COLLOQUIUM MATHEMATICUM

VOL. 99 2004 NO. 2

HAUSDORFF DIMENSION OF A FRACTAL INTERPOLATION FUNCTION

BY

GUANTIE DENG (Beijing)

Abstract. We obtain a lower bound for the Hausdorff dimension of the graph of a fractal interpolation function with interpolation points $\{(i/N, y_i) : i = 0, 1, \ldots, N\}$.

1. Introduction. Let $N > 1$ be an integer, $c \in (0,1)$, and $\{y_0, y_1, \ldots, y_N\}$ be real numbers, where $y_0 = y_N = 0$. Let $S_i : \mathbb{R}^2 \to \mathbb{R}^2$ ($i \in M = \{0, 1, \ldots, N-1\}$) denote the iterated function system (briefly, IFS) defined by

$$S_i \vec{x} = \vec{x} T_i + \vec{y}_i, \qquad \vec{x} \in \mathbb{R}^2,$$

which are two-dimensional affine transformations, where

$$\vec{y}_i = (i/N, y_i), \quad 0 \le i \le N, \qquad T_i = \begin{pmatrix} 1/N & a_i \\ 0 & c \end{pmatrix}, \quad i \in M,$$

and the a_i are determined by the conditions

$$S_i \vec{y}_0 = \vec{y}_i, \qquad S_i \vec{y}_N = \vec{y}_{i+1}, \qquad i \in M. \tag{1}$$

yielding $a_i = y_{i+1} - y_i$.

Barnsley ([1]) has showed that if $Nc > 1$, then there is a unique nonempty compact set $G \subset [0,1] \times \mathbb{R}$ such that

$$G = \bigcup_{i=0}^{N-1} S_i(G) \tag{2}$$

and G is the graph of a continuous function $f : [0,1] \to \mathbb{R}$ which interpolates the data $\{(i/N, y_i) : i = 0, 1, \ldots, N\}$. That is,

$$G = G(f, [0,1]) = \{(x, f(x)) : x \in [0,1]\}, \tag{3}$$

where

$$f(i/N) = y_i \quad \text{for } i = 0, 1, \ldots, N. \tag{4}$$

2000 *Mathematics Subject Classification*: 28A78, 28A80.

Key words and phrases: fractal, Hausdorff dimension, interpolation.

Supported by the National Natural Science Foundation of China No. 10371011 and 10071005.

The set G is called the *attractor* for the IFS $\{S_i : i \in M\}$, and the function f is called a *fractal interpolation function* (briefly, FIF) corresponding to the data $\{(i/N, y_i) : 0 \le i \le N\}$. The function f can be constructed by the following method: Let $f_0(x) \equiv 0$, and define $f_n : [0,1] \to \mathbb{R}$ by induction as follows:

$$\left(\frac{i+x}{N}, f_{n+1}\left(\frac{i+x}{N}\right)\right) = S_i(x, f_n(x)), \quad i \in M,\ x \in [0,1]. \tag{5}$$

Then f_n is continuous and piecewise linear on $[0,1]$, since

$$\begin{aligned} S_{i_1} \dots S_{i_n}(x,y) &= (x,y)T_{i_n} \dots T_{i_1} + \vec{y}_{i_n} T_{i_{n-1}} \dots T_{i_1} + \dots + \vec{y}_{i_2} T_{i_1} + \vec{y}_{i_1} \\ &= ((0.i_1 \dots i_n)_N + N^{-n}x, d(i_1,\dots,i_n)x + c^n y + e(i_1,\dots,i_n)), \\ &\qquad\qquad\qquad\qquad\qquad i_1,\dots,i_n \in M, \end{aligned}$$

where

$$(0.i_1 i_2 \dots i_n)_N = \frac{i_1}{N} + \frac{i_2}{N^2} + \dots + \frac{i_n}{N^n}$$

is an N-adic rational number, and

$$\begin{aligned} d(i_1,\dots,i_n) &= c^{n-1}a_{i_n} + N^{-1}d(i_1,\dots,i_{n-1}) \\ &= c^{n-1}a_{i_n} + N^{-1}c^{n-2}a_{i_{n-1}} + \dots + N^{-(n-1)}a_{i_1}, \\ e(i_1,\dots,i_n) &= c^{n-1}y_{i_n} + N^{-1}i_n d(i_1,\dots,i_{n-1}) + e(i_1,\dots,i_{n-1}), \\ e(i_1) &= y_{i_1}. \end{aligned}$$

Moreover,

$$\begin{aligned} f_n(i/N) &= y_i; \\ f_n((0.i_1\dots i_n)_N + N^{-n}x) &= d(i_1,\dots,i_n)x + e(i_1,\dots,i_n), \\ &\qquad\qquad x \in [0,1],\ i_1,\dots,i_n \in M; \\ f_n((0.i_1\dots i_n)_N + N^{-n}x) &- f_{n-1}((0.i_1\dots i_n)_N + N^{-n}x) \\ &= c^{n-1}(a_{i_n}x + y_{i_n}), \quad x \in [0,1],\ i_1,\dots,i_n \in M, \end{aligned}$$

and

$$|f_n(x) - f_{n-1}(x)| \le A_1 c^{n-1}, \tag{6}$$

where $A_1 = \max\{|y_0|, |y_1|, \dots, |y_N|\}$. Let

$$f(x) = \sum_{n=1}^{\infty} (f_n(x) - f_{n-1}(x)).$$

Then $\{f_n\}$ converges uniformly to f, so f is continuous on $[0,1]$; moreover, f satisfies (4) and

$$|f(x) - f_n(x)| \le A_1 c^n (1-c)^{-1}, \tag{7}$$

$$S_i(x, f(x)) = \left(\frac{x+i}{N}, f\left(\frac{x+i}{N}\right)\right), \tag{8}$$

so (2) and (3) hold. The function f is therefore a FIF corresponding to the data $\{(i/N, y_i) : 0 \le x \le N\}$.

For $G = G(f, [0,1])$, Barnsley has proved that if $Nc > 1$, the box dimension of G is $2 + (\log c)(\log N)^{-1}$. Since the box dimension is not less than the Hausdorff dimension $\dim_{\mathrm{H}} G$ ([2]), it is important to give a lower bound on the Hausdorff dimension. In this paper, a lower bound of the Hausdorff dimension of the graph of the FIF above is obtained. In particular, a lower bound on the Hausdorff dimension of the graph of the Weierstrass function is given.

THEOREM. *Let real numbers* $y_0, y_1, \ldots, y_N, c$ *and* N *satisfy*

$$y_0 = y_N = 0, \qquad 0 < c < 1, \qquad 0 < \beta \le \alpha, \qquad Nc > 1 + \frac{\alpha}{\beta}, \tag{9}$$

where

$$\begin{aligned} \alpha &= \max\{|y_{i+1} - y_i| : 0 \le i \le N-1\}, \\ \beta &= \min\{|y_{i+1} - y_i| : 0 \le i \le N-1\}, \\ 2\gamma &= \max\{|y_{i+1} + y_i - y_{j+1} - y_j| : i, j \in M\}, \\ \delta &= \frac{1}{Nc}\left(\beta - \frac{\alpha}{cN-1}\right). \end{aligned}$$

Then

$$2 + \frac{\log c}{\log N} \ge \dim_{\mathrm{H}} G \ge 1 + \left(1 + \frac{\log c}{\log N}\right)\frac{\log(mN^{-1})}{\log c} \tag{10}$$

for the graph $G = G(f, [0,1])$ *of the fractal interpolation function* f *such that* (2)–(4) *hold, where* $m = \min\{N, N_1\}$, *and* N_1 *is the minimal positive integer not less than* $1 + (\alpha + \gamma)\delta^{-1}$.

REMARK 1. Since the Hausdorff dimension of the graph of a continuous function is not less than 1, we can suppose, in the proof of the Theorem, that $m = N_1 < N$.

REMARK 2. If N is an even integer, $0<c<1$, $Nc>2$, $y_i = 2^{-1}(1-(-1)^i)$ $(0 \le i \le N)$, then

$$\alpha = \beta = 1, \qquad \gamma = 0, \qquad \delta = \frac{cN-2}{cN(cN-1)}$$

and N_1 is the minimal positive integer not less than $1 + \delta^{-1} = \frac{(cN)^2-2}{cN-1}$. The Weierstrass function

$$f(x) = \sum_{n=0}^{\infty} c^n g(2^{-1}N^{n+1}x) \tag{11}$$

satisfies (4) and (8), so (2) and (3) hold, where g is a continuous periodic function with period 1 and $g(x) = 1 - |2x-1|$ on $[0,1]$. So (11) is a FIF corresponding to the data $\{i/N, (1-(-1)^i)/2) : 0 \le i \le N\}$, and we obtain the following corollary:

COROLLARY. *The inequalities* (10) *hold for the graph* $G = G(f,[0,1])$ *of the Weierstrass function* f *defined by* (11).

REMARK 3. In the Corollary, if $c = N^{s-2}$ and $2 > s > 1 + \log 4/\log N$, then

$$s \ge \dim_{\mathrm{H}} G \ge s - \frac{s-1}{2-s}\,\frac{4}{N^{s-1}\log N},$$

which is an improvement of a result in [2].

2. Proof of the Theorem and Corollary. To prove the Theorem, we first give some notations: Let $E \subset \mathbb{R}$ be a Lebesgue measurable set, $|E|$ its Lebesgue measure, and let $G(f,E)$ denote the graph of f on E. Let $I = [a,b]$ be an interval with mid-point $O(I) = 2^{-1}(a+b)$. Let

$$I(i_1,\dots,i_n) = [(0.i_1\dots i_n)_N, (0.i_1\dots i_{n-1}(1+i_n))_N]$$

denote an N-adic rational interval, where $i_1,\dots,i_n \in M$. Let $\operatorname{sign} x$ denote the sign of x, $[x]$ the integer part of x, and ${}^{\#}F$ the cardinality of the set F.

Proof of the Theorem. First, we can suppose that $m < N$ (see Remark 1). For any $i_1,\dots,i_n \in M$, we have

$$Nc^n\delta < d(i_1,\dots,i_n)\operatorname{sign} a_{i_n} < \frac{\alpha c^n N}{Nc-1} \tag{12}$$

and

$$\begin{aligned} f_{n+1}(O(I(i_1,\dots,i_n,i'))) - f_{n+1}(O(I(i_1,\dots,i_n,i))) \\ = 2^{-1}c^n(y_{i'+1}+y_{i'}-y_{i+1}-y_i) + N^{-1}(i'-i)d(i_1,\dots,i_n). \end{aligned} \tag{13}$$

Let

$$N_i = \begin{cases} 0 & \text{if } a_i > 0,\\ N-1 & \text{if } a_i < 0. \end{cases}$$

Since $y_0 = y_N = 0$, we obtain

$$\begin{gathered} y_{N_i+1} + y_{N_i} - a_{N_i}\operatorname{sign} a_i = 0,\\ y_{N-N_i} + y_{N-1-N_i} - a_{N-1-N_i}\operatorname{sign} a_i = 0 \end{gathered}$$

and, by (13),

$$\begin{aligned} \inf f_{n+1}(I(i_1,\dots,i_n,i)) - \inf f_n(I(i_1,\dots,i_n)) \\ = N^{-1}(|i-N_{i_n}|+1/2)|d(i_1,\dots,i_n)| \\ + 2^{-1}c^n(y_{i+1}+y_i) - 2^{-1}|d(i_1,\dots,i_n,i)|, \end{aligned}$$

$$\begin{aligned}
&\sup f_{n+1}(I(i_1,\dots,i_n,i)) - \sup f_n(I(i_1,\dots,i_n)) \\
&\qquad = -N^{-1}(|i-N+1+N_{i_n}|+1/2)|d(i_1,\dots,i_n)| \\
&\qquad\quad + 2^{-1}c^n(y_{i+1}+y_i) + 2^{-1}|d(i_1,\dots,i_n,i)|.
\end{aligned}$$

Therefore, for $i'_1,\dots,i'_{i_n},i',i_1,\dots,i_n,i \in M$,

$$\begin{aligned}
&\inf f_{n+1}(I(i'_1,\dots,i'_n,i')) - \sup f_{n+1}(I(i_1,\dots,i_n,i)) \\
&\qquad \geq \inf f_n(I(i'_1,\dots,i'_n)) - \sup f_n(I(i_1,\dots,i_n)) \\
&\qquad\quad + N^{-1}(|i'-N_{i'_n}|+|i-N+1+N_{i_n}|)c^n\delta - c^n(\gamma+\alpha),
\end{aligned}$$

so, for $k \geq 1$, if $i, i+km \in M$,

$$\begin{aligned}
(14)\qquad &\inf f_{n+1}(I(i_1,\dots,i_n,N_{i_n}+(i+km)\operatorname{sign} a_{i_n})) \\
&\qquad - \sup f_{n+1}(I(i_1,\dots,i_n,N_{i_n}+i\operatorname{sign} a_{i_n})) \\
&\qquad > c^n((km-1)\delta-(\gamma+\alpha)) \geq 0.
\end{aligned}$$

If

$$(15)\qquad \inf f_n(I(i'_1,\dots,i'_n)) - \sup f_n(I(i_1,\dots,i_n)) \geq 0$$

and

$$|i'-N_{i'_n}|+|i-N+1+N_{i_n}| \geq m-1,$$

then

$$\begin{aligned}
(16)\qquad &\inf f_{n+1}(I(i'_1,\dots,i'_n,i')) - \sup f_{n+1}(I(i_1,\dots,i_n,i)) \\
&\qquad > (|i'-N_{i'_n}|+|i-N+1+N_{i_n}|)c^n\delta - c^n(\gamma+\alpha) \geq 0.
\end{aligned}$$

Let

$$\begin{aligned}
&F(i_1,\dots,i_n,p_1,b) \\
&\qquad = \{i_{n+1} : ([0,1]\times\{b\}) \cap G(f_{n+1}, I(i_1,\dots,i_n,i_{n+1})) \neq \emptyset, \\
&\qquad\quad i_{n+1} = N_{i_n} + (p_1+km)\operatorname{sign} a_{i_n},\ k=0,1,\dots,k_1\}
\end{aligned}$$

for $i_1,\dots,i_n \in M,\ 0 \leq p_1 \leq m-1,\ b \in \mathbb{R}$, where $k_1 = [(N-p_1-1)m^{-1}]$. Then, by (14),

$$^{\#}F(i_1,\dots,i_n,p_1,b) \leq 1,$$

and so

$$\begin{aligned}
&^{\#}\{i_{n+1} : ([0,1]\times\times\{b\}) \cap G(f_{n+1}, I(i_1,\dots,i_n,i_{n+1})) \neq \emptyset\} \\
&\qquad = {}^{\#}\Big(\bigcup_{p_1=0}^{m-1} F(i_1,\dots,i_n,p_1,b)\Big) \leq m.
\end{aligned}$$

Similarly, let

$$
\begin{aligned}
&F(i_1,\dots,i_n,p_1,p_2,b)\\
&\qquad=\{(i_{n+1},i_{n+2}):([0,1]\times\{b\})\cap G(f_{n+2},I(i_1,\dots,i_n,i_{n+1},i_{n+2}))\neq\emptyset,\\
&\qquad\qquad i_{n+1}=N_{i_n}+(p_1+km)\operatorname{sign}a_{i_n},\\
&\qquad\qquad i_{n+2}=N_{i_{n+1}}+(p_2(k)+jm)\operatorname{sign}a_{i_{n+1}},\\
&\qquad\qquad k,j=0,1,\dots,k_1\}
\end{aligned}
$$

for $i_1,\dots,i_n\in M,\ 0\le p_1,p_2\le m-1,\ b\in\mathbb{R}$, where

$$p_2(0)=p_2,\qquad p_2(k+1)=N-1-(p_2(k)+k_1m).$$

Then, by (14)–(16),

$$^{\#}F(i_1,\dots,i_n,p_1,p_2,b)\le 1,$$

and so

$$
\begin{aligned}
&^{\#}\{(i_{n+1},i_{n+2}):([0,1]\times\times\{b\})\cap G(f_{n+2},I(i_1,\dots,i_n,i_{n+1},i_{n+2}))\neq\emptyset\}\\
&\qquad\qquad={}^{\#}\Big(\bigcup_{p_1,p_2=0}^{m-1}F(i_1,i_2,\dots,i_n,p_1,b)\Big)\le m^2.
\end{aligned}
$$

By induction,

$$^{\#}\{(i_{n+1},\dots,i_{n+l}):([0,1]\times\{b\})\cap G(f_{n+l},I(i_1,\dots,i_{n+l}))\neq\emptyset\}\le m^l.$$

Let

$$P(a,b,\varepsilon)=\{x\in[0,1]:(x,f(x))\in([a,a+\varepsilon]\times[b,b+\varepsilon])\cap G(f,[0,1])\}$$

for $\varepsilon\in(0,N^{-1})$, $a\in[0,1-\varepsilon]$, $b\in\mathbb{R}$. Suppose that

$$N^{-n-1}<\varepsilon\le N^{-n},\qquad N^{-n-1}<c^{n+l}\le N^{-n},\qquad a\in I(i_1,\dots,i_n).$$

Then, for $x\in P(a,b,\varepsilon)$, we have

$$x\in I(i_1,\dots,i_{n-1},i_n)\cup I(i_1,\dots,i_{n-1},1+i_n),$$

$$|f_{n+l}(x)-b|\le\varepsilon+A_1(1-c)^{-1}c^{n+l}\le(1+A_1(1-c)^{-1})N^{-n}=A_2N^{-n}.$$

Since

$$|f_{n+l}(I(i_1,\dots,i_{n+l}))|=|d(i_1,\dots,i_{n+l})|\ge c^{n+l}N\delta\ge N^{-n}\delta,$$

it is clear that if $x\in P(a,b,\varepsilon)\cap I(i_1,\dots,i_{n+l})$, then

$$T\cap G(f_{n+l},I(i_1,\dots,i_{n+l})\neq\emptyset,$$

where

$$T=\bigcup_{p=-k_2}^{k_2}[0,1]\times\{b+p\delta N^{-n}\},\qquad k_2=[2+A_2\delta^{-1}].$$

Since

$$\#\{I : T \cap G(f_{n+l}, I) \neq \emptyset,\ I = I(i_1, \ldots, i_{n-1}, i_n, \ldots, i_{n+l}),\\ i_n, \ldots, i_{n+l} \in M\} \leq A_3 m^l,$$

where $A_3 = 2N(2k_2 + 1) + N$, we obtain

$$|P(a, b, \varepsilon)| \leq A_3 m^l N^{-n-l} \leq A_3 N^s \varepsilon^s,$$

where

$$s = \frac{\log m}{\log N} + \frac{\log(mN^{-1})}{\log c} = 1 + \left(1 + \frac{\log c}{\log N}\right) \frac{\log(mN^{-1})}{\log c}.$$

Frostman's Lemma in [3] implies that (10) holds. This completes the proof of the Theorem.

Proof of the Corollary. Since the Weierstrass function f satisfies

$$f\left(\frac{i + x}{N}\right) = (-1)^i + \frac{1 - (-1)^i}{2} + cf(x)$$

for $0 \leq i \leq N - 1$, $x \in [0, 1]$, it follows that (4) and (8) hold for $y_i = 2^{-1}(1 + (-1)^i)$, so (2) and (3) hold, and the inequalities (10) hold. This completes the proof of the Corollary.

REFERENCES

[1] M. Barnsley, *Fractals Everywhere*, Academic Press, London, 1988.
[2] K. J. Falconer, *Fractal Geometry—Mathematical Foundations and Applications*, Wiley, New York, 1990.
[3] J.-P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge Univ. Press, London, 1985.

Department of Mathematics
Beijing Normal University
100875 Beijing, P.R. China
E-mail: denggt@bnu.edu.cn

Received 28 September 2001;
revised 15 April 2004 (4112)