$ and $j = N-1$. We define

$$u(t) = \sin\left(\frac{t+\delta}{2}\right) - \sin\left(\frac{t}{2}\right).$$

Since $0 < t/2 < (t+\delta)/2 < \pi$, we get

$$2u'(t) = \cos\left(\frac{t+\delta}{2}\right) - \cos\left(\frac{t}{2}\right) < 0, \qquad u(t) > u((2j+2)\delta) = 0.$$

This leads to (2.11). If $n = 2N$ and $j \in \{1, \ldots, N-2\}$, or if $n = 2N+1$ and $j \in \{1, \ldots, N-1\}$, then $0 < t/2 < (t+\delta)/2 < \pi/2$, which also leads to (2.11). Hence, we get

$$F_n((2j+3)\delta) - F_n((2j+1)\delta) = \int_{(2j+1)\delta}^{(2j+3)\delta} D_n(t)\,dt \tag{2.12}$$

$$= \frac{1}{2} \int_{(2j+1)\delta}^{(2j+2)\delta} \sin(\pi t/\delta) \left[\frac{1}{\sin(t/2)} - \frac{1}{\sin((t+\delta)/2)}\right] dt \leq 0 \qquad \text{for } j = 1, \ldots, N-1.$$

From (2.10) and (2.12) we conclude that the first inequality of (2.9) holds.

We have

$$\int_0^{3\delta} D_n(t)\,dt \tag{2.13}$$

$$= \int_0^{\delta} \frac{\sin(\pi t/\delta)}{2\sin(t/2)}\,dt + \int_0^{\delta} \frac{\sin(\pi(t+\delta)/\delta)}{2\sin((t+\delta)/2)}\,dt + \int_0^{\delta} \frac{\sin(\pi(t+2\delta)/\delta)}{2\sin(t/2+\delta)}\,dt$$

$$= \frac{\delta}{2\pi}\int_0^{\pi} \sin(t)\left[\frac{1}{\sin(\delta t/(2\pi))} - \frac{1}{\sin(\delta t/(2\pi)+\delta/2)} + \frac{1}{\sin(\delta t/(2\pi)+\delta)}\right] dt.$$

Let $0 < x < \delta/2$ and

$$H(x) = \frac{1}{x} - \frac{1}{\sin(x)} + \frac{1}{\sin(x+\delta/2)} - \frac{1}{\sin(x+\delta)}.$$

Differentiation gives

$$H'(x) = \frac{1}{x}\left(f(x) - \frac{1}{x}\right) + \frac{1}{x+\delta} f(x+\delta) - \frac{1}{x+\delta/2} f(x+\delta/2),$$

where f is defined in (2.1). Applying Lemma 1 we see that H' is negative on $(0,\delta/2)$, which implies

$$H(x) > \frac{1}{\delta/2} - \frac{1}{\sin(\delta/2)} + \frac{1}{\sin(\delta)} - \frac{1}{\sin(3\delta/2)} = Q(\delta/2), \quad \text{say.}$$

Since

$$xQ'(x) = f(x) - 1/x + f(3x) - f(2x) < 0 \quad \text{for } 0 < x < \pi/6,$$

we conclude from $0 < \delta/2 = \pi/(2n+1) \le \pi/11$ that

$$Q(\delta/2) \ge Q(\pi/11) = 0.478\ldots$$

Hence, $H(x) > 0$ for $x \in (0,\delta/2)$. Using (2.13) we get

$$\mathrm{Si}(\pi) - \int_0^{3\pi/(n+1/2)} D_n(t)\,dt = \frac{\delta}{2\pi}\int_0^{\pi} \sin(t) H(\delta t/(2\pi))\,dt > 0.$$

This proves the right-hand side of (2.9). ∎

3. Main result. We are now in a position to establish the following sharpening of (1.2) and (1.3).

THEOREM. *For all integers $n \ge 1$ and real numbers x we have*

$$\left|\sum_{k=1}^{n} \frac{\sin((2k-1)x)}{k}\right| < \mathrm{Si}(\pi) = 1.8519\ldots \tag{3.1}$$

The upper bound is best possible.

Proof. Let $S_n(x)$ be defined in (2.5). Then for $x \in \mathbb{R}$ we have

$$|S_n(x+\pi)| = |S_n(x)|, \qquad S_n((\pi/2)-x) = S_n((\pi/2)+x).$$

The identity

$$\sin(x)S_n(x) = \sum_{k=1}^{n} \frac{\sin^2(kx)}{k(k+1)} + \frac{\sin^2(nx)}{n+1}$$

reveals that $S_n(x) \geq 0$ for $x \in [0,\pi]$. This implies that in order to prove (3.1) it suffices to show that

$$S_n(x) < \mathrm{Si}(\pi) \qquad (0 < x \leq \pi/2;\ n = 1,2,\ldots). \tag{3.2}$$

Obviously, (3.2) is valid for $n = 1$. Let p be the function given in (2.6). Then for $x \in (0,\pi/2]$ we get

$$S_2(x) = p(\sin(x)) < 1.08 < \mathrm{Si}(\pi),$$
$$S_3(x) = S_2(x) + \tfrac{1}{3}\sin(5x) \leq 1.08 + \tfrac{1}{3} = 1.413\ldots < \mathrm{Si}(\pi),$$
$$S_4(x) = S_2(x) + \tfrac{1}{3}\sin(5x) + \tfrac{1}{4}\sin(7x) < 1.08 + \tfrac{1}{3} + \tfrac{1}{4} = 1.663\ldots < \mathrm{Si}(\pi).$$

It remains to prove (3.2) for $n \geq 5$. We consider two cases.

Case 1: $0 < x < \pi/n$. Lemma 4 implies that $S_n(x) < \mathrm{Si}(\pi)$ for $x \in (0,\pi/n)$ and $n = 5,6,7$. Let $n \geq 8$. Then we conclude from (2.7), Lemma 3(ii), (iii), and (1.1) (with $M = \mathrm{Si}(\pi)$) that

$$S_n(x) < \cos(x) \sum_{k=1}^{n} \frac{\sin(2kx)}{k} < \mathrm{Si}(\pi).$$

Case 2: $\pi/n \leq x \leq \pi/2$. Let $D_n(t)$ be defined in (2.8). Applying (2.7), Lemma 3(i), and Lemma 5 we obtain

$$\begin{aligned} S_n(x) &\leq \cos(x) \sum_{k=1}^{n} \frac{\sin(2kx)}{k} + \frac{5}{6}\sin(x) \\ &= \cos(x)\Big(\int_0^{2x} D_n(t)\,dt - x\Big) + \frac{5}{6}\sin(x) \\ &\leq -x\cos(x) + \frac{5}{6}\sin(x) + \cos(x) \int_0^{3\pi/(n+1/2)} D_n(t)\,dt \\ &\leq -x\cos(x) + \frac{5}{6}\sin(x) + \cos(x)\mathrm{Si}(\pi) = \omega(x), \qquad \text{say.} \end{aligned} \tag{3.3}$$

Since $\omega'(x) = \sin(x)[x - \mathrm{Si}(\pi)] - \cos(x)/6 < 0$ for $x \in (0,\pi/2]$, we get

$$\omega(x) < \omega(0) = \mathrm{Si}(\pi) \qquad (0 < x \leq \pi/2), \tag{3.4}$$

so that (3.3) and (3.4) lead to $S_n(x) < \mathrm{Si}(\pi)$ for $x \in [\pi/n,\pi/2]$. This completes the proof of (3.1).

It remains to show that the upper bound $\mathrm{Si}(\pi)$ is sharp. We have

$$S_n(\pi/(2n)) = \cos(\pi/(2n)) \cdot \frac{1}{n}\sum_{k=1}^{n} \frac{\sin(k\pi/n)}{k/n} - \frac{\sin(\pi/(2n))}{\pi/(2n)} \cdot \frac{\pi}{2} \cdot \frac{1}{n}\sum_{k=1}^{n} \frac{\cos(k\pi/n)}{k}$$

and

$$\lim_{n\to\infty} S_n(\pi/(2n)) = 1 \cdot \int_0^1 \frac{\sin(\pi t)}{t}\,dt - 1 \cdot \frac{\pi}{2} \cdot 0 = \mathrm{Si}(\pi). \blacksquare$$

An application of (3.1) yields the following result.

COROLLARY. *For all integers $n \ge 1$ and real numbers x we have*

$$\left|\sum_{k=1}^{n} \frac{\sin((k-1)x) + \sin(kx)}{k}\right| \le 2\mathrm{Si}(\pi)\cos(x/2), \tag{3.5}$$

with equality holding if and only if $x = (2N+1)\pi$ $(N \in \mathbb{Z})$. The constant factor $2\mathrm{Si}(\pi)$ is best possible.

Proof. From (3.1) we obtain

$$\left|\sum_{k=1}^{n} \frac{\sin((k-1)x) + \sin(kx)}{k}\right| = \left|2\cos(x/2)\sum_{k=1}^{n} \frac{\sin((2k-1)x/2)}{k}\right| \le 2\mathrm{Si}(\pi)|\cos(x/2)|,$$

with equality holding if and only if $\cos(x/2) = 0$, that is, $x = (2N+1)\pi$ $(N \in \mathbb{Z})$. Furthermore, for $n \ge 2$ we get

$$\frac{1}{\cos(\pi/(2n))}\sum_{k=1}^{n} \frac{\sin((k-1)\pi/n) + \sin(k\pi/n)}{k} = 2S_n(\pi/(2n)).$$

Since $\lim_{n\to\infty} 2S_n(\pi/(2n)) = 2\mathrm{Si}(\pi)$, the factor $2\mathrm{Si}(\pi)$ in (3.5) is sharp. $\blacksquare$

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