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On isomorphism classes of $C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\alpha])$ spaces

by

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Abstract. We provide a complete isomorphic classification of the Banach spaces of continuous functions on the compact spaces $\mathbf{2}^{\mathfrak{m}} \oplus [0,\alpha]$, the topological sums of Cantor cubes $\mathbf{2}^{\mathfrak{m}}$, with $\mathfrak{m}$ smaller than the first sequential cardinal, and intervals of ordinal numbers $[0,\alpha]$. In particular, we prove that it is relatively consistent with ZFC that the only isomorphism classes of $C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\alpha])$ spaces with $\mathfrak{m} \geq \aleph_0$ and $\alpha \geq \omega_1$ are the trivial ones. This result leads to some elementary questions on large cardinals.

1. Introduction and statement of the main results. Given a compact Hausdorff topological space K, $C(K)$ stands for the Banach space of all continuous real-valued functions on K, equipped with the supremum norm. For a fixed cardinal number $\mathfrak{m}$, $\mathbf{2}^{\mathfrak{m}}$ denotes the product of $\mathfrak{m}$ copies of the two-point space $\mathbf{2}$, provided with the product topology.

If α is an ordinal number, $[0,\alpha]$ denotes the interval of ordinals $\{\xi : 0 \leq \xi \leq \alpha\}$ endowed with the order topology. As usual, we denote by $\aleph_0$, $\aleph_1$, ω and ω_1 the first infinite cardinal, the first uncountable cardinal, the first infinite ordinal and the first uncountable ordinal, respectively. The symbol $X \oplus Y$ will denote the Cartesian product of the Banach spaces X and Y, i.e. the space of all pairs (x,y), $x \in X$, $y \in Y$, with the norm $\|(x,y)\| = \max\{\|x\|, \|y\|\}$. We write $X \sim Y$ when the Banach spaces X and Y are isomorphic. By $X \hookrightarrow Y$ we mean that the Banach space Y contains a subspace isomorphic to the Banach space X. Other notations are standard and in conformity with [20].

This paper is concerned with the question of describing the isomorphism classes of $C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\alpha])$ spaces, where $\mathbf{2}^{\mathfrak{m}} \oplus [0,\alpha]$ is the topological sum of $\mathbf{2}^{\mathfrak{m}}$ and $[0,\alpha]$ for some cardinal $\mathfrak{m}$ and ordinal α, that is, of the family of $C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha])$ spaces. As we will see in the first three remarks below, the motivation for this work comes from some classical isomorphic

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classifications of $C(K)$ spaces closely connected with $C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha])$ spaces.

Remark 1.1. First of all, notice that this family of spaces includes the finite-dimensional spaces $\mathbb{R}^{\mathbf{2}^{\mathfrak{m}}+\alpha}$ as well as the $C([0,\alpha])$ spaces with $\alpha \geq \omega$. The isomorphic classification of $C([0,\alpha])$ spaces was accomplished by Bessaga and Pełczyński [2] in the case where $\omega \leq \alpha < \omega_1$; by Semadeni [19] in the case where $\omega_1 < \alpha \leq \omega_1\omega$; by Labbé [11] in the case where $\omega_1\omega < \alpha < \omega_1^{\omega}$; and independently by Kislyakov [10] and Gul'ko and Os'kin [8] in the general case.

Remark 1.2. For $\mathfrak{m} = \aleph_0$ and $\alpha < \omega_1$, it follows from the classical Milyutin theorem [20, Theorem 21.5.10] about the isomorphic classification of $C(K)$ spaces, with K compact metric uncountable, that every space of this family is isomorphic to $C(\mathbf{2}^{\aleph_0})$.

Remark 1.3. Assume $\mathfrak{m} \geq \aleph_0$ and $\alpha < \omega_1$. Then $\mathbf{2}^{\mathfrak{m}}$ is homeomorphic to the topological product $\mathbf{2}^{\mathfrak{m}} \times \mathbf{2}^{\aleph_0}$. Hence $C(\mathbf{2}^{\aleph_0})$ is isomorphic to a complemented subspace of $C(\mathbf{2}^{\mathfrak{m}})$. That is, there exists a Banach space Y such that $C(\mathbf{2}^{\mathfrak{m}})$ is isomorphic to $Y \oplus C(\mathbf{2}^{\aleph_0})$. Therefore, by the above mentioned Milyutin theorem we deduce

$$C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha]) \sim Y \oplus C(\mathbf{2}^{\aleph_0}) \oplus C([0,\alpha]) \sim Y \oplus C(\mathbf{2}^{\aleph_0}) \sim C(\mathbf{2}^{\mathfrak{m}}).$$

Consequently, if $\mathfrak{m}, \mathfrak{n} \geq \aleph_0$ and $\alpha, \beta < \omega_1$, then by [20, Corollary 8.2.7],

$$C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha]) \sim C(\mathbf{2}^{\mathfrak{n}}) \oplus C([0,\beta]) \quad \text{if and only if} \quad \mathfrak{m} = \mathfrak{n}.$$

Remark 1.4. Now we turn to the cases where both $\mathfrak{m}$ and α are large: $\mathfrak{m} \geq \aleph_0$ and $\alpha \geq \omega_1$. Suppose that $\mathfrak{n} \geq \aleph_0$ and $\beta \geq \omega$. Then $C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha]) \sim C(\mathbf{2}^{\mathfrak{n}}) \oplus C([0,\beta])$ implies that $\mathfrak{m} = \mathfrak{n}$. Indeed, assume that $\mathfrak{m} < \mathfrak{n}$ and let Γ and Λ be two sets of the same cardinality as α and β, respectively. According to [17, Proposition 5.2],

$$\Big(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}}\Big)_1 \oplus l_1(\Gamma) \sim \Big(\sum_{2^{\mathfrak{n}}} \oplus L^1[0,1]^{\mathfrak{n}}\Big)_1 \oplus l_1(\Lambda).$$

Recall that given a Banach space X, the *dimension* of X is the smallest cardinal δ for which there exists a subset of cardinality δ with linear span norm-dense in X. Pick a subspace H of $L^1[0,1]^{\mathfrak{n}}$ which is isomorphic to a Hilbert space of dimension $\mathfrak{n}$ [18, Proposition 1.5]. Hence

$$H \hookrightarrow \Big(\sum_{2^{\mathfrak{m}}} \oplus L^1[0,1]^{\mathfrak{m}}\Big)_1 \oplus l_1(\Gamma).$$

Since H contains no subspace isomorphic to l_1, by a standard gliding hump argument (see [3]) we infer that there exist a finite sum of $L^1[0,1]^{\mathfrak{m}}$ and

$1 \le p < \omega$ such that
$$H \hookrightarrow L^1[0,1]^{\mathfrak{m}} \oplus \cdots \oplus L^1[0,1]^{\mathfrak{m}} \oplus \mathbb{R}^p,$$
which is absurd, because the dimension of $L^1[0,1]^{\mathfrak{m}}$ is clearly $\mathfrak{m}$.

Remark 1.5. If $\mathfrak{m} \geq \aleph_0$ then $C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha]) \not\sim C(\mathbf{2}^{\mathfrak{m}})$ for every $\alpha \geq \omega_1$. Indeed, suppose $C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha]) \sim C(\mathbf{2}^{\mathfrak{m}})$ for some $\alpha \geq \omega_1$. Let Γ be the set of isolated points of $[0,\alpha]$ and denote by $C_0(\Gamma)$ the classical Banach space of all functions defined on Γ such that for every $\epsilon > 0$ the set $\{\gamma \in \Gamma : |f(\gamma)| \geq \epsilon\}$ is finite. Thus $C_0(\Gamma) \hookrightarrow C([0,\alpha]) \hookrightarrow C(\mathbf{2}^{\mathfrak{m}})$.

Now recall that a topological space K is said to satisfy the *countable chain condition* (ccc) if every uncountable family of open subsets of K contains two distinct sets with nonempty intersection. Since $C_0(\Gamma) \hookrightarrow C(\mathbf{2}^{\mathfrak{m}})$, it follows from [17, Theorem 4.5] that $\mathbf{2}^{\mathfrak{m}}$ would not satisfy the ccc, which is absurd by [6, Theorem 2.3.17].

In order to present a complete isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha])$ spaces, we will state a more general result on isomorphic classification of some Banach spaces. To do this, we recall that a Banach space X is said to have the *Mazur property* if every element of X^{**}, the bidual space of X, which is sequentially weak* continuous is weak* continuous and thus is an element of X. Such spaces were investigated in [5], [12] and also in [9] and [21] where they were called d-complete and μB-spaces, respectively. Section 2 is devoted to proving the following isomorphic classification theorem for $X \oplus C([0,\alpha])$ spaces:

Theorem 1.6. *Let X be a Banach space having the Mazur property and $\alpha, \beta \geq \omega_1$. If $X \oplus C([0,\alpha]) \sim X \oplus C([0,\beta])$ then $C([0,\alpha]) \sim C([0,\beta])$.*

Before applying Theorem 1.6, we need to recall a concept which had its origins in the study of continuity of functions on large Cartesian products. Following Noble [14] and Antonovskiĭ–Chudnovskiĭ [1], we say that a cardinal $\mathfrak{m}$ is *sequential* if there exists a sequentially continuous but not continuous real-valued function on $\mathbf{2}^{\mathfrak{m}}$. We recall that a function $f : \mathbf{2}^{\mathfrak{m}} \to \mathbb{R}$ is said to be *sequentially continuous* if $f(k_n)$ converges to $f(k)$ whenever the sequence $(k_n)_{n<\omega}$ converges to k in $\mathbf{2}^{\mathfrak{m}}$.

Remark 1.7. Important for us is a result due to Plebanek which states that $C(\mathbf{2}^{\mathfrak{m}})$ has the Mazur property for every nonsequential cardinal $\mathfrak{m}$ [15] (see also [16, Theorem 5.2.c]).

Remark 1.8. Mazur [13] showed that the first sequential cardinal $\mathfrak{s}$ is weakly inaccessible. Hence $\omega_1 < \mathfrak{s}$. Moreover, there are many weakly inaccessible cardinals before $\mathfrak{s}$ [4]. On the other hand, let $\mathfrak{m}_{\mathbb{R}}$ and $\mathfrak{m}_2$ denote the least real-valued measurable cardinal and two-valued measurable cardinal, respectively [7]. It is well-known that $\mathfrak{s} \le \mathfrak{m}_{\mathbb{R}}$; $\mathfrak{s} \le 2^{\aleph_0}$ or $\mathfrak{s} = \mathfrak{m}_2$; and $\mathfrak{s} = \mathfrak{m}_2$ under Martin's axiom [1], [7] and [13].

In particular, it is relatively consistent with ZFC that there exist no sequential cardinals [16]. Therefore, keeping in mind the above remarks, it is also consistent with ZFC that Corollary 1.9 completes the isomorphic classification of $C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha])$ spaces.

Corollary 1.9. *Suppose that* $\mathfrak{m}$ *is a nonsequential cardinal and* $\alpha, \beta \ge \omega_1$. *If* $C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\alpha]) \sim C(\mathbf{2}^{\mathfrak{m}}) \oplus C([0,\beta])$ *then* $C([0,\alpha]) \sim C([0,\beta])$.

Remark 1.10. As another direct application of Theorem 1.6 we get the isomorphic classification of $C(K) \oplus C([0,\alpha])$ spaces where $C(K)$ has the Mazur property and $\alpha \ge \omega_1$. This includes the cases where K is first-countable [16, Corollary 3.2], or the ω_1th Cantor derived set of K is empty [9, Theorem 4.1], or K is a Corson-compact [16, Corollary 3.4].

2. Proof of Theorem 1.6. As in [2], $C([0,\alpha])$ will be denoted by $\mathbb{R}^\alpha$ and we set $\mathbb{R}^\alpha_0 = \{f \in \mathbb{R}^\alpha : f(\alpha) = 0\}$. By [2, Lemma 1.2.1], $\mathbb{R}^\alpha \sim \mathbb{R}^\alpha_0$.

Since $\mathbb{R}^\alpha$ with $\alpha \ge \omega_1$ does not have the Mazur property [21, p. 49] and finite sums of Banach spaces with the Mazur property also have this property, it follows that Theorem 1.6 is an immediate consequence of Proposition 2.5 below.

A fundamental ingredient in the proof of Proposition 2.5 is Lemma 2.1, which generalizes the following result of Bessaga and Pełczyński [2, Lemma 2]:

$$\mathbb{R}^{\alpha^\omega} \not\hookrightarrow \mathbb{R}^\alpha, \qquad \forall \alpha \ge \omega.$$

Lemma 2.1. *Let* X *be an infinite-dimensional Banach space and* $\alpha \ge \omega$. *Then* $\mathbb{R}^{\alpha^\omega} \hookrightarrow X \oplus \mathbb{R}^\alpha$ *implies that* $\mathbb{R}^\alpha \hookrightarrow X^n$ *for some* $1 \le n < \omega$.

Proof. Assume that $\mathbb{R}^{\alpha^\omega} \hookrightarrow X \oplus \mathbb{R}^\alpha_0$ and consider the ordinal λ defined by

$$\lambda = \min\{\xi \le \alpha : \exists m, 1 \le m < \omega, \text{ with } \mathbb{R}^\alpha_0 \hookrightarrow X^m \oplus \mathbb{R}^\xi_0\}.$$

Thus there exists m, $1 \le m < \omega$, such that

$$\mathbb{R}^\alpha_0 \hookrightarrow X^m \oplus \mathbb{R}^\lambda_0. \tag{1}$$

We distinguish two cases:

Case 1: λ is finite. In this case, (1) yields $\mathbb{R}^\alpha_0 \hookrightarrow X^m \oplus \mathbb{R}^\lambda \hookrightarrow X^{m+1}$, and we are done.

Case 2: λ is infinite. Then again by (1),

$$\mathbb{R}^{\lambda^\omega} \hookrightarrow \mathbb{R}^{\alpha^\omega} \hookrightarrow X \oplus \mathbb{R}^\alpha_0 \hookrightarrow X^{m+1} \oplus \mathbb{R}^\lambda_0. \tag{2}$$

Notice that if $\mathbb{R}^\lambda \hookrightarrow X^{m+1} \oplus \mathbb{R}^\xi_0$ for some $\xi < \lambda$, then by (1) we would have

$$\mathbb{R}^\alpha_0 \hookrightarrow X^{2m+1} \oplus \mathbb{R}^\xi_0,$$

which is absurd by the choice of λ. Hence

$$\mathbb{R}^{\lambda} \not\hookrightarrow X^{m+1} \oplus \mathbb{R}_0^{\xi}, \qquad \forall \xi < \lambda. \tag{3}$$

According to (2) there are operators $\pi_1 : \mathbb{R}^{\lambda^\omega} \to X^{m+1}$ and $\pi_2 : \mathbb{R}^{\lambda^\omega} \to \mathbb{R}_0^{\lambda}$, and $a \in \mathbb{R}_+$, such that for every $f \in \mathbb{R}^{\lambda^\omega}$,

$$a\|f\| \le \max\{\|\pi_1(f)\|, \|\pi_2(f)\|\} \le \|f\|. \tag{4}$$

Fix an integer N and $\epsilon > 0$ such that $aN > 1$ and $1 + \epsilon < aN$. For every $0 \le \xi < \lambda$, write

$$\varDelta_\xi^1 = (\lambda^N \xi, \lambda^N(\xi + 1)].$$

Let Y_N be the subspace of $\mathbb{R}^{\lambda^\omega}$ given by

$$\{f \in \mathbb{R}^{\lambda^\omega} : f \text{ is constant on } \varDelta_\xi^1 \text{ for all } \xi \in [0,\lambda), \text{ and}$$
$$f(\xi) = 0 \text{ for all } \xi \in [\lambda^{N+1}, \lambda^\omega]\}.$$

Clearly, Y_N is isomorphic to $\mathbb{R}^\lambda$. Thus by (3), π_1 restricted to Y_N is not an isomorphism of Y_N into X^{m+1}. So there exists $f_1 \in Y_N$ such that $\|f_1\| = 1$ and $\|\pi_1(f_1)\| \le \epsilon/2$.

We may change f_1 to $-f_1$ and assume that there exists $\xi_1 \in [0, \lambda)$ such that $f_1(\gamma) = 1$ for all $\gamma \in (\lambda^N \xi_1, \lambda^N(\xi_1 + 1)]$.

Since $\pi_2(f_1) \in \mathbb{R}_0^\lambda$, there exists $\lambda_1 < \lambda$ such that for every $\gamma \in [\lambda_1 + 1, \lambda]$, we have $|\pi_2(f_1)(\gamma)| \le \epsilon/2$.

For the second step, for every $0 \le \xi < \lambda$, write

$$\varDelta_\xi^2 = (\lambda^N \xi_1 + \lambda^{N-1}\xi, \lambda^N \xi_1 + \lambda^{N-1}(\xi + 1)].$$

Let Y_{N-1} be the subspace of $\mathbb{R}^{\lambda^\omega}$ defined by

$$\{f \in \mathbb{R}^{\lambda^\omega} : f \text{ is constant on } \varDelta_\xi^2 \text{ for all } \xi \in [0,\lambda), \text{ and}$$
$$f(\xi) = 0 \text{ for all } \xi \notin (\lambda^N \xi_1, \lambda^N(\xi_1 + 1)]\}.$$

Denote by P_{λ_1} the natural projection of $\mathbb{R}_0^\lambda$ onto $\mathbb{R}_0^{\lambda_1}$ and define the operator $\pi_1 + P_{\lambda_1}\pi_2 : \mathbb{R}^{\lambda^\omega} \to X^{m+1} \oplus \mathbb{R}_0^{\lambda_1}$ by

$$(\pi_1 + P_{\lambda_1}\pi_2)(f) = (\pi_1(f), P_{\lambda_1}(\pi_2(f))), \qquad \forall f \in \mathbb{R}^{\lambda^\omega}.$$

Since Y_{N-1} is isomorphic to $\mathbb{R}^\lambda$, and since by (3), $X^{m+1} \oplus \mathbb{R}_0^{\lambda_1}$ contains no subspace isomorphic to $\mathbb{R}^\lambda$, it follows that $\pi_1 + P_{\lambda_1}\pi_2$ restricted to Y_{N-1} is not an isomorphism of Y_{N-1} into $X^{m+1} \oplus \mathbb{R}_0^{\lambda_1}$.

Hence there exists $f_2 \in Y_{N-1}$ such that $\|f_2\| = 1$, $\|\pi_1(f_2)\| \le \epsilon/2^2$ and $|\pi_2(f_2)(\gamma)| \le \epsilon/2^2$ for every $\gamma \in [0, \lambda_1]$.

Since $\pi_2(f_2) \in \mathbb{R}_0^\lambda$, pick $\lambda_2 \in [\lambda_1 + 1, \lambda)$ such that $|\pi_2(f_2)(\gamma)| \le \epsilon/2^2$ also for all $\gamma \in [\lambda_2 + 1, \lambda]$.

We may change f_2 to $-f_2$ and suppose that there exists $\xi_2 \in [0, \lambda)$ such that $f_2(\gamma) = 1$ for all $\gamma \in (\lambda^N \xi_1 + \lambda^{N-1}\xi_2, \lambda^N \xi_1 + \lambda^{N-1}(\xi_2 + 1)]$.

Repeating this procedure N times we will find

- $f_1, \ldots, f_N \in \mathbb{R}^{\lambda^\omega}$,

- $\xi_1 < \cdots < \xi_N < \lambda$,
- $\lambda_1 < \cdots < \lambda_N < \lambda$,

such that for every $1 \le k \le N$ and for every γ belonging to

$(\lambda^N \xi_1 + \lambda^{N-1}\xi_2 + \cdots + \lambda^{N-k+1}\xi_k, \lambda^N \xi_1 + \lambda^{N-1}\xi_2 + \cdots + \lambda^{N-k+1}(\xi_k + 1)]$

we have:

- $f_k(\gamma) = \|f_k\| = 1$,
- $\operatorname{supp} f_2 \subset f_1^{-1}(1)$, $\operatorname{supp} f_3 \subset f_2^{-1}(1), \ldots, \operatorname{supp} f_k \subset f_{k-1}^{-1}(1)$,
- $\|\pi_1(f_k)\| \le \epsilon/2^k$,
- $|\pi_2(f_k)(\gamma)| \le \epsilon/2^k$, $\forall \gamma \in [\lambda_k + 1, \lambda]$,
- $|\pi_2(f_k)(\gamma)| \le \epsilon/2^k$, $\forall \gamma \in [0, \lambda_{k-1}]$, $k > 1$.

Let $f = f_1 + \cdots + f_N$. Then it is obvious that $\|f\| = N$, $\|\pi_1(f)\| \le \epsilon$, and $\|\pi_2(f)\| \le 1+\epsilon$. Finally, by (4) we conclude that $aN \le 1+\epsilon$, which is absurd by the choice of ϵ. ∎

To state the next lemmas, we need to recall some Banach spaces introduced in [8] and [10]. Let us recall that an ordinal α is said to be *regular* if the smallest ordinal cofinal with α is equal to α. Otherwise α is said to be *singular*.

Let X be a Banach space and α a regular ordinal. We denote by X_α the space of all $x^{**} \in X^{**}$ having the following property: for any limit ordinal $\beta < \alpha$ and transfinite sequence $(f_\xi)_{\xi<\beta}$ of continuous linear functionals on X with $\sup\{\|f_\xi\| : \xi < \beta\} < \infty$ and $f_\xi(x) \xrightarrow{\xi\to\beta} 0$ for every $x \in X$, we have $x^{**}(f_\xi) \xrightarrow{\xi\to\beta} 0$.

From now on, if X is a Banach space, then cX denotes the canonical image of X in X^{**}.

Remark 2.2. Clearly $cX \subset X_\alpha \subset X_{\omega_1}$ for every regular ordinal $\alpha \ge \omega_1$. Moreover, it may easily be shown that if $X \sim Y$, then

$$\frac{X_\alpha}{cX} \sim \frac{Y_\alpha}{cY}.$$

Observe also that if X has the Mazur property, then $X_{\omega_1} = cX$.

Lemma 2.3. *Let X and Y be Banach spaces and α be a regular ordinal. Then there exists an isomorphism $\Phi : X^{**} \oplus Y^{**} \to (X \oplus Y)^{**}$ satisfying*

(i) $\Phi(cX \oplus cY) = c(X \oplus Y)$.
(ii) $\Phi(X_\alpha \oplus Y_\alpha) = (X \oplus Y)_\alpha$.

Proof. Let $T : (X \oplus Y)^* \to X^* \oplus Y^*$ be the isomorphism given by $T(z^*) = (z^*_{|X}, z^*_{|Y})$ for $z^{**} \in (X \oplus Y)^*$. Then the isomorphism $T^* : (X^* \oplus Y^*)^* \to (X \oplus Y)^{**}$ is given by $(T^* z^{**})(w^*) = z^{**}(Tw^*)$ for $z^{**} \in (X^* \oplus Y^*)^*$ and $w^* \in (X \oplus Y)^*$.

Consider also the isomorphism $L : X^{**} \oplus Y^{**} \to (X^* \oplus Y^*)^*$ defined by $L(x^{**}, y^{**})(x^*, y^*) = x^{**}(x^*) + y^{**}(y^*)$ for $x^{**} \in X^{**}$, $y^{**} \in Y^{**}$, $x^* \in X^*$ and $y^* \in Y^*$.

Put $\Phi = T^* L$. Then $\Phi(x^{**}, y^{**})(w^*) = x^{**}(w^*_{|X}) + y^{**}(w^*_{|Y})$ for $x^{**} \in X^{**}$, $y^{**} \in Y^{**}$ and $w^* \in (X \oplus Y)^*$. Now it is easy to see that (i) and (ii) hold. ■

The next lemma is a generalization of a result of Gul'ko and Os'kin [8] and independently of Kislyakov [10]. Let ξ be any ordinal and α a regular ordinal. The cardinality of ξ will be denoted by $\bar{\xi}$. Let Λ^α_ξ denote the subset of $[0, \xi]$ consisting of the nonisolated points that are not limit points for any set of cardinality smaller than $\bar{\alpha}$.

LEMMA 2.4. *Let α be an uncountable regular ordinal and $\xi \in [\alpha, \alpha^2]$, with $\xi = \alpha\xi' + \delta$ and $\xi', \delta \le \alpha$. Suppose that X is a Banach space satisfying $X_\alpha = cX$. Then*

$$\frac{(X \oplus \mathbb{R}^\xi)_\alpha}{c(X \oplus \mathbb{R}^\xi)} \sim C_0(\Lambda^\alpha_\xi).$$

Proof. Let Φ be as defined in Lemma 2.3. Then by using [10, Corollary 4.1], it can be easily checked that

$$\frac{(X \oplus \mathbb{R}^\xi)_\alpha}{c(X \oplus \mathbb{R}^\xi)} = \frac{\Phi(X_\alpha \oplus \mathbb{R}^\xi_\alpha)}{\Phi(cX \oplus c\mathbb{R}^\xi)} \sim \frac{X_\alpha \oplus \mathbb{R}^\xi_\alpha}{cX \oplus c\mathbb{R}^\xi} = \frac{cX \oplus \mathbb{R}^\xi_\alpha}{cX \oplus c\mathbb{R}^\xi} \sim \frac{\mathbb{R}^\xi_\alpha}{c\mathbb{R}^\xi} \sim C_0(\Lambda^\alpha_\xi). \ ■$$

PROPOSITION 2.5. *Suppose that $\omega_1 \le \alpha \le \beta$ and X is a Banach space satisfying*

- $\mathbb{R}^\alpha \not\hookrightarrow X^n$ *for every* $1 \le n < \omega$,
- $X_{\omega_1} = cX$.

Then $X \oplus \mathbb{R}^\alpha \sim X \oplus \mathbb{R}^\beta$ implies that $\mathbb{R}^\alpha \sim \mathbb{R}^\beta$.

Proof. First we will prove that $\bar{\alpha} = \bar{\beta}$. Suppose that $\bar{\alpha} < \bar{\beta}$. Then $\alpha^\omega < \beta$. Consequently,

$$\mathbb{R}^{\alpha^\omega} \hookrightarrow \mathbb{R}^\beta \hookrightarrow X \oplus \mathbb{R}^\beta \sim X \oplus \mathbb{R}^\alpha. \tag{5}$$

Therefore by Lemma 2.1, $\mathbb{R}^\alpha \hookrightarrow X^n$ for some $1 \le n < \omega$, contradicting our hypotheses.

Next let λ be the first ordinal of cardinality $\bar{\alpha}$. There are two cases:

CASE 1: λ is a singular ordinal or λ is a regular ordinal with $\lambda^2 \le \alpha$. If $\alpha^\omega \le \beta$, then (5) holds and again we obtain a contradiction. Thus $\beta < \alpha^\omega$ and by [10, Theorem 1], we conclude that $\mathbb{R}^\alpha \sim \mathbb{R}^\beta$.

CASE 2: λ is a regular ordinal with $\alpha < \lambda^2$. Thus $X_\lambda = cX$. Write $\alpha = \lambda\alpha' + \gamma$ with $\alpha', \gamma < \lambda$. If $\lambda^2 < \beta$, then $\mathbb{R}^{\lambda^2} \hookrightarrow \mathbb{R}^\beta \hookrightarrow X \oplus \mathbb{R}^\beta \sim X \oplus \mathbb{R}^\alpha$

and according to [10, Lemmas 1.4 and 2.4] we deduce

$$C_0(\Lambda^{\lambda}_{\lambda^2}) \sim \frac{\mathbb{R}^{\lambda^2}_{\lambda}}{c\mathbb{R}^{\lambda^2}} \hookrightarrow \frac{(X \oplus \mathbb{R}^{\alpha})_{\lambda}}{c(X \oplus \mathbb{R}^{\alpha})} \sim C_0(\Lambda^{\lambda}_{\alpha}).$$

Therefore by [10, Corollary 4.1], $\bar{\lambda} \le \overline{\alpha'}$, which is absurd. So we may assume that $\beta \le \lambda^2$. Write $\beta = \lambda\beta' + \delta$, with $\beta', \delta \le \alpha$. Then Lemma 2.4 yields

$$C_0(\Lambda^{\lambda}_{\alpha}) \sim \frac{(X \oplus \mathbb{R}^{\alpha})_{\lambda}}{c(X \oplus \mathbb{R}^{\alpha})} \sim \frac{(X \oplus \mathbb{R}^{\beta})_{\lambda}}{c(X \oplus \mathbb{R}^{\beta})} \sim C_0(\Lambda^{\lambda}_{\beta}).$$

Once again by [10, Corollary 4.1] we see that $\overline{\alpha'} = \overline{\beta'}$ and by [10, Theorem 2] we conclude that $\mathbb{R}^{\alpha} \sim \mathbb{R}^{\beta}$. ∎

3. Some questions. Corollary 1.9 leads naturally to the following question.

QUESTION 3.1. *Assume that $C(\mathbf{2}^{\mathfrak{m}})$ has the Mazur property. Does it follow that $\mathfrak{m}$ is not sequential?*

As pointed out by the referee, $C(\mathbf{2}^{\mathfrak{m}_2})$ does not have the Mazur property. Moreover, he noticed that $C(\mathbf{2}^{\mathfrak{m}_2})_{\lambda} \neq cC(\mathbf{2}^{\mathfrak{m}_2})$ for every $\omega_1 \le \lambda < \mathfrak{m}_2$. Indeed, let F be an $\mathfrak{m}_2$-complete ultrafilter and x^{**} the weak*-limit along the ultrafilter F of $\{c(p_{\alpha}) : \alpha \in \mathfrak{m}_2\} \subset C(\mathbf{2}^{\mathfrak{m}_2})^{**}$, where $p_{\alpha} : \mathbf{2}^{\mathfrak{m}_2} \to \mathbf{2}$ is the αth projection. Then $x^{**} \in C(\mathbf{2}^{\mathfrak{m}_2})_{\lambda} \setminus cC(\mathbf{2}^{\mathfrak{m}_2})$ for all $\lambda < \mathfrak{m}_2$.

However, we do not know the answer to the following question.

QUESTION 3.2. *Is it true that $C(\mathbf{2}^{\mathfrak{m}})_{\omega_1} = cC(\mathbf{2}^{\mathfrak{m}})$ whenever $\aleph_1 \le \mathfrak{m} < \mathfrak{m}_2$?*

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